1.15. The Theorem by Kronecker-Kapelly

Let us consider the system of m linear algebraic equations (SLAE) with n unknown variables:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = b_2 \\ \ldots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \ldots + a_{mn}x_n = b_m \end{cases}$$

If $A = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \\ a_{m1} & \ldots & a_{mn} \end{pmatrix}$, $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $B = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$, t_i is i-th column of matrix A, then the above system can be rewritten in the following equivalent forms:

$$AX = B \qquad \text{or} \qquad t_1x_1 + t_2x_2 + t_3x_3 + \ldots + t_nx_n = B.$$

Matrix A is called the matrix of the system, B is a column of right sides, X is a column of unknowns.

Definition. If $B \neq 0$ then the system is called inhomogeneous. In other case, i.e. $B = 0$, it is called homogeneous.

Definition. Any set of numbers $x_1 = \alpha_1, x_2 = \alpha_2, \ldots, x_n = \alpha_n$ is called a solution of the system if after substituting of these numbers in the system one obtains the identity.

Definition. If the system has a solution then it is called a compatible system.

Definition. If the system has no solutions then it is called an incompatible system.

Definition. If the system has the only solution then it is called a definite system.

Definition. If the system has more then one solution then it is called an indefinite system.

Definition. Matrix $A^* = \begin{pmatrix} a_{11} & \dots & a_{1n} & b_1 \\ \vdots & & & \vdots \\ a_{m1} & \dots & a_{mn} & b_m \end{pmatrix}$ is called an extended matrix of the system.

Note. To differ elements of the matrix *A* from elements of the matrix *B* the extended matrix A^* is usually written down as

$$A^* = \left(\begin{array}{ccc|c} a_{11} & \dots & a_{1n} & b_1 \\ \vdots & & & \vdots \\ a_{m1} & \dots & a_{mn} & b_m \end{array}\right).$$

Theorem (Theorem by Kronecker-Kapelly) In order to SLAE be compatible it is necessary and sufficient for the ranks of matrices *A* and A^* to be equal, i.e. $rg(A) = rg(A^*)$.

Proof. Necessity: SLAE has a solution $x_1 = \alpha_1, x_2 = \alpha_2, \dots, x_n = \alpha_n$. From the third record of system we have

$$t_1\alpha_1 + t_2\alpha_2 + t_3\alpha_3 + \dots + t_n\alpha_n = B,$$

i.e. *B* which is the last column of A^* is linear combination of the other columns of A^*. It means that *B* does not increase the number of linear independent columns of A^* with respect to *A*, so $rg(A) = rg(A^*)$.

Sufficiency: $rg(A) = rg(A^*) = r$. It means that basic minor of *A* can be chosen as basic minor of A^*. But from the theorem about basic rows and columns it means that *B* is a linear combination of the basic columns, i.e. of some columns of *A*:

$$t_{i_1}\alpha_{i_1} + t_{i_2}\alpha_{i_2} + \ldots + t_{i_r}\alpha_{i_r} = B.$$

Let us complete the sum from the left side of expression to full sum of columns by missing columns multiplied by zeros. Then according to the definition of the solution the coefficients of the obtained sum are solution of the system and the system is compatible. ***Theorem is proven.***

Note. It is simple to prove by means of the rule by Cramer that:

- If $rg(A) = rg(A^*) = n$ then system is definite;
- If $rg(A) = rg(A^*) < n$ then system is indefinite.

Let us demonstrate the second statement on the next example.

Example 1. Let us solve the system

$$\begin{cases} x_1 - 2x_2 + 3x_3 - x_4 = 4 \\ -2x_1 + 4x_2 - x_3 + 2x_4 = -3 \end{cases}.$$

Since the number of unknowns is greater than the number of equations then $rg(A) < n$ and if there are any solutions then the system is indefinite. Let us write down the extended matrix of the system

$$A^* = \left(\begin{array}{cccc|c} 1 & -2 & 3 & -1 & 4 \\ -2 & 4 & -1 & 2 & -3 \end{array}\right).$$

$$\begin{vmatrix} 1 & -2 \\ -2 & 4 \end{vmatrix} = 4 - 4 = 0,$$

but

$$\begin{vmatrix} 1 & 3 \\ -2 & -1 \end{vmatrix} = -1 + 6 = 5 \neq 0.$$

Thus $rg(A) = rg(A^*) = 2 < n = 4$ and system is compatible and indefinite.

Let us rewrite the system by leaving to the left only the unknowns corresponding to basic columns:

$$\begin{cases} x_1 + 3x_3 = 4 + 2x_2 + x_4 \\ -2x_1 - x_3 = -3 - 4x_2 - 2x_4 \end{cases}.$$

Since the determinant of the obtained system for variables x_1, x_3 is not equal to zero it can be solved by rule by Cramer.

$$x_1 = \frac{\begin{vmatrix} 4+2x_2+x_4 & 3 \\ -3-4x_2-2x_4 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 3 \\ -2 & -1 \end{vmatrix}} = \frac{-4-2x_2-x_4-3(-3-4x_2-2x_4)}{5} = 1+2x_2+x_4,$$

$$x_3 = \frac{\begin{vmatrix} 1 & 4+2x_2+x_4 \\ -2 & -3-4x_2-2x_4 \end{vmatrix}}{\begin{vmatrix} 1 & 3 \\ -2 & -1 \end{vmatrix}} = \frac{-3-4x_2-2x_4+2(4+2x_2+x_4)}{5} = 1,$$

x_2, x_4 are arbitrary.

Note, that some unknowns are expressed through the others. By assigning any values to x_2, x_4 we get a lot of particular solutions of this system.

Example 2. Let us solve the system

$$\begin{cases} x_1 - 2x_2 + 3x_3 - x_4 = 4 \\ -2x_1 + 4x_2 - x_3 + 2x_4 = -3 \\ -x_1 + 2x_2 + 2x_3 + x_4 = 1 \end{cases}.$$

We will write down the extended matrix of the system and by means of elementary row operations reduce this matrix to row echelon form. Since we work only with rows what is equivalent to elementary operations (summarizing, adding, subtracting, multiplying by nonzero numbers, changing of the order) on equations, the system stays the same.

$$A^* = \left(\begin{array}{cccc|c} 1 & -2 & 3 & -1 & 4 \\ -2 & 4 & -1 & 2 & -3 \\ -1 & 2 & 2 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & -2 & 3 & -1 & 4 \\ 0 & 0 & 5 & 0 & 5 \\ 0 & 0 & 5 & 0 & 5 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & -2 & 3 & -1 & 4 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Since $rg(A) = rg(A^*) = 2 < n = 4$ the system is compatible and indefinite.

It is appeared that the third equation in the initial system is linear combination of others equations. So to find solution it is enough to consider only the first two equations what was done in the Example 1.

To check the result we write down the system corresponding to the last extended matrix and compare solutions:

$$\begin{cases} x_1 - 2x_2 + 3x_3 - x_4 = 4 \\ 0x_1 + 0x_2 + 1x_3 + 0x_4 = 1 \end{cases} \Leftrightarrow$$

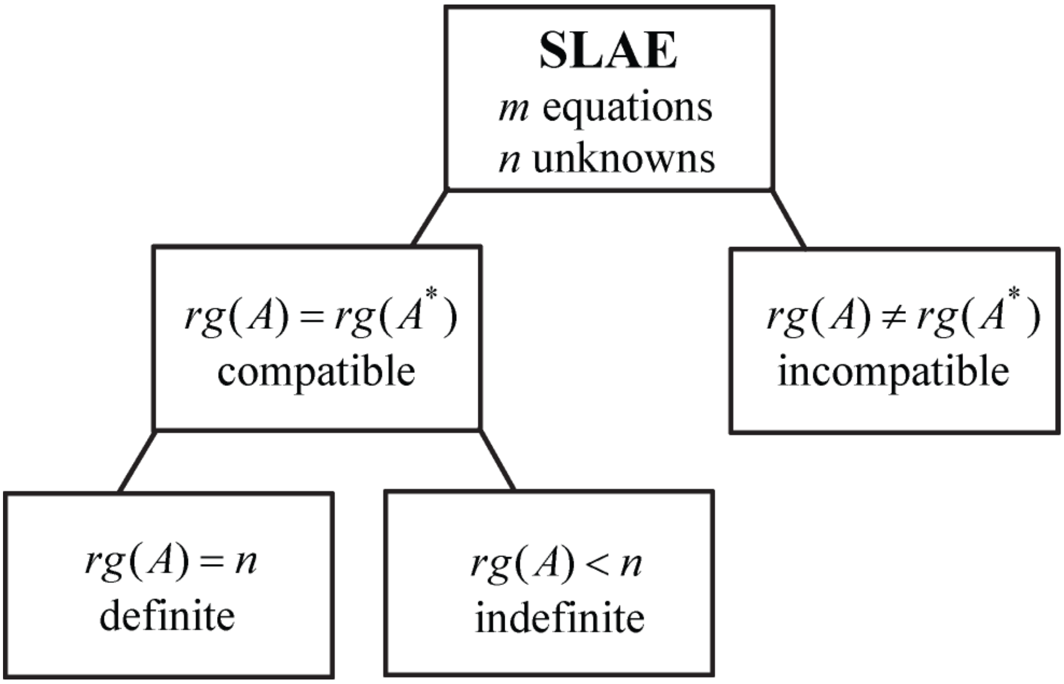
$$\Leftrightarrow \begin{cases} x_1 - 2x_2 + 3x_3 - x_4 = 4 \\ x_3 = 1 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x_1 = 4 + 2x_2 - 3x_3 + x_4 \\ x_3 = 1 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x_1 = 1 + 2x_2 + x_4 \\ x_3 = 1 \end{cases}, \ x_2, x_4 \text{ are arbitrary.}$$

Note 1. The solution of the indefinite system written as function of some arbitrary values is called *the general solution* of the system. Any solution calculated from general by substituting some certain values instead of arbitrary ones is called *the particular solution*.

Note 2. The plan to investigate the SLAE on compatibility can be described by the following diagram:



1.16. Homogeneous Systems
Construction of the Fundamental System of Solutions

Let us consider the homogeneous system of *m* linear algebraic equations with *n* unknown variables

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = 0 \\ \ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = 0 \end{cases} \quad \text{or} \quad AX = 0 \quad \text{or} \quad t_1x_1 + t_2x_2 + \ldots + t_nx_n = 0$$

Since $B = 0$ in the homogeneous system (HS) and zero column does not increase the number of linear independent columns in the extended matrix with respect to matrix of the system, *the homogeneous system is always compatible*.

Actually, It is obvious, since the homogeneous system always has a zero (trivial) solution. The question is when does it have nontrivial solution?

Theorem. For the homogeneous system to have nontrivial solution it is necessary and sufficient that $rg(A) < n$.

Proof. Necessity: If we have nontrivial solution then

$$t_1\alpha_1 + t_2\alpha_2 + t_3\alpha_3 + \ldots + t_n\alpha_n = 0, \ |\alpha_1| + |\alpha_2| + \ldots + |\alpha_n| \neq 0.$$

But it means that columns of the matrix *A* are linear dependent so $rg(A) \neq n$ and thus $rg(A) < n$.

Sufficiency: If $rg(A) < n$ then n columns of the matrix A are linear dependent and there is a set of numbers such that

$$|\alpha_1| + |\alpha_2| + \ldots + |\alpha_n| \neq 0 \text{ and } t_1\alpha_1 + t_2\alpha_2 + t_3\alpha_3 + \ldots + t_n\alpha_n = 0.$$

It means that this set of numbers is a nontrivial solution of the system. ***Theorem is proven.***

Note. It follows from the theorem, that for the homogeneous system of *n* equations with *n* variables to have nontrivial solution it is necessary and sufficient that the determinant of the system matrix is equal to zero, i.e. the homogeneous system with square matrix is indefinite if and only if $\det(A) = 0$.

So, if $rg(A) < n$ then the system $AX = 0$ is indefinite and has infinite number of solutions. But how many of them are linearly independent?

Note 1. When we say about the linear dependence of solutions we consider solutions as columns $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$ and investigate linear dependence of columns.

Note 2. Linear combination of the solutions of homogeneous system is also a solution of this system. Indeed, suppose Y_1, Y_2 are the solutions of the system $AX = 0$, i.e. $AY_1 = 0$, $AY_2 = 0$. Then

$$A(\alpha Y_1 + \beta Y_2) = A(\alpha Y_1) + A(\beta Y_2) = \alpha AY_1 + \beta AY_2 = \alpha 0 + \beta 0 = 0,$$

i.e. $\alpha Y_1 + \beta Y_2$ is also a solution.

Definition. Fundamental system of solutions (FSS) of the homogeneous system is any maximum set of linearly independent solutions.

Note. It follows from the definition that:

1) Only indefinite homogeneous systems have FSS.
2) Choice of the FSS is not unique.

Theorem (About Fundamental System of Solutions)

(i) If $r = rg(A) < n$ then the homogeneous system has a fundamental system of (*n*-*r*) solutions;

(ii) Any solution of the system is a linear combination solutions from FSS.

Proof. Suppose the basic minor stands in the upper left corner of the matrix *A*. Then the first *r* rows are linearly independent and all other rows (equations) are linear combination of the basic rows and, thus, do not contain helpful information to find a solution. So let us consider only the first *r* rows written in the following form:

$$\begin{pmatrix} a_{11} \\ \vdots \\ a_{r1} \end{pmatrix} x_1 + \begin{pmatrix} a_{12} \\ \vdots \\ a_{r2} \end{pmatrix} x_2 + \ldots + \begin{pmatrix} a_{1r} \\ \vdots \\ a_{rr} \end{pmatrix} x_r = -\begin{pmatrix} a_{1r+1} \\ \vdots \\ a_{rr+1} \end{pmatrix} x_{r+1} - \ldots - \begin{pmatrix} a_{1n} \\ \vdots \\ a_{rn} \end{pmatrix} x_n .$$

The determinant of the obtained system for the unknowns $x_1, x_2, \ldots, x_r$ is not equal to zero, i.e. it is basic minor, and we can find values of the unknowns $x_1, x_2, \ldots, x_r$ as functions of other unknowns by means of rule by Cramer. In this case substituting instead of unknowns $x_{r+1}, x_{r+2}, \ldots, x_n$ some values, we get particular solutions of the initial system. Let us consider the following set of (*n*-*r*) particular solutions:

$$\begin{matrix} x_{r+1}=1 \\ x_{r+2}=0 \\ x_{r+3}=0 \\ \vdots \\ x_n=0 \end{matrix} \Rightarrow X_1 = \begin{pmatrix} \alpha_{11} \\ \alpha_{12} \\ \vdots \\ \alpha_{1r} \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \begin{matrix} x_{r+1}=0 \\ x_{r+2}=1 \\ x_{r+3}=0 \\ \vdots \\ x_n=0 \end{matrix} \Rightarrow X_2 = \begin{pmatrix} \alpha_{21} \\ \alpha_{22} \\ \vdots \\ \alpha_{2r} \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \begin{matrix} x_{r+1}=0 \\ x_{r+2}=0 \\ x_{r+3}=1 \\ \vdots \\ x_n=0 \end{matrix} \Rightarrow X_3 = \begin{pmatrix} \alpha_{31} \\ \alpha_{32} \\ \vdots \\ \alpha_{3r} \\ 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix},$$

$$\ldots, \begin{matrix} x_{r+1}=0 \\ x_{r+2}=0 \\ x_{r+3}=0 \\ \vdots \\ x_n=1 \end{matrix} \Rightarrow X_{n-r} = \begin{pmatrix} \alpha_{n-r1} \\ \alpha_{n-r2} \\ \vdots \\ \alpha_{n-rr} \\ 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$

The matrix of the order *n* by (*n*-*r*) constructed on these columns has rank equal to (*n*-*r*) since there is unit matrix of the (*n*-*r*)th order in the bottom of it. It means that all these columns (solutions) are linearly independent.

Let us consider now an arbitrary solution of the system $X_0 = \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix}$. Then

$$Y = X_0 - q_{r+1}X_1 - q_{r+2}X_2 - \ldots - q_n X_{n-r} =$$

$$
= \begin{pmatrix} q_1 - & q_{r+1}\alpha_{11} & -q_{r+2}\alpha_{21} & -q_{r+3}\alpha_{31} & -\ldots - q_n\alpha_{n-r1} \\ q_2 - & q_{r+1}\alpha_{12} & -q_{r+2}\alpha_{22} & -q_{r+3}\alpha_{32} & -\ldots - q_n\alpha_{n-r2} \\ \vdots & & & & \\ q_r - & q_{r+1}\alpha_{1r} & -q_{r+2}\alpha_{2r} & -q_{r+3}\alpha_{3r} & -\ldots - q_n\alpha_{n-rr} \\ q_{r+1} - & q_{r+1}\cdot 1 & -q_{r+2}\cdot 0 & -q_{r+3}\cdot 0 & -\ldots \quad -q_n\cdot 0 \\ q_{r+2} - & q_{r+1}\cdot 0 & -q_{r+2}\cdot 1 & -q_{r+3}\cdot 0 & -\ldots \quad -q_n\cdot 0 \\ q_{r+3} - & q_{r+1}\cdot 0 & -q_{r+2}\cdot 0 & -q_{r+3}\cdot 1 & -\ldots \quad -q_n\cdot 0 \\ \vdots & & & & \\ q_n - & q_{r+1}\cdot 0 & -q_{r+2}\cdot 0 & -q_{r+3}\cdot 0 & -\ldots \quad -q_n\cdot 1 \end{pmatrix} =
$$

$$
= (\gamma_1 \quad \gamma_2 \quad \ldots \quad \gamma_r \quad 0 \quad \ldots \quad 0)^T
$$

is a solution as well, and thus

$$
\gamma_1 t_1 + \gamma_2 t_2 + \ldots + \gamma_r t_r + 0t_{r+1} + \ldots + 0t_n = \gamma_1 t_1 + \gamma_2 t_2 + \ldots + \gamma_r t_r = 0 .
$$

Since we obtained zero linear combination of the basic linearly independent columns then $\gamma_1 = 0, \gamma_2 = 0, \ldots, \gamma_r = 0$, i.e.

$$
Y = X_0 - q_{r+1}X_1 - q_{r+2}X_2 - \ldots - q_n X_{n-r} = 0
$$

and

$$
X_0 = q_{r+1}X_1 + q_{r+2}X_2 + \ldots + q_n X_{n-r} .
$$

It means that any other solution is linear combination of X_1, X_2, …X_{n-r} and can not increase number of linearly independent columns. Thus X_1, X_2, …X_{n-r} form FSS and any solution is a a linear combination of solutions from FSS.

Theorem is proven.

Theorem (about general solution of inhomogeneous system). General solution of the inhomogeneous system $AX = B$ is a sum of the particular solution of the inhomogeneous system and linear combination of solutions from the FSS of homogeneous system $AX = 0$.

Proof. Suppose X is an arbitrary solution and X_0 is some particular solution of the system $AX = B$.

Then $AY = A(X - X_0) = AX - AX_0 = B - B = 0$ and $Y = X - X_0$ is the solution of the homogeneous system and thus equal to the linear combination of solutions of the FSS.

Thus, $X = Y + X_0$ is a sum of the particular solution of the inhomogeneous system and linear combination of the FSS. *Theorem is proven.*

1.17. Method by Gauss (Method of Sequential Elimination of the Unknown Variables) Method by Jordan-Gauss

Method by Gauss is used to solve the system of the linear algebraic equations with arbitrary numbers of equations and unknowns.

It includes sequential elimination of the variables from equations (i.e. vanishing of its coefficient in the equations) according to the following scheme:

Step 1. Form the extended matrix of the system.

Step 2.

- Choose the leading equation and the leading variable (its coefficient in the leading equation has to be nonzero). Put the row of this equation on the first place. Eliminate the leading variable from the other rows below the leading one (i.e. from other equations) by the elementary row operations.
- Then choose new leading equation and new leading variable. Put the row of this equation on the second place and eliminate new leading variable from all other rows below this one.
- Then choose new equation and new variable and so on.
- After such manipulations the obtained matrix with columns rewritten in the order of the chosen leading variables has the row echelon (or trapezoidal) form.

Note. It is preferred to choose the leading variables in the natural order to get exactly row echelon form of the system matrix.

Step 3. Determine the ranks of the system matrix A and the extended matrix A^* and write down the system corresponding to the obtained extended matrix.

Step 4.

- If $rg(A) = rg(A^*) = n$, where n is a number of unknowns, then the system has the only solution which can be calculated from the obtained system.
- If $rg(A) = rg(A^*) < n$ then choose basic minor of the triangular form (for example, consisting of the columns of leading variables). Unknown variables whose coefficients correspond to this minor are called *basic (or main) variables*. All other are called *free (or independent) variables*. Solve the system expressing the main variables through the free ones. Start from the last equation. The obtained equalities are the general solution of the system. Assigning any values to free variables one gets the particular solution of the system.

Example. Let us solve the system of linear equations by the method by Gauss:

$$\begin{cases} x_1 + x_2 - x_3 - x_4 = -1 \\ 2x_1 - 3x_2 - 2x_3 + 3x_4 = 8 \\ x_1 - x_2 - x_3 + x_4 = 3 \\ 3x_1 + x_2 - 3x_3 - 2x_4 = -1 \end{cases}$$

Let us write down the extended matrix of the system and carry out the transformations:

$$\left(\begin{array}{cccc|c} 1 & 1 & -1 & -1 & -1 \\ 2 & -3 & -2 & 3 & 8 \\ 1 & -1 & -1 & 1 & 3 \\ 3 & 1 & -3 & -2 & -1 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1 & -1 & -1 & -1 \\ 0 & -5 & 0 & 5 & 10 \\ 0 & -2 & 0 & 2 & 4 \\ 0 & -2 & 0 & 1 & 2 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1 & -1 & -1 & -1 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & -2 & 0 & 1 & 2 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{cccc|c} 1 & 1 & -1 & -1 & -1 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & -6 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1 & -1 & -1 & -1 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1 & -1 & -1 & -1 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

We got the matrix in the row echelon form. $rg(A) = rg(A^*) = 3$. Number of variables is equal to 4. Thus the system is compatible and indefinite. We should choose main and free variables.

$\begin{vmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{vmatrix} = 1 \neq 0$. So, it can be chosen as basic minor and variables x_1, x_2, x_4 are main, x_3 is free.

Let us write down the system:

$$\begin{cases} 1 \cdot x_1 + 1 \cdot x_2 - 1 \cdot x_3 - 1 \cdot x_4 = -1 \\ 0 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 - 1 \cdot x_4 = -2 \\ 0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 + 1 \cdot x_4 = 2 \end{cases} \Leftrightarrow \begin{cases} x_1 + x_2 - x_3 - x_4 = -1 \\ x_2 - x_4 = -2 \\ x_4 = 2 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x_1 = -1 - x_2 + x_3 + x_4 = 1 + x_3 \\ x_2 = -2 + x_4 = 0 \\ x_4 = 2 \end{cases}$$

So, answer is

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_3 + 1 \\ 0 \\ x_3 \\ 2 \end{pmatrix} = \begin{pmatrix} x_3 \\ 0 \\ x_3 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} x_3 + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 2 \end{pmatrix}, x_3 \in R.$$

Notice, that according to the theorem about general solution of inhomogeneous system, the column at x_3 is a solution of the homogeneous system and free column is a particular solution of the inhomogeneous system. Let us check the result:

$$\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} : \begin{cases} 1 + 0 - 1 - 0 = 0 \\ 2 \cdot 1 - 3 \cdot 0 - 2 \cdot 1 + 3 \cdot 0 = 0 \\ 1 - 0 - 1 + 0 = 0 \\ 3 \cdot 1 + 0 - 3 \cdot 1 - 2 \cdot 0 = 0 \end{cases} \qquad \begin{pmatrix} 1 \\ 0 \\ 0 \\ 2 \end{pmatrix} : \begin{cases} 1 + 0 - 0 - 2 = -1 \\ 2 \cdot 1 - 3 \cdot 0 - 2 \cdot 0 + 3 \cdot 2 = 8 \\ 1 - 0 - 0 + 2 = 3 \\ 3 \cdot 1 + 0 - 3 \cdot 0 - 2 \cdot 2 = -1 \end{cases}$$

Note. A modification of the method by Gauss where the leading variable is eliminated not only from the below rows (equations) but from all other rows (equations) is called *the method by Jordan-Gauss*. In this method the extended matrix is reduced to *the row reduced echelon form.*

Example. Let us solve the homogeneous system of equation and find its fundamental system of solutions. Since the difference between the matrix of the system and extended matrix is in zero column we will work only with matrix of the system.

$$\begin{cases} x_1+x_2-x_3-x_4+x_5=0 \\ 2x_1-3x_2-x_3+3x_4-x_5=0 \\ x_1-x_2-x_3+x_4=0 \end{cases} \Rightarrow$$

$$\begin{pmatrix} 1 & 1 & -1 & -1 & 1 \\ 2 & -3 & -1 & 3 & -1 \\ 1 & -1 & -1 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -1 & -1 & 1 \\ 0 & -5 & 1 & 5 & -3 \\ 0 & -2 & 0 & 2 & -1 \end{pmatrix} \sim$$

~[Add to the second row the third one multiplied by (–3)] ~

$$\sim \begin{pmatrix} 1 & 1 & -1 & -1 & 1 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & -2 & 0 & 2 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -2 & 0 & 1 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 2 & 0 & -1 \end{pmatrix} \sim$$

~[Add to the first row the third one multiplied by 1] ~

$$\sim \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 2 & 0 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & -2 & 0 & 1 \end{pmatrix}.$$

$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \neq 0$. So the leading variables x_1, x_2, x_5 are main and x_3, x_4 are free.

New system is $\begin{cases} x_1=0 \\ x_2+x_3-x_4=0 \\ -2x_3+x_5=0 \end{cases} \Leftrightarrow \begin{cases} x_1=0 \\ x_2=-x_3+x_4 \\ x_5=2x_3 \end{cases}$ and the general solution is

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 2 \end{pmatrix} x_3 + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} x_4, \quad x_3, x_4 \in R. \text{ Here } FSS = \left\{ \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$