Practice 1. (Linear Algebra)

Topic: Rank of the matrix.

Example. Calculate the rank of the matrix: $A=\begin{pmatrix} 0 & 2 & -4 \\ -1 & -4 & 5 \\ 3 & 1 & 7 \\ 0 & 5 & -10 \\ 2 & 3 & 0 \end{pmatrix}$.

Solution. Let us interchange the 1st and the 2nd rows.

$$A\sim\begin{pmatrix} -1 & -4 & 5 \\ 0 & 2 & -4 \\ 3 & 1 & 7 \\ 0 & 5 & -10 \\ 2 & 3 & 0 \end{pmatrix}\sim\left\|\begin{matrix} e_2\cdot\frac{1}{2}\to e_2 \\ e_1\cdot 3+e_3\to e_3 \\ e_4\cdot\frac{1}{5}\to e_4 \\ e_1\cdot 2+e_5\to e_5 \end{matrix}\right\|\sim\begin{pmatrix} -1 & -4 & 5 \\ 0 & 1 & -2 \\ 0 & -11 & 22 \\ 0 & 1 & -2 \\ 0 & -5 & 10 \end{pmatrix}\sim\left\|\begin{matrix} e_3\cdot(\frac{-1}{11})\to e_3 \\ e_5\cdot(-\frac{1}{5})\to e_5 \end{matrix}\right\|\sim$$

$$\begin{pmatrix} -1 & -4 & 5 \\ 0 & 1 & -2 \end{pmatrix}.$$

The number of non-zero rows of the transformed matrix equivalent to the initial one is 2. Therefore *rang*(A)=2.

Present given matrices in row echelon form and define their ranks

1. $\begin{pmatrix} 2 & -1 & 5 & 6 \\ 1 & 1 & 3 & 5 \\ 1 & -5 & 1 & -3 \end{pmatrix}$. 2. $\begin{pmatrix} 1 & 2 & -1 & 1 & -3 \\ 3 & -1 & 1 & 6 & 11 \\ 1 & -1 & -1 & 4 & -3 \end{pmatrix}$.

Topic: Investigation of system compatibility

While solving systems of m linear equations with n unknowns at the same time with the matrix of the given system $A=\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$ we also should consider a matrix obtained from A by adding the column of free terms. This matrix is called an *extended* matrix of the given system and marked as

$$\tilde{A}=\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{pmatrix}.$$

Obviously, $rang(\tilde{A})$ is either equal to $rang(A)$ or greater on 1. The investigation of compatibility of SLAE of m equations with n unknowns is carried out using the theorem by **Kronecker – Kapelly.**

Theorem. In order to let SLAE of m equations with n unknowns be compatible it is necessary and enough for the rank of the extended matrix to be equal to the rank of the initial matrix, i.e. rang($\tilde{A}$)=rang(A)=r.

It may be shown that if $r = n$ then the system has the single solution, and if $r < n$, the system has an uncountable set of solutions, i.e. it is indefinite. The mentioned above reasonings are presented on fig.1.1:

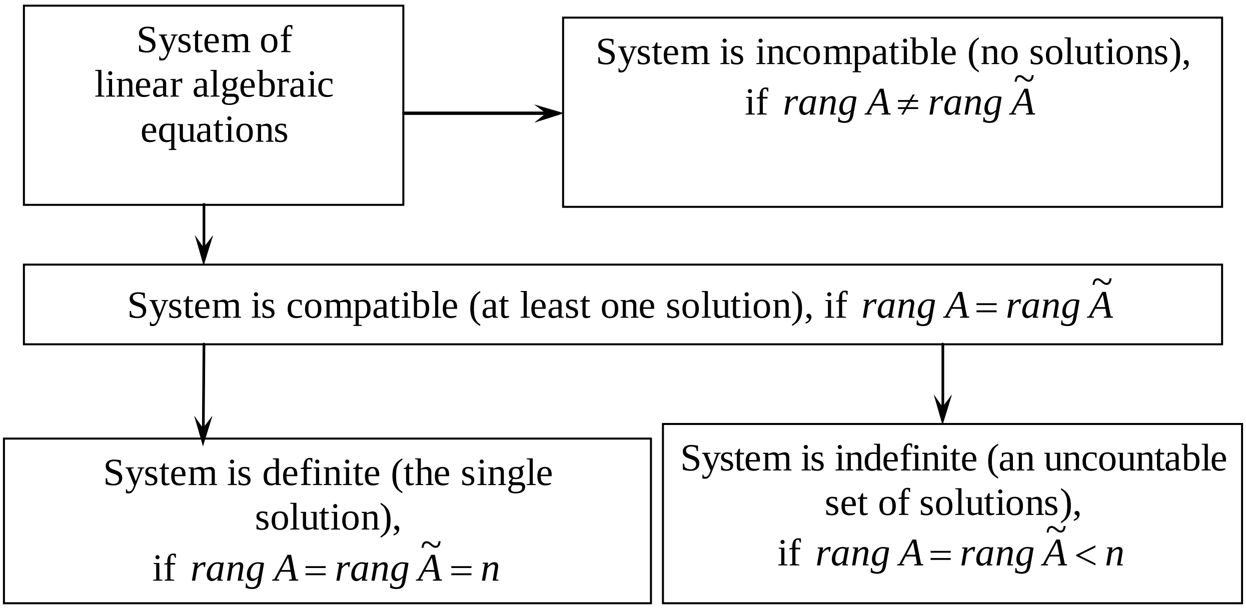

Figure 1.1

Example. Investigate the *compatibility* of the given system:

$$\begin{cases} 2x_1 - 3x_2 + 5x_3 + 7x_4 = 1 \\ 4x_1 - 6x_2 + 2x_3 + 3x_4 = 2 \\ 2x_1 - 3x_2 - 11x_3 - 15x_4 = 1 \end{cases}.$$

Solution. Let us investigate the *compatibility* of the given system:

$$\tilde{A} = \left(\begin{array}{cccc|c} 2 & -3 & 5 & 7 & 1 \\ 4 & -6 & 2 & 3 & 2 \\ 2 & -3 & -11 & -15 & 1 \end{array}\right) \sim \begin{array}{c} e_2 + e_1(-2) \to e_2 \\ e_3 + e_1(-1) \to e_3 \end{array} \sim \left(\begin{array}{ccccc} 2 & -3 & 5 & 7 & 1 \\ 0 & 0 & -8 & -11 & 0 \\ 0 & 0 & -16 & -22 & 0 \end{array}\right)$$

$$\sim \begin{array}{c} e_3 + e_2(-2) \to e_3 \\ e_2(-1) \to e_2 \end{array} \sim \sim \left(\begin{array}{cccc|c} 2 & -3 & 5 & 7 & 1 \\ 0 & 0 & 8 & 11 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 2 & -3 & 5 & 7 & 1 \\ 0 & 0 & 8 & 11 & 0 \\ & & & & \end{array}\right)$$

It is obvious that the system is compatible because the ranks of the main matrix and the extended one are equal, i.e. $rang(A)=rang(\tilde{A})=2$.

Investigate given systems on *compatibility*.

1. $$\begin{cases} x_1 - 2x_2 + 3x_3 - 4x_4 = 4 \\ x_2 - x_3 + x_4 = -3 \\ x_1 + 3x_2 - 3x_4 = 1 \\ -7x_2 + 3x_3 + x_4 = -3 \end{cases}$$

2. $$\begin{cases} x_1 + 2x_2 - x_3 + x_4 = 1; \\ 3x_1 + x_2 + 2x_3 - x_4 = 2; \\ 2x_1 + 3x_2 - x_3 + 3x_4 = 0; \\ 4x_1 + 2x_2 + 2x_3 + x_4 = 1. \end{cases}$$