Distance Learning. Higher Mathematics.
Period: 17 March 2020 – 2 April 2020

An indefinite integral. Methods of integrating.
6.1 Antiderivative, properties of an indefinite integral.

The function $F(x)$ is called an *antiderivative* of the function $f(x)$ in the interval (a,b), if $F(x)$ is differentiable $\forall x \in (a,b)$ and $F'(x) = f(x)$.

1°. If $F(x)$ is an antiderivative in the interval (a,b) then $F(x)+C$, where C is any constant , is also an antiderivative.

2°. If $F_1(x)$ и $F_2(x)$ are any two antiderivatives, then $F_1(x) - F_2(x) = C$, whence $F_1(x) = F_2(x) + C$.

The set of all antiderivatives $F(x)$ of a function $f(x)$ is called *an indefinite integral* and is designated by the symbol

$$\int f(x)dx = F(x) + C.$$

Properties of an indefinite integral

1^0. $\left(\int f(x)dx\right)' = f(x).$

2^0. $d\left(\int f(x)dx\right) = f(x)dx.$

3^0. $\int dF(x) = F(x) + C.$

4^0. $\int Cf(x)dx = C\int f(x)dx.$

5^0. $\int (u \pm v)dx = \int udx \pm \int vdx.$

The table of basic indefinite integrals

1. $\int dx = x + C.$
2. $\int x^n dx = \dfrac{x^{n+1}}{n+1} + C \quad (n \neq -1).$
3. $\int \dfrac{dx}{x} = \ln|x| + C \quad (x \neq 0).$
4. $\int a^x dx = \dfrac{a^x}{\ln a} + C \quad (0 < a \neq 1).$

13. $\int \dfrac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left|x + \sqrt{x^2 \pm a^2}\right| + C.$
14. $\int \operatorname{tg} x dx = -\ln|\cos x| + C.$
15. $\int \operatorname{ctg} x dx = \ln|\sin x| + C.$
16. $\int \dfrac{dx}{x^2 + a^2} = \dfrac{1}{a}\operatorname{arctg}\dfrac{x}{a} + C.$

5. $\int e^{x}dx = e^{x} + C.$

6. $\int \cos x dx = \sin x + C.$

7. $\int \sin x dx = -\cos x + C.$

8. $\int \frac{dx}{\cos^2 x} = \mathrm{tg} x + C.$

9. $\int \frac{dx}{\sin^2 x} = -\mathrm{ctg} x + C.$

10. $\int \frac{dx}{\sqrt{1-x^2}} = \begin{cases} \arcsin x + C \\ -\arccos x + C \end{cases}$

11. $\int \frac{dx}{1+x^2} = \begin{cases} \mathrm{arctg} x + C \\ -\mathrm{arcctg} x + C \end{cases}.$

12. $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln \left| \frac{a+x}{a-x} \right| + C.$

17. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \frac{x}{a} + C.$

18. $\int \frac{dx}{\sin x} = \ln \left| \mathrm{tg} \frac{x}{2} \right| + C.$

19. $\int \frac{\mathrm{dx}}{\cos \mathrm{x}} = \ln \left| \mathrm{tg}\left(\frac{\mathrm{x}}{2} + \frac{\pi}{4} \right) \right| + \mathrm{C}.$

20. $\int shx \; dx = chx + C.$

21. $\int \mathrm{ch}\, x dx = \mathrm{sh}\, x + C.$

22. $\int \frac{dx}{\mathrm{ch}^2 x} = \mathrm{th}\, x + C.$

23. $\int \frac{dx}{\mathrm{sh}^2 x} = -\mathrm{cth}\, x + C.$

While integrating the functions we rarely have an opportunity directly use the basic formulas. As a rule, the integrand has to be transformed in order to convert the integral to the tabular. Some examples of such transformations are presented below:

Example 1.

$$\int \frac{(1+x)^2}{x\sqrt{x}} dx = \int \frac{1+2x+x^2}{x^{3/2}} dx = \int x^{-3/2} dx + 2\int x^{-1/2} dx + \int x^{1/2} dx =$$

$$= -2x^{-1/2} + 4x^{1/2} + \frac{2}{3} x^{3/2} + C.$$

Example 2.

$$\int \frac{\left(1+2x^2\right)dx}{x^2\left(1+x^2\right)} = \int \frac{\left(1+x^2\right)+x^2}{x^2\left(1+x^2\right)} dx = \int \frac{dx}{x^2} + \int \frac{dx}{1+x^2} = -\frac{1}{x} + arctgx + C.$$

Example 3.

$$\int \frac{dx}{\left(x^2-1\right)\left(1+x^2\right)} = \frac{1}{2} \int \frac{2}{\left(x^2-1\right)\left(1+x^2\right)} dx = \left\| x^2 + 1 - \left(x^2 - 1\right) \equiv 2 \right\| =$$

$$= \frac{1}{2} \int \frac{x^2 + 1 - \left(x^2 - 1\right)}{\left(x^2-1\right)\left(1+x^2\right)} dx = \frac{1}{2} \int \frac{dx}{x^2 - 1} - \frac{1}{2} \int \frac{dx}{1+x^2} = \frac{1}{2} \ln \left| \frac{1-x}{1+x} \right| - \frac{1}{2} arctgx + C.$$

Example 4.

$$\int \frac{dx}{\sin^2 x\cos^2 x} = \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x\cos^2 x}dx = \int \frac{dx}{\cos^2 x} + \int \frac{dx}{\sin^2 x} = tgx - ctgx + C.$$

Example 5.

$$\int \frac{dx}{1-\cos 2x} = \frac{1}{2}\int \frac{dx}{\sin^2 x} = -\frac{1}{2}ctgx + C.$$

Example 6. $J = \int \frac{dx}{\sqrt{x+1} + \sqrt{x-1}}$.

Solution. Getting rid of irrationality in a denominator, we get the following:

$$J = \frac{1}{2}\int\left(\sqrt{x+1} - \sqrt{x-1}\right)dx = \frac{1}{2}\int (x+1)^{1/2} d(x+1) - \frac{1}{2}\int (x-1)^{1/2} d(x-1) =$$

$$= \frac{1}{3}\sqrt{(x+1)^3} - \sqrt{(x-1)^3} + C.$$

Example 7.

$$I = \int x^3 \sqrt[3]{1+x^2}\,dx = \left\| xdx = \frac{1}{2}d\left(x^2\right) \right\| = \frac{1}{2}\int x^2 \sqrt[3]{1+x^2}\,d\left(x^2\right) =$$

$$\frac{1}{2}\int \left(x^2+1-1\right)\sqrt[3]{1+x^2}\,d\left(x^2+1\right) = \frac{1}{2}\int \left(x^2+1\right)\sqrt[3]{1+x^2}\,d\left(x^2+1\right) - \frac{1}{2}\int \sqrt[3]{1+x^2}\,d\left(x^2+1\right) =$$

$$= \frac{1}{2}\int \left(x^2+1\right)^{4/3} d\left(x^2+1\right) - \frac{1}{2}\int \left(x^2+1\right)^{1/3} d\left(x^2+1\right) = \frac{3}{14}\left(x^2+1\right)^{7/3} - \frac{3}{8}\left(x^2+1\right)^{4/3} + C$$

.

Note. While integrating the same function the results can be different by their shape. Actually they are either identical, or differ on some constant value.

The theorem (about invariance of the formulas of integrating). The shape of the formulas of integrating remains invariable in spite of whether variable of integrating is an independent variable or some differentiable function, i.e. if $\int f(x)dx = F(x) + C$ then $\int f(\varphi(x))d\varphi(x) = F(\varphi(x)) + C$.

This theorem allows converting of many integrals to the tabular.

Examples.

1. $\int xe^{x^2}dx = \left\| xdx = \frac{1}{2}d\left(x^2\right) \right\| = \frac{1}{2}\int e^{x^2} d\left(x^2\right) = \frac{1}{2}e^{x^2} + C.$

2. $\int \sin^5 x \cos x dx = \left\| \cos x dx = d(\sin x) \right\| = \int \sin^5 x d(\sin x) = \frac{\sin^6 x}{6} + C.$

3. $\int \frac{(arctgx)^2}{1+x^2} dx = \left\| \frac{dx}{1+x^2} = d(arctgx) \right\| = \int (arctgx)^2 d(arctgx) = \frac{(arctgx)^3}{3} + C.$

4. $\int \frac{xdx}{\sqrt{4-x^4}} = \frac{1}{2} \int \frac{d(x^2)}{\sqrt{2^2 - (x^2)^2}} = \frac{1}{2} \arcsin \frac{x^2}{2} + C.$

5. $\int \frac{dx}{x \ln x} = \left\| \frac{dx}{x} = d(\ln x) \right\| = \int \frac{d(\ln x)}{\ln x} = \ln|\ln x| + C.$

6. $\int \frac{e^x dx}{\sqrt{4-e^{2x}}} = \left\| e^x dx = d(e^x) \right\| = \int \frac{d(e^x)}{\sqrt{2^2 - (e^x)^2}} = \arcsin \frac{e^x}{2} + C.$

7. $\int \frac{\sin x \cos x dx}{\sqrt{a^2 \sin^2 x + b^2 \cos^2 x}} = \left\| 2(a^2 - b^2) \sin x \cos x dx = d(a^2 \sin^2 x + b^2 \cos^2 x) \right\| =$

$$= \frac{1}{2(a^2-b^2)} \int \frac{d(a^2 \sin^2 x + b^2 \cos^2 x)}{\sqrt{a^2 \sin^2 x + b^2 \cos^2 x}} = \frac{\sqrt{a^2 \sin^2 x + b^2 \cos^2 x}}{(a^2 - b^2)} + C, \ a \neq b.$$

8. $\int \frac{\cos x dx}{\sqrt{2+\cos 2x}} = \left\| \begin{matrix} \cos x dx = d(\sin x) \\ \cos 2x = 1 - 2\sin^2 x \\ 2 + \cos 2x = 3 - 2\sin^2 x \end{matrix} \right\| = \int \frac{d(\sin x)}{\sqrt{3 - 2\sin^2 x}} =$

$$= \frac{1}{\sqrt{2}} \int \frac{d(\sqrt{2}\sin x)}{\sqrt{3-2\sin^2 x}} = \frac{1}{\sqrt{2}} \arcsin \frac{\sqrt{2}\sin x}{\sqrt{3}} + C.$$

9. $\int \frac{2^x 5^x}{25^x - 4^x} dx = \int \frac{2^x 5^x}{4^x \left(\left(\frac{25}{4} \right)^x - 1 \right)} dx = \int \frac{\left(\frac{5}{2} \right)^x}{\left(\left(\frac{5}{2} \right)^{2x} - 1 \right)} dx =$

$$= \frac{1}{\ln \frac{5}{2}} \int \frac{d\left(\left(\frac{5}{2} \right)^x \right)}{\left(\left(\frac{5}{2} \right)^{2x} - 1 \right)} = \frac{1}{2 \ln \frac{5}{2}} \ln \left| \frac{\left(\frac{5}{2} \right)^x - 1}{\left(\frac{5}{2} \right)^x + 1} \right| + C.$$

10. $\int \frac{\sin x\cos xdx}{\sin^4 x+\cos^4 x}=\int \frac{\cos^2 xtgxdx}{\cos^4 x\left(1+tg^4 x\right)}=\int \frac{tgxdx}{\cos^2 x\left(1+tg^4 x\right)}=$

$$=\left\| \frac{tgxdx}{\cos^2 x}=\frac{1}{2}d\left(tg^2 x\right)\right\| =\frac{1}{2}\int \frac{d\left(tg^2 x\right)}{\left(1+tg^4 x\right)}=\frac{1}{2}arctg\left(tg^2 x\right)+C.$$

6.2 Methods of integrating

6.2.1 Integrating by *Substitution* (Change of variable)

The method of ***substitution*** is one of the main methods for calculating indefinite integrals. Let the function $x=\varphi(t)$be continuously differentiable and monotone, then

$$\int f(x)dx=\int f(\varphi(t))\varphi'(t)dt.$$

This method is present in several examples below:

Examples.

1. $I=\int \frac{3x-1}{4x^2-4x+17}dx.$

Solution. Let’s allocate a full quadrate in the denominator.

$4x^2-4x+17=4\left(x^2-x+\frac{1}{4}\right)+16=4\left(x-\frac{1}{2}\right)^2+16.$ Then

$$I=\frac{1}{4}\int \frac{3x-1}{\left(x-\frac{1}{2}\right)^2+4}dx=\left\| \begin{array}{l} x-\frac{1}{2}=u \\ x=u+\frac{1}{2} \\ dx=du \end{array}\right\| .=\frac{1}{4}\int \frac{3\left(u+\frac{1}{2}\right)-1}{u^2+4}du=$$

$$=\frac{3}{4}\int \frac{udu}{u^2+4}+\frac{1}{8}\int \frac{du}{u^2+4}=\frac{3}{8}\ln\left(u^2+4\right)+\frac{1}{16}arctg\frac{u}{2}+C=$$

$$=\frac{3}{8}\ln\left(x^2-x+\frac{17}{4}\right)+\frac{1}{16}arctg\frac{2x-1}{4}+C.$$

2. $I=\int \frac{2x+5}{\sqrt{x^2+6x+2}}dx.$

Solution. Let's transform the subduplicate allocating the full quadrate:

$x^2+6x+2=(x+3)^2-7$, then

$$I=\int\frac{2x+5}{\sqrt{(x+3)^2-7}}dx=\left\|\begin{matrix}x+3=u\\x=u-3\\dx=du\end{matrix}\right\|.=\int\frac{2(u-3)+5}{\sqrt{u^2-7}}du=\int\frac{d\left(u^2-7\right)}{\sqrt{u^2-7}}-\int\frac{du}{\sqrt{u^2-7}}=$$

$$\frac{2}{\sqrt{u^2-7}}-\ln\left|u+\sqrt{u^2-7}\right|+C=\frac{2}{\sqrt{x^2+6x+2}}-\ln\left|x+3+\sqrt{x^2+6x+2}\right|+C.$$

3. $I=\int\frac{dx}{\left(x^2+1\right)^{3/2}}$.

Solution. *The 1st way* (change of variable):

$$I=\left\|\begin{matrix}x=tgt\quad dx=\dfrac{dt}{\cos^2 t}\\ \left(x^2+1\right)^{3/2}=\dfrac{1}{\cos^3 t}\end{matrix}\right\|$$

$$I=\int\cos tdt=\sin t+C=\left\|\begin{matrix}1+ctg^2 t=\dfrac{1}{\sin^2 t}\\ \sin t=\sqrt{\dfrac{tg^2 t}{1+tg^2 t}}=\dfrac{x}{\sqrt{1+x^2}}\end{matrix}\right\|=\frac{x}{\sqrt{1+x^2}}+C.$$

The 2nd way (direct calculus)

$$I=\int\frac{dx}{x^3\left(1+\frac{1}{x^2}\right)^{3/2}}=-\frac{1}{2}\int\left(1+\frac{1}{x^2}\right)^{-3/2}d\left(1+\frac{1}{x^2}\right)=$$

$$=-\frac{1}{2}(-2)\left(1+\frac{1}{x^2}\right)^{-\frac{1}{2}}+C=\frac{x}{\sqrt{1+x^2}}+C.$$

4. $\int\frac{dx}{\sin^2 x+6\sin x\cos x-16\cos^2 x}=\int\frac{dx}{\cos^2 x\left(\frac{\sin^2 x}{\cos^2 x}+6\frac{\sin x}{\cos x}-16\right)}=$

$$=\int\frac{d(\mathrm{tg}\,x)}{\mathrm{tg}^2 x+6\,\mathrm{tg}\,x-16}=\left|t=\mathrm{tg}\,x\right|==\int\frac{dt}{(t+3)^2-25}=\frac{1}{10}\ln\left|\frac{t-2}{t+8}\right|+C=$$

$$=\frac{1}{10}\ln\left|\frac{\mathrm{tg}\,x-2}{\mathrm{tg}\,x+8}\right|+C.$$

5. $\int \frac{\cos^3 x}{2+\sin x}dx \left| t=\sin x \right| = \int \frac{(1-t^2)dt}{2+t} = \int \left(-t+2-\frac{3}{2+t}\right)dt =$

$= -\frac{t^2}{2} + 2t - 3\ln(2+t) + C = -\frac{\sin^2 x}{2} + 2\sin x - 3\ln(2+\sin x) + C.$

The integrals of the form $\int R\left(x^{\frac{m}{n}}, x^{\frac{l}{s}}, ..., x^{\frac{p}{r}}\right)dx$, where R is a rational function of its arguments are calculated by the substitute of the variable $x = t^k$ (k is a common denominator of the fractions), allowing to avoid irrationality.

6. $I = \int \frac{dx}{\sqrt{x}(x+1)}$.

Solution. In the given example $k = 2$, therefore it is necessary to make a substitution $x = t^2$. Then

$$I = \int \frac{2tdt}{t(t^2+1)} = 2\int \frac{dt}{t^2+1} = 2\,\mathrm{arctg}\,t + C = 2\,arctg\sqrt{x} + C.$$

This integral can be also calculated directly.

The 2nd way:

$$I = \int \frac{dx}{\sqrt{x}(x+1)} = \left\| \frac{dx}{\sqrt{x}} = 2d(\sqrt{x}) \right\| = 2\int \frac{d(\sqrt{x})}{(\sqrt{x})^2+1} = 2\,arctg\sqrt{x} + C.$$

7. $\int \frac{dx}{\sqrt[4]{x}+\sqrt{x}} \left| \begin{matrix} x = t^4 \\ \sqrt[4]{x} = t \\ dx = 4t^3\,dt \end{matrix} \right| = \int \frac{4t^3\,dt}{t+t^2} = 4\int \frac{t^2\,dt}{t+1} = 4\int \frac{(t^2-1)+1}{t+1}dt =$

$= 4\left(\int (t-1)dt + \int \frac{dt}{t+1}\right) = 4\left(\frac{(t-1)^2}{2} + \ln(t+1)\right) + C =$

$= 4\left(\frac{(\sqrt[4]{x}-1)^2}{2} + \ln(\sqrt[4]{x}+1)\right) + C.$

While integrating the expressions like $R\left(x, \sqrt{x^2 \pm a^2}\right)$ we use the following substitutions:

a) $\int R\left(x, \sqrt{a^2 + x^2}\right)dx$ $\left| \begin{matrix} x = a \cdot tgt; & dx = \dfrac{a}{\cos^2 t} dt \\ x = a \cdot sht; & dx = a \cdot chtdt \end{matrix} \right|$

b) $\int R\left(x, \sqrt{a^2 - x^2}\right)dx$ $\left| \begin{matrix} x = a \sin t; & dx = a \cos tdt \\ x = a \cos t; & dx = -a \sin tdt \end{matrix} \right|$

c) $\int R\left(x, \sqrt{x^2 - a^2}\right)dt$ $\left| \begin{matrix} x = \dfrac{a}{\sin t}; & dx = -\dfrac{a \cos t}{\sin^2 t} dt \\ x = a \cdot cht; & dx = a \cdot shtdt \end{matrix} \right|$.

8.

$$\int x^2 \sqrt{1 - x^2}\, dx = \int \sin^2 t \sqrt{1 - \sin^2 t} \cos tdt = \int \sin^2 t \cos^2 tdt = \frac{1}{4}\int \sin^2 2tdt =$$

$$= \frac{1}{4}\int \frac{1 - \cos 4t}{2} dt = \frac{1}{8}\left(t - \frac{\sin 4t}{4}\right) + C = \frac{1}{8}\left(\arcsin x - \frac{\sin(4\arcsin x)}{4}\right) + C.$$

9. $I = \int \dfrac{x^2\, dx}{\sqrt{x^2 - 1}} = \left\| \begin{matrix} x = cht \\ dx = shtdt \end{matrix} \right\| = \int \dfrac{ch^2 t\; shtdt}{\sqrt{ch^2 t - 1}} = \int \dfrac{ch^2 t\; shtdt}{sht} = \int ch^2 tdt =$

$$\int \frac{1 + ch2t}{2} dt = \frac{1}{2}\left(t + \frac{1}{2} sh2t\right) + C = \frac{1}{2}\left(arcchx + \frac{1}{2} sh(2\, arcchx)\right) + C.$$

10. $J = \int \dfrac{x^5 dx}{\sqrt{1 - x^2}}$.

Solution. *The 1st way* (change of variable)

$$J = \int \frac{x^5 dx}{\sqrt{1 - x^2}} \left\| \begin{matrix} x = sin\,t, \quad dx = cos\,tdt \\ \sqrt{1 - x^2} = cos\,t \end{matrix} \right\| = \int sin^5\, tdt = -\int \left(1 - cos^2\, t\right)^2 d\, cos\, t =$$

$$= -\int \left(1 - 2cos^2 t + cos^4 t\right) d\, cos\,t = -cos\,t + \frac{2}{3}cos^3 t - \frac{1}{5}cos^5 t =$$

$$= \left\| \begin{matrix} x = sin\,t \\ cos\,t = \sqrt{1 - x^2} \end{matrix} \right\| = -\sqrt{1 - x^2}\left(1 - \frac{2}{3}\left(1 - x^2\right) + \frac{1}{5}\left(1 - x^2\right)^2\right) =$$

$$= -\frac{1}{15}\sqrt{1 - x^2}\left(8 + 4x^2 + 3x^4\right) + C.$$

The 2nd way.

$$J = \int \frac{x^5\, dx}{\sqrt{1 - x^2}} = \left\| \begin{matrix} t^2 = 1 - x^2, \quad 2tdt = -2xdx \\ x^4 = 1 - 2t^2 + t^4, \; x^5 dx = -\left(1 - 2t^2 + t^4\right)tdt \end{matrix} \right\| =$$

$$= -\int \frac{t\left(1-2t^2+t^4\right)}{t}dt = -\int \left(1-2t^2+t^4\right)dt = -\left(t-\frac{2}{3}t^3+\frac{1}{5}t^5\right)+C =$$

$$-\frac{1}{15}\left(8+4x^2+3x^4\right)\sqrt{1-x^2}+C.$$

11.

$$I = \int \frac{\sin x dx}{\cos x\sqrt{1+\sin^2 x}} = \left\| \begin{matrix} \cos x = t \\ -\sin x dx = dt \end{matrix} \right\| = -\int \frac{dt}{t\sqrt{2-t^2}} = -\int \frac{d\left(\frac{\sqrt{2}}{t}\right)}{\sqrt{\left(\frac{\sqrt{2}}{t}\right)^2-1}} =$$

$$\frac{1}{\sqrt{2}}\ln\left|\frac{\sqrt{2}}{t}+\sqrt{\frac{2}{t^2}-1}\right|+C = \frac{1}{\sqrt{2}}\ln\left|\frac{\sqrt{2}+\sqrt{2-\cos^2 x}}{\cos x}\right|+C.$$

While calculating the integrals like $\int \frac{dx}{a+b\cos x+c\sin x}$ we use the *universal trigonometric substitution* $tg\frac{x}{2}=t$ or $x=2\,arctgt$. Then:

$$\sin x = \frac{2tg\frac{x}{2}}{1+tg^2\frac{x}{2}} = \frac{2t}{1+t^2},\ \cos x = \frac{1-tg^2\frac{x}{2}}{1+tg^2\frac{x}{2}} = \frac{1-t^2}{1+t^2},\ dx = \frac{2dt}{1+t^2}$$

12. $I = \int \frac{dx}{9+8\cos x+\sin x}$.

Solution. Taking into consideration the formulas above, we get:

$$I = \int \frac{2dt}{t^2+2t+17} = 2\int \frac{d(t+1)}{(t+1)^2+4^2} == \frac{1}{2}\text{arctg}\frac{t+1}{4}+C = \frac{1}{2}\text{arctg}\frac{tg\frac{x}{2}+1}{4}+C$$

The integrals like

$$\text{a)} \int \frac{P_m(x)dx}{\sqrt{ax^2+bx+c}},\ \text{b)} \int \frac{dx}{(x-\alpha)^n\sqrt{ax^2+bx+c}}.$$

The integral б) by the substitution $x-\alpha=\frac{1}{z}$ is converted to the integral a). The integral a) is calculated by the method of undetermined coefficients according to the following formula::

$$\int \frac{P_m(x)dx}{\sqrt{ax^2+bx+c}} = Q_{m-1}(x)\sqrt{ax^2+bx+c}+A\int \frac{dx}{\sqrt{ax^2+bx+c}}$$

where P_m is an *m*-degree polynomial, Q_{m-1} is an *m-1*-degree polynomial with undetermined coefficients, A – indefinite const.

13. $I=\int \frac{\left(x^2+1\right)dx}{\sqrt{-x^2+3x-2}}$.

Solution.

$$\int \frac{\left(x^2+1\right)dx}{\sqrt{-x^2+3x-2}}=(ax+b)\sqrt{-x^2+3x-2}+A\int \frac{dx}{\sqrt{-x^2+3x-2}}.$$

Let us differentiate the both parts of the obtained equality:

$$\frac{\left(x^2+1\right)}{\sqrt{-x^2+3x-2}}=(ax+b)\frac{-2x+3}{2\sqrt{-x^2+3x-2}}+a\sqrt{-x^2+3x-2}+$$

$$+\frac{A}{\sqrt{-x^2+3x-2}}.$$

Let’s multiply both parts of the equality on $\sqrt{-x^2+3x-2}$.

$x^2+1=-ax^2+\frac{3}{2}ax-bx+\frac{3}{2}b-ax^2+3ax-2a+A$ or

$$x^2+1=-2ax^2+x\left(\frac{3}{2}a-b+3a\right)+\frac{3}{2}b-2a+A.$$

Let's equate factors at identical degrees *x*:

$$-2a=1 \Rightarrow a=-\frac{1}{2}.$$

$$\frac{9}{2}a-b=0 \Rightarrow b=\frac{9}{2}a=\frac{-9}{2\cdot 2}=-\frac{9}{4}.$$

$$\frac{3}{2}b-2a+A=1 \Rightarrow A=1+2a-\frac{3}{2}b=1-1+\frac{27}{8}=\frac{27}{8}.$$

Then

$$I=-\frac{1}{4}(2x+9)\sqrt{-x^2+3x-2}+\frac{27}{8}\int \frac{dx}{\sqrt{\frac{1}{4}-\left(x-\frac{3}{2}\right)^2}}=-\frac{2x+9}{4}\sqrt{-x^2+3x-2}+$$

$$+\frac{27}{8}\arcsin(2x-3)+C.$$

The integrals like $\int R\left(x,\sqrt[m]{\frac{ax+b}{a_1x+b_1}},...,\sqrt[p]{\frac{ax+b}{a_1x+b_1}}\right)dx,$ where *R* is a rational function of its arguments, are calculated by the substitution

$\frac{ax+b}{a_1x+b_1}=t^k$ (where k is the least common multiply of the fractions) allowing to avoid the irrationality.

14. $I=\int\frac{dx}{\sqrt[4]{(x-1)^3(x+2)^5}}$.

Solution. Let's transform the denominator:

$$\sqrt[4]{(x-1)^3(x+2)^5}=\sqrt[4]{\frac{(x-1)^3(x-1)^5(x+2)^5}{(x-1)^5}}=\sqrt[4]{(x-1)^8\left(\frac{x+2}{x-1}\right)^5}=$$

$$=(x-1)^2\,\frac{x+2}{x-1}\sqrt[4]{\frac{x+2}{x-1}}.$$

Then

$$I=\int\frac{dx}{\frac{x+2}{x-1}(x-1)^2\sqrt[4]{\frac{x+2}{x-1}}}=\left\|\begin{array}{ll}\frac{x+2}{x-1}=t^4, & \frac{(x-1)-(x+2)}{(x-1)^2}dx=4t^3\,dt\\ \frac{-3\,dx}{(x-1)^2}=4t^3\,dt, & \frac{dx}{(x-1)^2}=-\frac{4}{3}t^3\,dt\end{array}\right\|=$$

$$=-\frac{4}{3}\int\frac{t^3\,dt}{t^4\cdot t}=\frac{4}{3t}+C=\frac{4}{3\sqrt[4]{\frac{x+2}{x-1}}}+C=\frac{4}{3}\sqrt[4]{\frac{x-1}{x+2}}+C.$$

6.2.1 Method of integrating in parts

Let the functions $u=u(x), v=v(x)$ have the continuous derivatives, then the following formula of *integrating in parts* is valid:

$$\boxed{\int udv=uv-\int vdu}.$$

Note. The formula doesn't produce the final result but only transforms the problem from calculating the integral $\int udv$ to calculating the integral $\int vdu$, that at the *successful* choice of *u* and *v* can be much more simple.

It should be mentioned there are no common methods for the right choice of *u* and *v*, but we can try to give some recommendations for certain cases.

As a rule, the method of integrating in parts is used in the case when the integrand includes the product of rational and transcendental functions and other methods are unsuitable.

For example, $\int P_n(x)\cos\alpha x dx$, $\int P_n(x)\sin\alpha x dx$, $\int P_n(x)e^{\alpha x}dx$, $\int x^k \, arctgx dx$, $\int x^k \ln x dx$, etc.

If the integrand has a look like $P_n(x)\cos\alpha x$, $P_n(x)\sin\alpha x$, $P_n(x)e^{\alpha x}$, the polynomial $P_n(x)$ is selected as *u*.

If the integrand is a product of logarithmic or inverse trigonometric functions and a polynomial, the mentioned functions are selected as *u*.

Examples.

1.

$$\int x\sin x dx \left| \begin{array}{l} u = x, \quad du = dx \\ dv = \sin x dx \\ v = -\cos x \end{array} \right| = -x\cos x + \int \cos x dx = -x\cos x + \sin x + C.$$

2.

$$\int x\,arctgx dx = \left\| \begin{array}{l} u = arctgx;\ du = \dfrac{dx}{1+x^2} \\ dv = xdx;\ \ v = \dfrac{x^2}{2} \end{array} \right\| = arctgx \cdot \frac{x^2}{2} - \frac{1}{2}\int x^2 \frac{dx}{1+x^2} =$$

$$= \frac{x^2}{2}\operatorname{arctg} x - \frac{1}{2}\int \frac{(x^2+1)-1}{1+x^2}dx = \frac{x^2}{2}\operatorname{arctg} x - \frac{1}{2}\int dx + \frac{1}{2}\int \frac{dx}{1+x^2} =$$

$$\frac{x^2}{2}\operatorname{arctg} x - \frac{x}{2} + \frac{1}{2}\operatorname{arctg} x + C.$$

In some cases due to the method of integrating in parts we manage to obtain the equation concerning the initial parameter.

3. $I = \int \sqrt{a^2 + x^2}\,dx$.

Solution. *The 1*st *way* (integrating in parts).

$$I=\int\sqrt{a^2+x^2}\,dx=\left\|\begin{matrix} u=\sqrt{a^2+x^2}, & du=\dfrac{xdx}{\sqrt{a^2+x^2}} \\ dv=dx, & v=x\end{matrix}\right\|=$$

$$x\sqrt{a^2+x^2}-\int\frac{x^2\,dx}{\sqrt{a^2+x^2}}=x\sqrt{a^2+x^2}-\int\frac{\left(x^2+a^2\right)-a^2}{\sqrt{a^2+x^2}}\,dx=$$

$$x\sqrt{a^2+x^2}-\underbrace{\int\sqrt{a^2+x^2}\,dx}_{I}+a^2\int\frac{dx}{\sqrt{a^2+x^2}}=$$

$$=x\sqrt{a^2+x^2}-I+a^2\ln\left(x+\sqrt{x^2+a^2}\right)+2C$$

Thus, we obtain the equation concerning the initial parameter, i.e. concerning I. Solving this equation, we obtain:

$$2I=x\sqrt{a^2+x^2}+a^2\ln\left(x+\sqrt{x^2+a^2}\right)+2C$$

$$I=\frac{1}{2}\left(x\sqrt{a^2+x^2}+a^2\ln\left(x+\sqrt{x^2+a^2}\right)\right)+C$$

The 2nd way (integrating by method of change of variable).

$$I=\int\sqrt{a^2+x^2}\,dx=\left\|\begin{matrix} x=asht \\ dx=achtdt \\ \sqrt{a^2+x^2}=acht\end{matrix}\right\|=a^2\int ch^2\,tdt=$$

$$=\frac{a^2}{2}\int(1+ch2t)dt=\frac{a^2t}{2}+\frac{a^2}{4}sh2t+c\left\|\begin{matrix} sht=\dfrac{x}{a}=\dfrac{e^t-e^{-t}}{2} \\ e^t=\dfrac{x\pm\sqrt{a^2+x^2}}{a}, \quad e^t>0, \\ t=\ln\left(x+\sqrt{x^2+a^2}\right)-\ln a \\ sh2t=2\,sh\,t\,ch\,t=\dfrac{2x\sqrt{a^2+x^2}}{a^2}\end{matrix}\right\|=$$

$$=\frac{x}{2}\sqrt{a^2+x^2}+\frac{a^2}{2}\ln\left(x+\sqrt{a^2+x^2}\right)+C.$$

4.

$$I=\int\frac{dx}{\cos^3 x}=\left\|\begin{matrix} u=\frac{1}{\cos x}, & du=\frac{\sin x}{\cos^2 x}dx \\ dv=\frac{dx}{\cos^2 x}, & v=tgx \end{matrix}\right\|=\frac{1}{\cos x}tgx-\int tgx\frac{\sin x}{\cos^2 x}dx=$$

$$=\frac{tgx}{\cos x}-\int\frac{\sin^2 x}{\cos^3 x}dx=\frac{tgx}{\cos x}-\int\frac{1-\cos^2 x}{\cos^3 x}dx=\frac{tgx}{\cos x}-\int\frac{dx}{\cos^3 x}+\int\frac{dx}{\cos x}=$$

$$=\frac{tgx}{\cos x}-I+\ln\left|tg\left(\frac{x}{2}+\frac{\pi}{4}\right)\right|+2C$$

Similarly, like in the example 3, we obtain the equation concerning I, which solution gives us the following:

$$I=\frac{1}{2}\left(\frac{tgx}{\cos x}+\ln\left|tg\left(\frac{x}{2}+\frac{\pi}{4}\right)\right|\right)+C.$$

5. $I=\int e^{ax}\cos nxdx$

$$\int e^{ax}\cos nxdx=\left\|\begin{matrix} u=e^{ax}, & du=ae^{ax}dx \\ dv=\cos nxdx, & v=\frac{\sin nx}{n} \end{matrix}\right\|=e^{ax}\frac{\sin nx}{n}-\frac{a}{n}\int e^{ax}\sin nxdx=$$

$$=\left\|\begin{matrix} u=e^{ax}, & du=ae^{ax}dx \\ dv=\sin nxdx, & v=-\frac{\cos nx}{n} \end{matrix}\right\|=$$

$$=e^{ax}\frac{\sin nx}{n}-\frac{a}{n}\left(-e^{ax}\frac{\cos nx}{n}+\frac{a}{n}\underbrace{\int e^{ax}\cos nxdx}_{I}\right)=$$

$$=\frac{e^{ax}}{n}\left(\sin nx+\frac{a\cos nx}{n}\right)-\frac{a^2}{n^2}I+C.$$

$$I\left(1+\frac{a^2}{n^2}\right)=\frac{e^{ax}}{n^2}(n\sin nx+a\cos nx)+C.$$

$$I=\frac{e^{ax}}{a^2+n^2}(n\sin nx+a\cos nx)+\frac{C}{a^2+n^2}n^2=$$

$$=\frac{e^{ax}}{a^2+n^2}(n\sin nx+a\cos nx)+C_1.$$

The method of integrating in parts is combined with the method of integrating by substitution of variables.

For example,

6.

$$I=\int \sin^2\left(\sqrt{x}\right)dx=\left\| \begin{matrix} x=t^2 \\ dx=2tdt \end{matrix} \right\| =2\int t\sin^2 tdt=2\int t\frac{1-\cos 2t}{2}dt=$$

$$=\frac{t^2}{2}-\int t\cos 2tdt=\left\| \begin{matrix} u=t, \quad du=dt \\ dv=\cos 2tdt, \quad v=\dfrac{\sin 2t}{2} \end{matrix} \right\| =\frac{t^2}{2}-t\frac{\sin 2t}{2}+\frac{1}{2}\int \sin 2tdt=$$

$$=\frac{t^2}{2}-t\frac{\sin 2t}{2}-\frac{1}{4}\cos 2t+C=\frac{\sqrt{x}}{2}\left(\sqrt{x}-\sin 2\sqrt{x}\right)-\frac{1}{4}\cos 2\sqrt{x}+C.$$

6.2.3 Integration of rational fractions

The antiderivative function exists for any continuous function (it can be strictly proved). However, the problem of finding the analytical expression of the antiderivative function in the elementary functions, i.e. in final combination of elementary function, has an exact solution only in some separate cases. Here is the table of "similar" integrals, but the integrals in the first line can be represented in the elementary functions, and at the second – those, can't be integrated in the elementary functions.

$\int \frac{dx}{\sin x}$	$\int e^{-\sqrt{x}}dx$	$\int \frac{\ln xdx}{x}$	$\int \frac{dx}{\sqrt{x^2+1}}$	$\int e^{\arcsin x}dx$	$\int \sqrt{tgx}dx$
$\int \frac{\sin xdx}{x}$	$\int e^{-x^2}dx$	$\int \frac{dx}{\ln x}$	$\int \frac{dx}{\sqrt{x^3+1}}$	$\int e^{arctgx}dx$	$\int \sqrt{\sin xdx}$

Only rather small class of functions can be integrated in the elementary functions. For example, Chebishev has proved the theorem the integral from the differential polynomial $\int x^m\left(a+bx^n\right)^p dx$, where n, m, p are the rational numbers, can be expressed through the elementary functions only in three following cases:

1. p is an integer (Consider $x=t^N$, where N is a common denominator of the fractions m and n)

2. $\frac{m+1}{n}$ is an integer. (Consider $a+bx^{n}=t^{N}$, where N is a common denominator of the fraction p).

3. $\frac{m+1}{n}+p$ is an integer. (Substitute $b+ax^{-n}=t^{N}$, where N is common denominator of the fraction p).

The rational fractions belong to the functions, which integrals can be expressed through the elementary functions. Rational fraction we call the following relation:

$$R(x)=\frac{P_n(x)}{Q_m(x)}=\frac{a_0x^n+a_1x^{n-1}+...+a_{n-1}x+a_n}{b_0x^m+b_1x^{m-1}+...+b_{m-1}x+b_m}.$$

Any rational fraction can be represented as a sum of a polynomial and elementary fractions. Elementary fractions are the fractions of the four following types:

a) $\frac{A}{x-a}$.

b) $\frac{A}{(x-a)^n}$.

c) $\frac{Ax+B}{x^2+px+q}$, where $(p^2-4q)<0$.

d) $\frac{Ax+B}{\left(x^2+px+q\right)^n}$.

Finding the integrals of the rational fractions should be carried out according to the following scheme:

1. If $n \geq m$ (the improper fraction), we should secure an integer part, presenting the integrand as a sum of the integer part of a polynomial and proper rational fraction.

2. Disclose the denominator of a proper rational fraction $Q_m(x)$ on the factors, correspondent to real and couple of complex roots, i.e. factors like $(x-a)^k,\left(x^2+px+q\right)^r$ where $(p^2-4q)<0$.

3. Disclose a proper rational fraction on simplest according to the following theorem:

The theorem. If

$$Q_m(x)=b_0(x-a)^{\alpha}(x-b)^{\beta}...\left(x^2+px+q\right)^{\mu}...\left(x^2+lx+s\right)^{\nu},$$

then proper irreducible rational fraction $R(x)=\frac{P_n(x)}{Q_m(x)}$ can be represented the following way:

$$R(x)=\frac{P_n(x)}{Q_m(x)}=\frac{A}{(x-a)^{\alpha}}+\frac{A_1}{(x-a)^{\alpha-1}}+\ldots+\frac{A_{\alpha-1}}{(x-a)}+\frac{B}{(x-b)^{\beta}}+\frac{B_1}{(x-b)^{\beta-1}}+.+$$

$$+\frac{Mx+N}{\left(x^2+px+q\right)^{\mu}}+\frac{M_1x+N_1}{\left(x^2+px+q\right)^{\mu-1}}+\ldots+\frac{M_{\mu-1}x+N_{\mu-1}}{\left(x^2+px+q\right)}+\ldots+$$

$$+\frac{Px+Q}{\left(x^2+lx+s\right)^{\nu}}+\frac{P_1x+N_1}{\left(x^2+lx+s\right)^{\nu-1}}+\ldots+\frac{P_{\nu-1}x+N_{\nu-1}}{\left(x^2+lx+s\right)}.$$

The coefficients $A, A_1,..., B, B_1,...$ can be defined according to the following. The obtained equation is an identity, therefore reducing the fractions to the common denominator we obtain the identical polynomials in the numerators on the right and left. Equating the coefficients of the common degrees x we obtain a system of equations for defining the undetermined coefficients $A, A_1,..., B, B_1,...$.

Alongside with the mentioned we can use the next remark for defining the undetermined coefficients: as the polynomials, obtained in the left and right sides of the equation, after reducing to the common denominator are identically equal, their values are equal at any x. Giving x concrete values we'll get the equations for defining the unknown coefficients. It's convenient to choose real roots of the denominator. On practice we can use both methods for finding the undetermined coefficients simultaneously.

4. The integrals of the simplest rational fractions are calculated according to the following formulas:

a) $\int\frac{Adx}{x-a}=A\ln|x-a|+C$

b) $\int\frac{A}{(x-a)^n}dx=A\int(x-a)^{-n}d(x-a)=\frac{A}{(1-n)(x-a)^{n-1}}+C,\ n\neq 1$

c) $\int\frac{Ax+B}{x^2+px+q}dx=\int\frac{\frac{A}{2}(2x+p)-\frac{A}{2}p+B}{x^2+px+q}dx=\frac{A}{2}\ln\left(x^2+px+q\right)+$

$$+\left(B-\frac{A}{2}p\right)\int\frac{d\left(x+{}^{p}\!/\!_{2}\right)}{\left(x+\frac{p}{2}\right)^2+q-\frac{p^2}{4}}=\frac{A}{2}\ln\left(x^2+px+q\right)+$$

$$\frac{B-\frac{A}{2}p}{\sqrt{q-\frac{p^2}{4}}}\,arctg\frac{x+\frac{p}{2}}{\sqrt{q-\frac{p^2}{4}}}+C,\text{where }(p^2-4q)<0.$$

While calculating the integrals of fractions of the 4th type we have to make the following conversions of the integrand:

e)

$$\int\frac{Ax+B}{\left(x^2+px+q\right)^n}dx=\int\frac{\frac{A}{2}(2x+p)-\frac{A}{2}p+B}{\left(x^2+px+q\right)^n}dx=\frac{A}{2}\int\frac{d\left(x^2+px+q\right)}{\left(x^2+px+q\right)^n}+$$

$$+\left(B-\frac{A}{2}p\right)\int\frac{dx}{\left(x^2+px+q\right)^n}=\frac{A}{2}\frac{\left(x^2+px+q\right)^{-n+1}}{1-n}+\left(B-\frac{A}{2}p\right)\cdot$$

$$\cdot\int\frac{dx}{\left(\left(x+\frac{p}{2}\right)^2+\left(q-\frac{p^2}{4}\right)\right)^n}=\left\|\begin{array}{l}x+\frac{p}{2}=t\\ dx=dt\\ q-\frac{p^2}{4}=a^2\end{array}\right\|=$$

$$\frac{A}{2}\frac{\left(x^2+px+q\right)^{-n+1}}{1-n}+\left(B-\frac{A}{2}p\right)\cdot$$

$$\cdot\int\frac{dt}{\left(t^2+a^2\right)^n},\text{ where } q-\frac{p^2}{4}>0.$$

For calculating the integral $I_n=\int\frac{dt}{\left(t^2+a^2\right)^n}$, $n\in N$ we use either the method allowing to represent the integral I_n through the integral I_{n-1}, or substitution $t=a\cdot tg\,z$.

There are some **examples** of integrating the rational fractions.

1. $I=\int \frac{x^3+1}{x^3-x^2}\,dx.$

Solution. 1) The fraction $\frac{x^3+1}{x^3-x^2}$ is improper, as degree of the numerator is equal to degree of the denominator. Let's secure the integer part.

$$\frac{x^3+1}{x^3-x^2}=\frac{\left(x^3-x^2\right)+(1+x^2)}{x^3-x^2}=1+\frac{x^2+1}{x^3-x^2}.$$

Then the initial integral will be converted to the following two integrals:

$$I=\int\left(1+\frac{x^2+1}{x^3-x^2}\right)dx=\int dx+\int\frac{1+x^2}{x^2(x-1)}\,dx.$$

2) Denominator of the fraction of the second integral has real multiple roots and can be represented as a product $x^2(x-1)$.

Let's disclose the integrand on the simplest fractions.

$$\frac{x^2+1}{x^2(x-1)}=\frac{A}{x^2}+\frac{B}{x}+\frac{C}{x-1}.$$

Let's reduce to the common denominator and equate the numerators:

$$1+x^2=A(x-1)+B(x-1)x+Cx^2.$$

For finding the coefficients at first let's assume $x=1$ and get $2=C$, then considering $x=0$ we find the coefficient $A=-1$. Equating the x coefficients, we find B from the equation: $A-B=0$, $B=-1$.

Therefore, $I=x-\int\frac{dx}{x^2}-\int\frac{dx}{x}+2\int\frac{dx}{x-1}=x+\frac{1}{x}+\ln\left|\frac{(x-1)^2}{x}\right|+C.$

2. $\int\frac{dx}{x^3-1}=\int\frac{dx}{(x-1)\left(x^2+x+1\right)}$;

Solution. Let's disclose the fraction on the simplest:

$$\frac{1}{(x-1)\left(x^2+x+1\right)}=\frac{A}{x-1}+\frac{Bx+c}{x^2+x+1};$$

then $1=A\left(x^2+x+1\right)+(Bx+c)(x-1)$

Let $x=1$ then $1=3A,\ A=1/3$.

Equating factors at identical degrees x (for example, the 1st and the 2nd) we obtain the set of equations for finding the rest coefficients:

$$x^1 \cdot 0 = A + C - B, \quad C = -2/3$$

$$x^2 \cdot 0 = A + B \Rightarrow B = -1/3.$$

Then

$$\int \frac{dx}{x^3-1} = \frac{1}{3}\int \frac{d(x-1)}{x-1} - \frac{1}{3}\int \frac{x+2}{x^2+x+1}dx = \frac{1}{3}\ln|x-1| - \frac{1}{3}\int \frac{\frac{1}{2}(2x+1) - \frac{1}{2} + 2}{x^2+x+1}dx =$$

$$= \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln\left(x^2+x+1\right) - \frac{1}{2}\int \frac{d(x+1/2)}{(x+1/2)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} =$$

$$= \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln\left(x^2+x+1\right) - \frac{1}{\sqrt{3}}\, arctg\frac{2x+1}{\sqrt{3}} + C.$$

3. $I = \int \frac{dx}{\left(x^2+9\right)^2}$.

Solution. The integrand is the simplest rational fraction of the 4th type. Here are two ways for calculating the integral of this fraction.

The 1st way

$$\int \frac{dx}{\left(x^2+9\right)^2} = \frac{1}{9}\int \frac{\left(9+x^2\right) - x^2}{\left(x^2+9\right)^2}dx = \frac{1}{9}\int \frac{dx}{x^2+9} - \frac{1}{9}\int x\frac{xdx}{\left(x^2+9\right)^2} =$$

$$= \left\| \begin{array}{l} u = x \quad dv = \dfrac{xdx}{\left(x^2+9\right)^2} \\ du = dx \quad v = \dfrac{1}{2}\displaystyle\int \frac{d\left(x^2+9\right)}{\left(x^2+9\right)^2} = \\ = -\dfrac{1}{2\left(x^2+9\right)} \end{array} \right\| = \frac{1}{9}\int \frac{dx}{x^2+9} - \frac{1}{9}\left(-\frac{x}{2\left(x^2+9\right)} + \frac{1}{2}\int \frac{dx}{\left(x^2+9\right)}\right) =$$

$$= \frac{1}{18}\int \frac{dx}{x^2+9} + \frac{1}{18}\cdot\frac{x}{x^2+9} = \frac{1}{54}\, arctg\frac{x}{3} + \frac{1}{18}\cdot\frac{x}{x^2+9} + C.$$

The 2nd way

$$\int \frac{dx}{\left(x^2+9\right)^2} = \left\| \begin{matrix} x = 3tgz & tgz = \frac{x}{3} \\ dx = \frac{3dz}{\cos^2 z} & \end{matrix} \right\| = 3\int \frac{dz}{\cos^2 z\left(9tg^2 z+9\right)^2} = \frac{1}{27}\int \cos^2 zdz =$$

$$\frac{1}{27}\int \frac{1+\cos 2z}{2}dz = \frac{1}{54}z + \frac{\sin 2z}{27\cdot 2} + C = \frac{1}{54}z + \frac{2\sin z\cos z}{27\cdot 2} + C = \frac{1}{54}z + \frac{tgz}{54\left(1+tg^2 z\right)} +$$

$$+C = \frac{1}{54}arctg\frac{x}{3} + \frac{1}{18}\cdot\frac{x}{x^2+9} + C.$$

Note. While calculating the integrals of rational functions sometimes we can avoid disclosing them on the simplest using other methods, for example:

1. $I = \int \frac{t^2}{\left(t^2-1\right)^2}dt.$

Solution. This integral can be calculated using the method of integrating in parts:

$$u = t,\ dv = \frac{tdt}{\left(t^2-1\right)^2} = \frac{1}{2}\frac{d\left(t^2-1\right)}{\left(t^2-1\right)^2};\ \ du = dt,\ v = \frac{1}{2}\int \frac{d\left(t^2-1\right)}{\left(t^2-1\right)^2} = -\frac{1}{2\left(t^2-1\right)}$$

we obtain

$$I = -\frac{t}{2\left(t^2-1\right)} + \frac{1}{2}\int \frac{dt}{t^2-1} = -\frac{t}{2\left(t^2-1\right)} + \frac{1}{4}\ln\left|\frac{t+1}{t-1}\right| - C.$$

2. $I = \int tg^5 xdx$**.**

Solution. *The 1st way*:

$$\int tg^5 xdx = \left\| \begin{matrix} tgx = t \\ x = arctgt \\ dx = \frac{dt}{1+t^2} \end{matrix} \right\| = \int \frac{t^5}{1+t^2}dt = \left\| tdt = \frac{1}{2}d\left(t^2\right) \right\| = \frac{1}{2}\int \frac{t^4 d\left(t^2\right)}{1+t^2} = \left\| t^2 = u \right\| =$$

$$= \frac{1}{2}\int \frac{u^2 du}{1+u} = \frac{1}{2}\int \frac{u^2-1+1}{1+u}du = \frac{1}{2}\left[\int \frac{u^2-1}{1+u}du + \int \frac{du}{1+u}\right] =$$

$$\frac{1}{2}\int(u-1)du+\frac{1}{2}\ln|1+u| = \frac{1}{2}\frac{(u-1)^2}{2}+\frac{1}{2}\ln|u+1|+C = \left\|\begin{matrix} u=t^2, \\ t=tgx, \\ u=tg^2x \end{matrix}\right\| =$$

$$=\frac{1}{4}\left(tg^2x-1\right)^2+\frac{1}{2}\ln\left|tg^2x+1\right|+C=\frac{1}{4}\left(tg^2x-1\right)^2-\frac{1}{2}\ln\left(\cos^2 x\right)+C.$$

The 2nd way:

$$\int tg^5 xdx=\int tg^3 x\left(\frac{1`}{\cos^2 x}-1\right)dx=\int tg^3 xdtg\,x-\int tg^3 xdx=$$

$$\frac{tg^4 x}{4}-\int tg\,x\left(\frac{1}{\cos^2 x}-1\right)dx=\frac{tg^4 x}{4}-\frac{tg^2 x}{2}-ln|cos\,x|+C.$$

3.

$$\int\frac{dt}{\left(t^2-1\right)\left(t^2+4\right)}=\left\|t^2+4-\left(t^2-1\right)\equiv 5\right\|=\frac{1}{5}\int\frac{t^2+4-\left(t^2-1\right)}{\left(t^2-1\right)\left(t^2+4\right)}dt=\frac{1}{5}\int\frac{dt}{t^2-1}-$$

$$-\frac{1}{5}\int\frac{dt}{t^2+4}=\frac{1}{5}\ln\left|\frac{t-1}{t+1}\right|-\frac{1}{10}arctg\frac{t}{2}+C.$$