Practice 3 (Linear Algebra)

Topic: Solution of a Homogeneous SLAE

Example. Solve the following homogeneous system of equations by the Gaussian method

$$\begin{cases} x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 = 0 \\ 2x_1 + 3x_2 + 4x_3 + 5x_4 + x_5 = 0 \\ 3x_1 + 4x_2 + 5x_3 + x_4 + 2x_5 = 0 \\ x_1 + 3x_2 + 5x_3 + 12x_4 + 9x_5 = 0 \\ 4x_1 + 5x_2 + 6x_3 - 3x_4 + 3x_5 = 0 \end{cases}$$

and find its fundamental system of solutions.

Solution. Let us write down the extended matrix of the system and carry out elementary transformations:

$$\tilde{A} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 1 & 2 \\ 1 & 3 & 5 & 12 & 9 \\ 4 & 5 & 6 & -3 & 3 \end{pmatrix} \sim \left\| \begin{matrix} e_1(-2)+e_2 \to e_2 \\ e_1(-3)+e_3 \to e_3 \\ e_4 - e_1 \to e_4 \\ e_1(-4)+e_5 \to e_5 \end{matrix} \right\| \sim \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & -1 & -2 & -3 & -9 \\ 0 & -2 & -4 & -11 & -13 \\ 0 & 1 & 2 & 8 & 4 \\ 0 & -3 & -6 & -19 & -17 \end{pmatrix} \sim$$

$$\sim \left\| \begin{matrix} e_2(-2)+e_3 \to e_3 \\ e_4 + e_2 \to e_4 \\ e_2(-3)+e_5 \to e_5 \end{matrix} \right\| \sim \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & -1 & -2 & -3 & -9 \\ 0 & 0 & 0 & -5 & 5 \\ 0 & 0 & 0 & 5 & -5 \\ 0 & 0 & 0 & -10 & 10 \end{pmatrix} \sim \left\| \begin{matrix} e_2(-1) \to e_2 \\ e_3 \frac{1}{5} \to e_3 \\ e_4 + e_3 \to e_4 \\ e_3(-2)+e_5 \to e_5 \end{matrix} \right\| \sim$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 9 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 9 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}.$$

The rank of the matrix r=3, i.e. it is less then the number of the unknowns n=5. Let us select as a basic minor the 1st, the 2nd ant the 4th columns, thus the variables x_1, x_2, x_4 are basic and x_3, x_5 are free. Let us write down the system of equations corresponding to the transformed matrix.

$$\begin{cases} x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 = 0 \\ \qquad x_2 + 2x_3 + 3x_4 + 9x_5 = 0 \\ \qquad\qquad -x_4 + x_5 = 0 \end{cases}.$$

Let us transpose the items including the free variables to the right and carry out the backward way of the Gaussian method:

$$\begin{cases} x_4 = x_5 \\ x_2 = -2x_3 - 3x_4 - 9x_5 = -2x_3 - 12x_5 \\ x_1 = -2x_2 - 3x_3 - 4x_4 - 5x_5 = -2(-2x_3 - 12x_5) - 3x_3 - 4x_5 - 5x_5 = x_3 + 15x_5 \end{cases}$$

The general solution of the system is as follows:

$x_1 = x_3 + 15x_5$

$x_2 = -2x_3 - 12x_5$.

$x_4 = x_5$

In order to get the fundamental set of solutions $\vec{e}_1, \vec{e}_2$ we can assign to the free variables the values $x_3 = 1, x_5 = 0$ and $x_3 = 0, x_5 = 1$. The fundamental set of solutions is presented in the table below:

	x_1	x_2	x_3	x_4	x_5
$\vec{e}_1$	1	-2	1	0	0
$\vec{e}_2$	15	-12	0	1	1

The general solution of the system can be presented as $X = C_1\vec{e}_1 + C_2\vec{e}_2$.

Solve the following homogeneous systems of equations:

1. $\begin{cases} 4x_1 - 3x_2 + 3x_3 = 0; \\ -x_1 + 2x_2 + 3x_3 + 2x_4 = 0; \\ x_1 - 2x_2 + x_3 - x_4 = 0; \\ 3x_1 - x_2 + 2x_3 + x_4 = 0. \end{cases}$

2. $\begin{cases} x_1 + 3x_2 + 5x_3 - 2x_4 = 0; \\ 2x_1 + 7x_2 + 3x_3 + x_4 = 0; \\ x_1 + 5x_2 + 9x_3 + 8x_4 = 0; \\ 5x_1 + 18x_2 - 4x_3 + 5x_4 = 0. \end{cases}$