Matrix Representation of Linear Operators

6.1. Consider the linear mapping $F: \mathbf{R}^2 \to \mathbf{R}^2$ defined by $F(x, y) = (3x + 4y, \quad 2x - 5y)$ and the following bases of $\mathbf{R}^2$:

$$E = \{e_1, e_2\} = \{(1, 0), \ (0, 1)\} \qquad \text{and} \qquad S = \{u_1, u_2\} = \{(1, 2), \ (2, 3)\}$$

(a) Find the matrix A representing F relative to the basis E.

(b) Find the matrix B representing F relative to the basis S.

(a) Because E is the usual basis, the rows of A are simply the coefficients in the components of $F(x, y)$; that is, using $(a, b) = ae_1 + be_2$, we have

$$\begin{aligned} F(e_1) &= F(1, 0) = (3, 2) \quad = 3e_1 + 2e_2 \\ F(e_2) &= F(0, 1) = (4, -5) = 4e_1 - 5e_2 \end{aligned} \qquad \text{and so} \qquad A = \begin{bmatrix} 3 & 4 \\ 2 & -5 \end{bmatrix}$$

Note that the coefficients of the basis vectors are written as columns in the matrix representation.

(b) First find $F(u_1)$ and write it as a linear combination of the basis vectors u_1 and u_2. We have

$$F(u_1) = F(1, 2) = (11, -8) = x(1, 2) + y(2, 3), \qquad \text{and so} \qquad \begin{aligned} x + 2y &= 11 \\ 2x + 3y &= -8 \end{aligned}$$

Solve the system to obtain $x = -49$, $y = 30$. Therefore,

$$F(u_1) = -49u_1 + 30u_2$$

Next find $F(u_2)$ and write it as a linear combination of the basis vectors u_1 and u_2. We have

$$F(u_2) = F(2, 3) = (18, -11) = x(1, 2) + y(2, 3), \qquad \text{and so} \qquad \begin{aligned} x + 2y &= 18 \\ 2x + 3y &= -11 \end{aligned}$$

Solve for x and y to obtain $x = -76$, $y = 47$. Hence,

$$F(u_2) = -76u_1 + 47u_2$$

Write the coefficients of u_1 and u_2 as columns to obtain $B = \begin{bmatrix} -49 & -76 \\ 30 & 47 \end{bmatrix}$

(b′) Alternatively, one can first find the coordinates of an arbitrary vector (a, b) in $\mathbf{R}^2$ relative to the basis S. We have

$$(a, b) = x(1, 2) + y(2, 3) = (x + 2y, \ 2x + 3y), \qquad \text{and so} \qquad \begin{aligned} x + 2y &= a \\ 2x + 3y &= b \end{aligned}$$

Solve for x and y in terms of a and b to get $x = -3a + 2b$, $y = 2a - b$. Thus,

$$(a, b) = (-3a + 2b)u_1 + (2a - b)u_2$$

Then use the formula for (a, b) to find the coordinates of $F(u_1)$ and $F(u_2)$ relative to S:

$$\begin{aligned} F(u_1) &= F(1, 2) = (11, -8) \quad = -49u_1 + 30u_2 \\ F(u_2) &= F(2, 3) = (18, -11) = -76u_1 + 47u_2 \end{aligned} \qquad \text{and so} \qquad B = \begin{bmatrix} -49 & -76 \\ 30 & 47 \end{bmatrix}$$

6.2. Consider the following linear operator G on $\mathbf{R}^2$ and basis S:

$$G(x, y) = (2x - 7y, \ 4x + 3y) \qquad \text{and} \qquad S = \{u_1, u_2\} = \{(1, 3), \ (2, 5)\}$$

(a) Find the matrix representation $[G]_S$ of G relative to S.

(b) Verify $[G]_S[v]_S = [G(v)]_S$ for the vector $v = (4, -3)$ in $\mathbf{R}^2$.

First find the coordinates of an arbitrary vector $v = (a, b)$ in $\mathbf{R}^2$ relative to the basis S. We have

$$\begin{bmatrix} a \\ b \end{bmatrix} = x\begin{bmatrix} 1 \\ 3 \end{bmatrix} + y\begin{bmatrix} 2 \\ 5 \end{bmatrix}, \qquad \text{and so} \qquad \begin{aligned} x + 2y &= a \\ 3x + 5y &= b \end{aligned}$$

Solve for x and y in terms of a and b to get $x = -5a + 2b, \; y = 3a - b$. Thus,

$$(a, b) = (-5a + 2b)u_1 + (3a - b)u_2, \qquad \text{and so} \qquad [v] = [-5a + 2b, \quad 3a - b]^T$$

(a) Using the formula for (a, b) and $G(x, y) = (2x - 7y, \; 4x + 3y)$, we have

$$\begin{aligned} G(u_1) &= G(1, 3) = (-19, 13) = 121u_1 - 70u_2 \\ G(u_2) &= G(2, 5) = (-31, 23) = 201u_1 - 116u_2 \end{aligned} \qquad \text{and so} \qquad [G]_S = \begin{bmatrix} 121 & 201 \\ -70 & -116 \end{bmatrix}$$

(We emphasize that the coefficients of u_1 and u_2 are written as columns, not rows, in the matrix representation.)

(b) Use the formula $(a, b) = (-5a + 2b)u_1 + (3a - b)u_2$ to get

$$\begin{aligned} v &= (4, -3) = -26u_1 + 15u_2 \\ G(v) &= G(4, -3) = (20, 7) = -131u_1 + 80u_2 \end{aligned}$$

Then $\qquad [v]_S = [-26, 15]^T \qquad \text{and} \qquad [G(v)]_S = [-131, 80]^T$

Accordingly,

$$[G]_S[v]_S = \begin{bmatrix} 121 & 201 \\ -70 & -116 \end{bmatrix} \begin{bmatrix} -26 \\ 15 \end{bmatrix} = \begin{bmatrix} -131 \\ 80 \end{bmatrix} = [G(v)]_S$$

6.3. Consider the following 2×2 matrix A and basis S of $\mathbf{R}^2$:

$$A = \begin{bmatrix} 2 & 4 \\ 5 & 6 \end{bmatrix} \qquad \text{and} \qquad S = \{u_1, u_2\} = \left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \; \begin{bmatrix} 3 \\ -7 \end{bmatrix} \right\}$$

The matrix A defines a linear operator on $\mathbf{R}^2$. Find the matrix B that represents the mapping A relative to the basis S.

First find the coordinates of an arbitrary vector $(a, b)^T$ with respect to the basis S. We have

$$\begin{bmatrix} a \\ b \end{bmatrix} = x \begin{bmatrix} 1 \\ -2 \end{bmatrix} + y \begin{bmatrix} 3 \\ -7 \end{bmatrix} \qquad \text{or} \qquad \begin{aligned} x + 3y &= a \\ -2x - 7y &= b \end{aligned}$$

Solve for x and y in terms of a and b to obtain $x = 7a + 3b, \; y = -2a - b$. Thus,

$$(a, b)^T = (7a + 3b)u_1 + (-2a - b)u_2$$

Then use the formula for $(a, b)^T$ to find the coordinates of Au_1 and Au_2 relative to the basis S:

$$\begin{aligned} Au_1 &= \begin{bmatrix} 2 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -6 \\ -7 \end{bmatrix} = -63u_1 + 19u_2 \\ Au_2 &= \begin{bmatrix} 2 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ -7 \end{bmatrix} = \begin{bmatrix} -22 \\ -27 \end{bmatrix} = -235u_1 + 71u_2 \end{aligned}$$

Writing the coordinates as columns yields

$$B = \begin{bmatrix} -63 & -235 \\ 19 & 71 \end{bmatrix}$$

6.4. Find the matrix representation of each of the following linear operators F on $\mathbf{R}^3$ relative to the usual basis $E = \{e_1, e_2, e_3\}$ of $\mathbf{R}^3$; that is, find $[F] = [F]_E$:

(a) F defined by $F(x, y, z) = (x + 2y - 3z, \; 4x - 5y - 6z, \; 7x + 8y + 9z)$.

(b) F defined by the 3×3 matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 5 & 5 & 5 \end{bmatrix}$.

(c) F defined by $F(e_1) = (1, 3, 5), F(e_2) = (2, 4, 6), \; F(e_3) = (7, 7, 7)$. (Theorem 5.2 states that a linear map is completely defined by its action on the vectors in a basis.)

(a) Because E is the usual basis, simply write the coefficients of the components of $F(x, y, z)$ as rows:

$$[F] = \begin{bmatrix} 1 & 2 & -3 \\ 4 & -5 & -6 \\ 7 & 8 & 9 \end{bmatrix}$$

(b) Because E is the usual basis, $[F] = A$, the matrix A itself.

(c) Here

$$\begin{aligned} F(e_1) &= (1,3,5) = \ \ e_1 + 3e_2 + 5e_3 \\ F(e_2) &= (2,4,6) = 2e_1 + 4e_2 + 6e_3 \\ F(e_3) &= (7,7,7) = 7e_1 + 7e_2 + 7e_3 \end{aligned} \qquad \text{and so} \qquad [F] = \begin{bmatrix} 1 & 2 & 7 \\ 3 & 4 & 7 \\ 5 & 6 & 7 \end{bmatrix}$$

That is, the columns of $[F]$ are the images of the usual basis vectors.

6.5. Let G be the linear operator on $\mathbf{R}^3$ defined by $G(x, y, z) = (2y + z,\ x - 4y,\ 3x)$.

(a) Find the matrix representation of G relative to the basis

$$S = \{w_1, w_2, w_3\} = \{(1,1,1), \quad (1,1,0), \quad (1,0,0)\}$$

(b) Verify that $[G][v] = [G(v)]$ for any vector v in $\mathbf{R}^3$.

First find the coordinates of an arbitrary vector $(a, b, c) \in \mathbf{R}^3$ with respect to the basis S. Write (a, b, c) as a linear combination of w_1, w_2, w_3 using unknown scalars x, y, and z:

$$(a, b, c) = x(1,1,1) + y(1,1,0) + z(1,0,0) = (x + y + z,\ x + y,\ x)$$

Set corresponding components equal to each other to obtain the system of equations

$$x + y + z = a, \qquad x + y = b, \qquad x = c$$

Solve the system for x, y, z in terms of a, b, c to find $x = c,\ \ y = b - c,\ \ z = a - b$. Thus,

$$(a, b, c) = cw_1 + (b - c)w_2 + (a - b)w_3, \qquad \text{or equivalently,} \qquad [(a, b, c)] = [c,\ b - c,\ a - b]^T$$

(a) Because $G(x, y, z) = (2y + z,\ x - 4y,\ 3x)$,

$$\begin{aligned} G(w_1) &= G(1,1,1) = (3, -3, 3) = 3w_1 - 6x_2 + 6x_3 \\ G(w_2) &= G(1,1,0) = (2, -3, 3) = 3w_1 - 6w_2 + 5w_3 \\ G(w_3) &= G(1,0,0) = (0, 1, 3) = 3w_1 - 2w_2 - w_3 \end{aligned}$$

Write the coordinates $G(w_1)$, $G(w_2)$, $G(w_3)$ as columns to get

$$[G] = \begin{bmatrix} 3 & 3 & 3 \\ -6 & -6 & -2 \\ 6 & 5 & -1 \end{bmatrix}$$

(b) Write $G(v)$ as a linear combination of w_1, w_2, w_3, where $v = (a, b, c)$ is an arbitrary vector in $\mathbf{R}^3$,

$$G(v) = G(a, b, c) = (2b + c,\ a - 4b,\ 3a) = 3aw_1 + (-2a - 4b)w_2 + (-a + 6b + c)w_3$$

or equivalently,

$$[G(v)] = [3a,\ -2a - 4b,\ -a + 6b + c]^T$$

Accordingly,

$$[G][v] = \begin{bmatrix} 3 & 3 & 3 \\ -6 & -6 & -2 \\ 6 & 5 & -1 \end{bmatrix} \begin{bmatrix} c \\ b - c \\ a - b \end{bmatrix} = \begin{bmatrix} 3a \\ -2a - 4b \\ -a + 6b + c \end{bmatrix} = [G(v)]$$

6.6. Consider the following 3×3 matrix A and basis S of $\mathbf{R}^3$:

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 3 & -1 & 0 \\ 1 & 4 & -2 \end{bmatrix} \qquad \text{and} \qquad S = \{u_1, u_2, u_3\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$$

The matrix A defines a linear operator on $\mathbf{R}^3$. Find the matrix B that represents the mapping A relative to the basis S. (Recall that A represents itself relative to the usual basis of $\mathbf{R}^3$.)

First find the coordinates of an arbitrary vector (a, b, c) in $\mathbf{R}^3$ with respect to the basis S. We have

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = x\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + z\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \text{or} \quad \begin{array}{r} x + \quad z = a \\ x + y + 2z = b \\ x + y + 3z = c \end{array}$$

Solve for x, y, z in terms of a, b, c to get

$$x = a + b - c, \quad y = -a + 2b - c, \quad z = c - b$$

thus, $\quad (a, b, c)^T = (a + b - c)u_1 + (-a + 2b - c)u_2 + (c - b)u_3$

Then use the formula for $(a, b, c)^T$ to find the coordinates of Au_1, Au_2, Au_3 relative to the basis S:

$$\begin{array}{ll} A(u_1) = A(1,1,1)^T = (0,2,3)^T & = -u_1 + u_2 + u_3 \\ A(u_2) = A(1,1,0)^T = (-1,-1,2)^T & = -4u_1 - 3u_2 + 3u_3 \\ A(u_3) = A(1,2,3)^T = (0,1,3)^T & = -2u_1 - u_2 + 2u_3 \end{array} \quad \text{so} \quad B = \begin{bmatrix} -1 & -4 & -2 \\ 1 & -3 & -1 \\ 1 & 3 & 2 \end{bmatrix}$$

6.7. For each of the following linear transformations (operators) L on $\mathbf{R}^2$, find the matrix A that represents L (relative to the usual basis of $\mathbf{R}^2$):

(a) L is defined by $L(1,0) = (2,4)$ and $L(0,1) = (5,8)$.

(b) L is the rotation in $\mathbf{R}^2$ counterclockwise by $90°$.

(c) L is the reflection in $\mathbf{R}^2$ about the line $y = -x$.

(a) Because $\{(1,0), (0,1)\}$ is the usual basis of $\mathbf{R}^2$, write their images under L as columns to get

$$A = \begin{bmatrix} 2 & 5 \\ 4 & 8 \end{bmatrix}$$

(b) Under the rotation L, we have $L(1,0) = (0,1)$ and $L(0,1) = (-1,0)$. Thus,

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

(c) Under the reflection L, we have $L(1,0) = (0,-1)$ and $L(0,1) = (-1,0)$. Thus,

$$A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

6.8. The set $S = \{e^{3t}, te^{3t}, t^2e^{3t}\}$ is a basis of a vector space V of functions $f: \mathbf{R} \to \mathbf{R}$. Let $\mathbf{D}$ be the differential operator on V; that is, $\mathbf{D}(f) = df/dt$. Find the matrix representation of $\mathbf{D}$ relative to the basis S.

Find the image of each basis function:

$$\begin{array}{lll} \mathbf{D}(e^{3t}) & = 3e^{3t} & = 3(e^{3t}) + 0(te^{3t}) + 0(t^2e^{3t}) \\ \mathbf{D}(te^{3t}) & = e^{3t} + 3te^{3t} & = 1(e^{3t}) + 3(te^{3t}) + 0(t^2e^{3t}) \\ \mathbf{D}(t^2e^{3t}) & = 2te^{3t} + 3t^2e^{3t} & = 0(e^{3t}) + 2(te^{3t}) + 3(t^2e^{3t}) \end{array} \quad \text{and thus,} \quad [\mathbf{D}] = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

Change of Basis

The coordinate vector $[v]_S$ in this section will always denote a column vector; that is,

$$[v]_S = [a_1, a_2, \ldots, a_n]^T$$

6.13. Consider the following bases of $\mathbf{R}^2$:

$$E = \{e_1, e_2\} = \{(1,0), (0,1)\} \quad \text{and} \quad S = \{u_1, u_2\} = \{(1,3), (1,4)\}$$

(a) Find the change-of-basis matrix P from the usual basis E to S.

(b) Find the change-of-basis matrix Q from S back to E.

(c) Find the coordinate vector $[v]$ of $v = (5, -3)$ relative to S.

(a) Because E is the usual basis, simply write the basis vectors in S as columns: $P = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}$

(b) **Method 1.** Use the definition of the change-of-basis matrix. That is, express each vector in E as a linear combination of the vectors in S. We do this by first finding the coordinates of an arbitrary vector $v = (a, b)$ relative to S. We have

$$(a, b) = x(1, 3) + y(1, 4) = (x + y, 3x + 4y) \qquad \text{or} \qquad \begin{aligned} x + \ \ y &= a \\ 3x + 4y &= b \end{aligned}$$

Solve for x and y to obtain $x = 4a - b$, $y = -3a + b$. Thus,

$$v = (4a - b)u_1 + (-3a + b)u_2 \qquad \text{and} \qquad [v]_S = [(a, b)]_S = [4a - b, \ -3a + b]^T$$

Using the above formula for $[v]_S$ and writing the coordinates of the e_i as columns yields

$$\begin{aligned} e_1 &= (1, 0) = \ \ 4u_1 - 3u_2 \\ e_2 &= (0, 1) = -u_1 + \ \ u_2 \end{aligned} \qquad \text{and} \qquad Q = \begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix}$$

Method 2. Because $Q = P^{-1}$, find P^{-1}, say by using the formula for the inverse of a 2×2 matrix. Thus,

$$P^{-1} = \begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix}$$

(c) **Method 1.** Write v as a linear combination of the vectors in S, say by using the above formula for $v = (a, b)$. We have $v = (5, -3) = 23u_1 - 18u_2$, and so $[v]_S = [23, -18]^T$.

Method 2. Use, from Theorem 6.6, the fact that $[v]_S = P^{-1}[v]_E$ and the fact that $[v]_E = [5, -3]^T$:

$$[v]_S = P^{-1}[v]_E = \begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -3 \end{bmatrix} = \begin{bmatrix} 23 \\ -18 \end{bmatrix}$$

6.14. The vectors $u_1 = (1, 2, 0)$, $u_2 = (1, 3, 2)$, $u_3 = (0, 1, 3)$ form a basis S of $\mathbf{R}^3$. Find

(a) The change-of-basis matrix P from the usual basis $E = \{e_1, e_2, e_3\}$ to S.

(b) The change-of-basis matrix Q from S back to E.

(a) Because E is the usual basis, simply write the basis vectors of S as columns: $P = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & 1 \\ 0 & 2 & 3 \end{bmatrix}$

(b) **Method 1.** Express each basis vector of E as a linear combination of the basis vectors of S by first finding the coordinates of an arbitrary vector $v = (a, b, c)$ relative to the basis S. We have

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = x \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + y \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} + z \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \qquad \text{or} \qquad \begin{aligned} x + \ \ y \qquad &= a \\ 2x + 3y + \ \ z &= b \\ 2y + 3z &= c \end{aligned}$$

Solve for x, y, z to get $x = 7a - 3b + c$, $y = -6a + 3b - c$, $z = 4a - 2b + c$. Thus,

$$v = (a, b, c) = (7a - 3b + c)u_1 + (-6a + 3b - c)u_2 + (4a - 2b + c)u_3$$

or $\qquad [v]_S = [(a, b, c)]_S = [7a - 3b + c, \ -6a + 3b - c, \ 4a - 2b + c]^T$

Using the above formula for $[v]_S$ and then writing the coordinates of the e_i as columns yields

$$\begin{aligned} e_1 &= (1, 0, 0) = \ \ 7u_1 - 6u_2 + 4u_3 \\ e_2 &= (0, 1, 0) = -3u_1 + 3u_2 - 2u_3 \\ e_3 &= (0, 0, 1) = \ \ \ u_1 - \ \ u_2 + \ \ u_3 \end{aligned} \qquad \text{and} \qquad Q = \begin{bmatrix} 7 & -3 & 1 \\ -6 & 3 & -1 \\ 4 & -2 & 1 \end{bmatrix}$$

Method 2. Find P^{-1} by row reducing $M = [P, I]$ to the form $[I, P^{-1}]$:

$$M = \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 & 1 & 0 \\ 0 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 2 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 4 & -2 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7 & -3 & 1 \\ 0 & 1 & 0 & -6 & 3 & -1 \\ 0 & 0 & 1 & 4 & -2 & 1 \end{array}\right] = [I, P^{-1}]$$

Thus, $Q = P^{-1} = \begin{bmatrix} 7 & -3 & 1 \\ -6 & 3 & -1 \\ 4 & -2 & 1 \end{bmatrix}$.

6.15. Suppose the x-axis and y-axis in the plane $\mathbf{R}^2$ are rotated counterclockwise 45° so that the new x'-axis and y'-axis are along the line $y = x$ and the line $y = -x$, respectively.

(a) Find the change-of-basis matrix P.

(b) Find the coordinates of the point $A(5, 6)$ under the given rotation.

(a) The unit vectors in the direction of the new x'- and y'-axes are

$$u_1 = (\tfrac{1}{2}\sqrt{2}, \tfrac{1}{2}\sqrt{2}) \qquad \text{and} \qquad u_2 = (-\tfrac{1}{2}\sqrt{2}, \tfrac{1}{2}\sqrt{2})$$

(The unit vectors in the direction of the original x and y axes are the usual basis of $\mathbf{R}^2$.) Thus, write the coordinates of u_1 and u_2 as columns to obtain

$$P = \begin{bmatrix} \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{bmatrix}$$

(b) Multiply the coordinates of the point by P^{-1}:

$$\begin{bmatrix} \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \\ -\frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} \frac{11}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \end{bmatrix}$$

(Because P is orthogonal, P^{-1} is simply the transpose of P.)

6.16. The vectors $u_1 = (1, 1, 0)$, $u_2 = (0, 1, 1)$, $u_3 = (1, 2, 2)$ form a basis S of $\mathbf{R}^3$. Find the coordinates of an arbitrary vector $v = (a, b, c)$ relative to the basis S.

Method 1. Express v as a linear combination of u_1, u_2, u_3 using unknowns x, y, z. We have

$$(a, b, c) = x(1, 1, 0) + y(0, 1, 1) + z(1, 2, 2) = (x + z,\ \ x + y + 2z,\ \ y + 2z)$$

this yields the system

$$\begin{aligned} x + \quad\ \ z &= a \\ x + y + 2z &= b \\ y + 2z &= c \end{aligned} \qquad \text{or} \qquad \begin{aligned} x + \quad\ z &= a \\ y + \ z &= -a + b \\ y + 2z &= c \end{aligned} \qquad \text{or} \qquad \begin{aligned} x + \quad z &= a \\ y + z &= -a + b \\ z &= a - b + c \end{aligned}$$

Solving by back-substitution yields $x = b - c$, $y = -2a + 2b - c$, $z = a - b + c$. Thus,

$$[v]_S = [b - c,\ \ -2a + 2b - c,\ \ a - b + c]^T$$

Method 2. Find P^{-1} by row reducing $M = [P, I]$ to the form $[I, P^{-1}]$, where P is the change-of-basis matrix from the usual basis E to S or, in other words, the matrix whose columns are the basis vectors of S.

We have

$$M = \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -2 & 2 & -1 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right] = [I, P^{-1}]$$

Thus, $$P^{-1} = \begin{bmatrix} 0 & 1 & -1 \\ -2 & 2 & -1 \\ 1 & -1 & 1 \end{bmatrix} \text{ and } [v]_S = P^{-1}[v]_E = \begin{bmatrix} 0 & 1 & -1 \\ -2 & 2 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} b - c \\ -2a + 2b - c \\ a - b + c \end{bmatrix}$$

6.17. Consider the following bases of $\mathbf{R}^2$:

$$S = \{u_1, u_2\} = \{(1, -2),\ (3, -4)\} \qquad \text{and} \qquad S' = \{v_1, v_2\} = \{(1, 3),\ (3, 8)\}$$

(a) Find the coordinates of $v = (a, b)$ relative to the basis S.

(b) Find the change-of-basis matrix P from S to S'.

(c) Find the coordinates of $v = (a, b)$ relative to the basis S'.
(d) Find the change-of-basis matrix Q from S' back to S.
(e) Verify $Q = P^{-1}$.
(f) Show that, for any vector $v = (a, b)$ in $\mathbf{R}^2$, $P^{-1}[v]_S = [v]_{S'}$.

(a) Let $v = xu_1 + yu_2$ for unknowns x and y; that is,

$$\begin{bmatrix} a \\ b \end{bmatrix} = x\begin{bmatrix} 1 \\ -2 \end{bmatrix} + y\begin{bmatrix} 3 \\ -4 \end{bmatrix} \quad \text{or} \quad \begin{aligned} x + 3y &= a \\ -2x - 4y &= b \end{aligned} \quad \text{or} \quad \begin{aligned} x + 3y &= a \\ 2y &= 2a + b \end{aligned}$$

Solve for x and y in terms of a and b to get $x = -2a - \frac{3}{2}b$ and $y = a + \frac{1}{2}b$. Thus,

$$(a, b) = (-2a - \tfrac{3}{2})u_1 + (a + \tfrac{1}{2}b)u_2 \quad \text{or} \quad [(a, b)]_S = [-2a - \tfrac{3}{2}b, \ a + \tfrac{1}{2}b]^T$$

(b) Use part (a) to write each of the basis vectors v_1 and v_2 of S' as a linear combination of the basis vectors u_1 and u_2 of S; that is,

$$v_1 = (1, 3) = (-2 - \tfrac{9}{2})u_1 + (1 + \tfrac{3}{2})u_2 = -\tfrac{13}{2}u_1 + \tfrac{5}{2}u_2$$
$$v_2 = (3, 8) = (-6 - 12)u_1 + (3 + 4)u_2 = -18u_1 + 7u_2$$

Then P is the matrix whose columns are the coordinates of v_1 and v_2 relative to the basis S; that is,

$$P = \begin{bmatrix} -\frac{13}{2} & -18 \\ \frac{5}{2} & 7 \end{bmatrix}$$

(c) Let $v = xv_1 + yv_2$ for unknown scalars x and y:

$$\begin{bmatrix} a \\ b \end{bmatrix} = x\begin{bmatrix} 1 \\ 3 \end{bmatrix} + y\begin{bmatrix} 3 \\ 8 \end{bmatrix} \quad \text{or} \quad \begin{aligned} x + 3y &= a \\ 3x + 8y &= b \end{aligned} \quad \text{or} \quad \begin{aligned} x + 3y &= a \\ -y &= b - 3a \end{aligned}$$

Solve for x and y to get $x = -8a + 3b$ and $y = 3a - b$. Thus,

$$(a, b) = (-8a + 3b)v_1 + (3a - b)v_2 \quad \text{or} \quad [(a, b)]_{S'} = [-8a + 3b, \quad 3a - b]^T$$

(d) Use part (c) to express each of the basis vectors u_1 and u_2 of S as a linear combination of the basis vectors v_1 and v_2 of S':

$$u_1 = (1, -2) = (-8 - 6)v_1 + (3 + 2)v_2 = -14v_1 + 5v_2$$
$$u_2 = (3, -4) = (-24 - 12)v_1 + (9 + 4)v_2 = -36v_1 + 13v_2$$

Write the coordinates of u_1 and u_2 relative to S' as columns to obtain $Q = \begin{bmatrix} -14 & -36 \\ 5 & 13 \end{bmatrix}$.

(e) $QP = \begin{bmatrix} -14 & -36 \\ 5 & 13 \end{bmatrix}\begin{bmatrix} -\frac{13}{2} & -18 \\ \frac{5}{2} & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

(f) Use parts (a), (c), and (d) to obtain

$$P^{-1}[v]_S = Q[v]_S = \begin{bmatrix} -14 & -36 \\ 5 & 13 \end{bmatrix}\begin{bmatrix} -2a - \frac{3}{2}b \\ a + \frac{1}{2}b \end{bmatrix} = \begin{bmatrix} -8a + 3b \\ 3a - b \end{bmatrix} = [v]_{S'}$$

Linear Operators and Change of Basis

6.23. Consider the linear transformation F on $\mathbf{R}^2$ defined by $F(x, y) = (5x - y, \ 2x + y)$ and the following bases of $\mathbf{R}^2$:

$$E = \{e_1, e_2\} = \{(1, 0), \ (0, 1)\} \qquad \text{and} \qquad S = \{u_1, u_2\} = \{(1, 4), \ (2, 7)\}$$

(a) Find the change-of-basis matrix P from E to S and the change-of-basis matrix Q from S back to E.
(b) Find the matrix A that represents F in the basis E.
(c) Find the matrix B that represents F in the basis S.

(a) Because E is the usual basis, simply write the vectors in S as columns to obtain the change-of-basis matrix P. Recall, also, that $Q = P^{-1}$. Thus,

$$P = \begin{bmatrix} 1 & 2 \\ 4 & 7 \end{bmatrix} \quad \text{and} \quad Q = P^{-1} = \begin{bmatrix} -7 & 2 \\ 4 & -1 \end{bmatrix}$$

(b) Write the coefficients of x and y in $F(x,y) = (5x - y,\ 2x + y)$ as rows to get

$$A = \begin{bmatrix} 5 & -1 \\ 2 & 1 \end{bmatrix}$$

(c) **Method 1.** Find the coordinates of $F(u_1)$ and $F(u_2)$ relative to the basis S. This may be done by first finding the coordinates of an arbitrary vector (a,b) in $\mathbf{R}^2$ relative to the basis S. We have

$$(a,b) = x(1,4) + y(2,7) = (x + 2y,\ 4x + 7y), \quad \text{and so} \quad \begin{matrix} x + 2y = a \\ 4x + 7y = b \end{matrix}$$

Solve for x and y in terms of a and b to get $x = -7a + 2b,\ \ y = 4a - b$. Then

$$(a,b) = (-7a + 2b)u_1 + (4a - b)u_2$$

Now use the formula for (a,b) to obtain

$$\begin{matrix} F(u_1) = F(1,4) = (1,6) & = 5u_1 - 2u_2 \\ F(u_2) = F(2,7) = (3,11) = & u_1 + \ u_2 \end{matrix} \quad \text{and so} \quad B = \begin{bmatrix} 5 & 1 \\ -2 & 1 \end{bmatrix}$$

Method 2. By Theorem 6.7, $B = P^{-1}AP$. Thus,

$$B = P^{-1}AP = \begin{bmatrix} -7 & 2 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} 5 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 4 & 7 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ -2 & 1 \end{bmatrix}$$

6.24. Let $A = \begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix}$. Find the matrix B that represents the linear operator A relative to the basis $S = \{u_1, u_2\} = \{[1,3]^T,\ [2,5]^T\}$. [Recall A defines a linear operator $A{:}\,\mathbf{R}^2 \to \mathbf{R}^2$ relative to the usual basis E of $\mathbf{R}^2$].

Method 1. Find the coordinates of $A(u_1)$ and $A(u_2)$ relative to the basis S by first finding the coordinates of an arbitrary vector $[a,b]^T$ in $\mathbf{R}^2$ relative to the basis S. By Problem 6.2,

$$[a,b]^T = (-5a + 2b)u_1 + (3a - b)u_2$$

Using the formula for $[a,b]^T$, we obtain

$$A(u_1) = \begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 11 \\ 1 \end{bmatrix} = -53u_1 + 32u_2$$

and

$$A(u_2) = \begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 19 \\ 3 \end{bmatrix} = -89u_1 + 54u_2$$

Thus,

$$B = \begin{bmatrix} -53 & -89 \\ 32 & 54 \end{bmatrix}$$

Method 2. Use $B = P^{-1}AP$, where P is the change-of-basis matrix from the usual basis E to S. Thus, simply write the vectors in S (as columns) to obtain the change-of-basis matrix P and then use the formula for P^{-1}. This gives

$$P = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \quad \text{and} \quad P^{-1} = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$$

Then

$$B = P^{-1}AP = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} -53 & -89 \\ 32 & 54 \end{bmatrix}$$

6.25. Let $A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 5 & -4 \\ 1 & -2 & 2 \end{bmatrix}$. Find the matrix B that represents the linear operator A relative to the basis

$$S = \{u_1, u_2, u_3\} = \{[1,1,0]^T,\quad [0,1,1]^T,\quad [1,2,2]^T\}$$

[Recall A that defines a linear operator $A: \mathbf{R}^3 \to \mathbf{R}^3$ relative to the usual basis E of $\mathbf{R}^3$.]

Method 1. Find the coordinates of $A(u_1)$, $A(u_2)$, $A(u_3)$ relative to the basis S by first finding the coordinates of an arbitrary vector $v = (a, b, c)$ in $\mathbf{R}^3$ relative to the basis S. By Problem 6.16,

$$[v]_S = (b - c)u_1 + (-2a + 2b - c)u_2 + (a - b + c)u_3$$

Using this formula for $[a, b, c]^T$, we obtain

$$A(u_1) = [4, 7, -1]^T = 8u_1 + 7u_2 - 4u_3, \qquad A(u_2) = [4, 1, 0]^T = u_1 - 6u_2 + 3u_3$$
$$A(u_3) = [9, 4, 1]^T = 3u_1 - 11u_2 + 6u_3$$

Writing the coefficients of u_1, u_2, u_3 as columns yields

$$B = \begin{bmatrix} 8 & 1 & 3 \\ 7 & -6 & -11 \\ -4 & 3 & 6 \end{bmatrix}$$

Method 2. Use $B = P^{-1}AP$, where P is the change-of-basis matrix from the usual basis E to S. The matrix P (whose columns are simply the vectors in S) and P^{-1} appear in Problem 6.16. Thus,

$$B = P^{-1}AP = \begin{bmatrix} 0 & 1 & -1 \\ -2 & 2 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 \\ 2 & 5 & -4 \\ 1 & -2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 1 & 3 \\ 7 & -6 & -11 \\ -4 & 3 & 6 \end{bmatrix}$$

Similarity of Matrices

6.27. Let $A = \begin{bmatrix} 4 & -2 \\ 3 & 6 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

(a) Find $B = P^{-1}AP$. (b) Verify $\text{tr}(B) = \text{tr}(A)$. (c) Verify $\det(B) = \det(A)$.

(a) First find P^{-1} using the formula for the inverse of a 2×2 matrix. We have

$$P^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

Then

$$B = P^{-1}AP = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 4 & -2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 25 & 30 \\ -\frac{27}{2} & -15 \end{bmatrix}$$

(b) $\text{tr}(A) = 4 + 6 = 10$ and $\text{tr}(B) = 25 - 15 = 10$. Hence, $\text{tr}(B) = \text{tr}(A)$.

(c) $\det(A) = 24 + 6 = 30$ and $\det(B) = -375 + 405 = 30$. Hence, $\det(B) = \det(A)$.

6.28. Find the trace of each of the linear transformations F on $\mathbf{R}^3$ in Problem 6.4.

Find the trace (sum of the diagonal elements) of any matrix representation of F such as the matrix representation $[F] = [F]_E$ of F relative to the usual basis E given in Problem 6.4.

(a) $\text{tr}(F) = \text{tr}([F]) = 1 - 5 + 9 = 5$.

(b) $\text{tr}(F) = \text{tr}([F]) = 1 + 3 + 5 = 9$.

(c) $\text{tr}(F) = \text{tr}([F]) = 1 + 4 + 7 = 12$.

Matrix Representations of General Linear Mappings

6.31. Let $F: \mathbf{R}^3 \to \mathbf{R}^2$ be the linear map defined by $F(x, y, z) = (3x + 2y - 4z,\; x - 5y + 3z)$.

(a) Find the matrix of F in the following bases of $\mathbf{R}^3$ and $\mathbf{R}^2$:

$$S = \{w_1, w_2, w_3\} = \{(1, 1, 1),\; (1, 1, 0),\; (1, 0, 0)\} \quad \text{and} \quad S' = \{u_1, u_2\} = \{(1, 3),\; (2, 5)\}$$

(b) Verify Theorem 6.10: The action of F is preserved by its matrix representation; that is, for any v in $\mathbf{R}^3$, we have $[F]_{S,S'}[v]_S = [F(v)]_{S'}$.

(a) From Problem 6.2, $(a, b) = (-5a + 2b)u_1 + (3a - b)u_2$. Thus,

$$F(w_1) = F(1,1,1) = (1,-1) = -7u_1 + 4u_2$$
$$F(w_2) = F(1,1,0) = (5,-4) = -33u_1 + 19u_2$$
$$F(w_3) = F(1,0,0) = (3,1) = -13u_1 + 8u_2$$

Write the coordinates of $F(w_1)$, $F(w_2), F(w_3)$ as columns to get

$$[F]_{S,S'} = \begin{bmatrix} -7 & -33 & 13 \\ 4 & 19 & 8 \end{bmatrix}$$

(b) If $v = (x,y,z)$, then, by Problem 6.5, $v = zw_1 + (y-z)w_2 + (x-y)w_3$. Also,

$$F(v) = (3x + 2y - 4z,\ x - 5y + 3z) = (-13x - 20y + 26z)u_1 + (8x + 11y - 15z)u_2$$

Hence, $\quad [v]_S = (z,\ y-z,\ x-y)^T \quad$ and $\quad [F(v)]_{S'} = \begin{bmatrix} -13x - 20y + 26z \\ 8x + 11y - 15z \end{bmatrix}$

Thus, $\quad [F]_{S,S'}[v]_S = \begin{bmatrix} -7 & -33 & -13 \\ 4 & 19 & 8 \end{bmatrix} \begin{bmatrix} z \\ y-x \\ x-y \end{bmatrix} = \begin{bmatrix} -13x - 20y + 26z \\ 8x + 11y - 15z \end{bmatrix} = [F(v)]_{S'}$

6.32. Let $F: \mathbf{R}^n \to \mathbf{R}^m$ be the linear mapping defined as follows:

$$F(x_1, x_2, \ldots, x_n) = (a_{11}x_1 + \cdots + a_{1n}x_n,\ a_{21}x_1 + \cdots + a_{2n}x_n,\ \ldots,\ a_{m1}x_1 + \cdots + a_{mn}x_n)$$

(a) Show that the rows of the matrix $[F]$ representing F relative to the usual bases of $\mathbf{R}^n$ and $\mathbf{R}^m$ are the coefficients of the x_i in the components of $F(x_1, \ldots, x_n)$.

(b) Find the matrix representation of each of the following linear mappings relative to the usual basis of $\mathbf{R}^n$:

(i) $F: \mathbf{R}^2 \to \mathbf{R}^3$ defined by $F(x,y) = (3x - y,\ \ 2x + 4y,\ \ 5x - 6y)$.

(ii) $F: \mathbf{R}^4 \to \mathbf{R}^2$ defined by $F(x,y,s,t) = (3x - 4y + 2s - 5t,\ \ 5x + 7y - s - 2t)$.

(iii) $F: \mathbf{R}^3 \to \mathbf{R}^4$ defined by $F(x,y,z) = (2x + 3y - 8z,\ \ x + y + z,\ \ 4x - 5z,\ \ 6y)$.

(a) We have

$$\begin{array}{l} F(1,0,\ldots,0) = (a_{11}, a_{21}, \ldots, a_{m1}) \\ F(0,1,\ldots,0) = (a_{12}, a_{22}, \ldots, a_{m2}) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ F(0,0,\ldots,1) = (a_{1n}, a_{2n}, \ldots, a_{mn}) \end{array} \quad \text{and thus,} \quad [F] = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}$$

(b) By part (a), we need only look at the coefficients of the unknown $x, y, \ldots$ in $F(x, y, \ldots)$. Thus,

$$\text{(i)} \quad [F] = \begin{bmatrix} 3 & -1 \\ 2 & 4 \\ 5 & -6 \end{bmatrix}, \qquad \text{(ii)} \quad [F] = \begin{bmatrix} 3 & -4 & 2 & -5 \\ 5 & 7 & -1 & -2 \end{bmatrix}, \qquad \text{(iii)} \quad [F] = \begin{bmatrix} 2 & 3 & -8 \\ 1 & 1 & 1 \\ 4 & 0 & -5 \\ 0 & 6 & 0 \end{bmatrix}$$

6.33. Let $A = \begin{bmatrix} 2 & 5 & -3 \\ 1 & -4 & 7 \end{bmatrix}$. Recall that A determines a mapping $F: \mathbf{R}^3 \to \mathbf{R}^2$ defined by $F(v) = Av$, where vectors are written as columns. Find the matrix $[F]$ that represents the mapping relative to the following bases of $\mathbf{R}^3$ and $\mathbf{R}^2$:

(a) The usual bases of $\mathbf{R}^3$ and of $\mathbf{R}^2$.

(b) $S = \{w_1, w_2, w_3\} = \{(1,1,1),\ (1,1,0),\ (1,0,0)\}$ and $S' = \{u_1, u_2\} = \{(1,3),\ (2,5)\}$.

(a) Relative to the usual bases, $[F]$ is the matrix A.

(b) From Problem 6.2, $(a,b) = (-5a + 2b)u_1 + (3a - b)u_2$. Thus,

$$F(w_1) = \begin{bmatrix} 2 & 5 & -3 \\ 1 & -4 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} = -12u_1 + 8u_2$$

$$F(w_2) = \begin{bmatrix} 2 & 5 & -3 \\ 1 & -4 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ -3 \end{bmatrix} = -41u_1 + 24u_2$$

$$F(w_3) = \begin{bmatrix} 2 & 5 & -3 \\ 1 & -4 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} = -8u_1 + 5u_2$$

Writing the coefficients of $F(w_1)$, $F(w_2)$, $F(w_3)$ as columns yields $[F] = \begin{bmatrix} -12 & -41 & -8 \\ 8 & 24 & 5 \end{bmatrix}$.

6.34. Consider the linear transformation T on $\mathbf{R}^2$ defined by $T(x, y) = (2x - 3y, \quad x + 4y)$ and the following bases of $\mathbf{R}^2$:

$$E = \{e_1, e_2\} = \{(1, 0), \ (0, 1)\} \qquad \text{and} \qquad S = \{u_1, u_2\} = \{(1, 3), \ (2, 5)\}$$

(a) Find the matrix A representing T relative to the bases E and S.

(b) Find the matrix B representing T relative to the bases S and E.

(We can view T as a linear mapping from one space into another, each having its own basis.)

(a) From Problem 6.2, $(a, b) = (-5a + 2b)u_1 + (3a - b)u_2$. Hence,

$$\begin{aligned} T(e_1) &= T(1, 0) = (2, 1) &= -8u_1 + 5u_2 \\ T(e_2) &= T(0, 1) = (-3, 4) &= 23u_1 - 13u_2 \end{aligned} \qquad \text{and so} \qquad A = \begin{bmatrix} -8 & 23 \\ 5 & -13 \end{bmatrix}$$

(b) We have

$$\begin{aligned} T(u_1) &= T(1, 3) = (-7, 13) &= -7e_1 + 13e_2 \\ T(u_2) &= T(2, 5) = (-11, 22) &= -11e_1 + 22e_2 \end{aligned} \qquad \text{and so} \qquad B = \begin{bmatrix} -7 & -11 \\ 13 & 22 \end{bmatrix}$$

6.35. How are the matrices A and B in Problem 6.34 related?

By Theorem 6.12, the matrices A and B are equivalent to each other; that is, there exist nonsingular matrices P and Q such that $B = Q^{-1}AP$, where P is the change-of-basis matrix from S to E, and Q is the change-of-basis matrix from E to S. Thus,

$$P = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}, \qquad Q = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}, \qquad Q^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$$

and

$$Q^{-1}AP = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} -8 & -23 \\ 5 & -13 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} -7 & -11 \\ 13 & 22 \end{bmatrix} = B$$

6.36. Prove Theorem 6.14: Let $F: V \to U$ be linear and, say, $\operatorname{rank}(F) = r$. Then there exist bases V and of U such that the matrix representation of F has the following form, where I_r is the r-square identity matrix:

$$A = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

Suppose $\dim V = m$ and $\dim U = n$. Let W be the kernel of F and U' the image of F. We are given that rank $(F) = r$. Hence, the dimension of the kernel of F is $m - r$. Let $\{w_1, \ldots, w_{m-r}\}$ be a basis of the kernel of F and extend this to a basis of V:

$$\{v_1, \ldots, v_r, w_1, \ldots, w_{m-r}\}$$

Set

$$u_1 = F(v_1), \ u_2 = F(v_2), \ \ldots, \ u_r = F(v_r)$$