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Differential equation-chapter II

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Lecture topic. 2.6. Homogeneous Linear Equations of the Second Order with Constant Coefficients

Let us consider an equation

$$y'' + py' + qy = 0, \tag{2.16}$$

where $p = const$ and $q = const$.

It is easy to note that not each function can be solution of this equation. For example, it is obvious that functions which look like x^{α}, $\tan x$, $\ln x$ do not satisfy it. To satisfy the equation expressions for y' and y'' should be similar to expression of itself function $y(x)$. The exponent function corresponds to these requirements. That is why we will search for solution of the equation (2.16) in the form

$$y = e^{\lambda x}, \tag{2.17}$$

where $\lambda = const$. Substituting (2.17) into equation (2.16) we get

$$\lambda^2 e^{\lambda x} + p\lambda e^{\lambda x} + qe^{\lambda x} = 0,$$

since $e^{\lambda x} \neq 0$ we have

$$\lambda^2 + p\lambda + q = 0. \tag{2.18}$$

So, in order to the function $y = e^{\lambda x}$ be solution of the equation (2.16) the number λ must satisfy obtained square equation (2.18). This equation is called **characteristic equation** of differential equation (2.16).

Let λ_1 and λ_2 be roots of the characteristic equation. There are the following possible cases:

I. Roots λ_1 and λ_2 are real and different. Then we obtain two particular solutions:

$$y_1 = e^{\lambda_1 x} \text{ and } y_2 = e^{\lambda_2 x}.$$

They are linearly independent because

$$\frac{y_2}{y_1} = \frac{e^{\lambda_2 x}}{e^{\lambda_1 x}} = e^{(\lambda_2 - \lambda_1)x} \neq const.$$

For this reason we can write down the general solution at once:

$$y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}.$$

Example 1. Find the general solution of the equation

$$y'' - 5y' + 6y = 0.$$

Solution. Let us construct the appropriate characteristic equation

$$\lambda^2 - 5\lambda + 6 = 0,$$

whence

$$\lambda_1 = 2,\ \lambda_2 = 3.$$

It means that the general solution has the following form:

$$y = C_1 e^{2x} + C_2 e^{3x}.$$

Example 2. Find the general solution of the equation $y'' + y' = 0$.

Solution. The characteristic equation of the equation has the form:

$$\lambda^2 + \lambda = 0,$$

from this it follows, that $\lambda_1 = 0$, $\lambda_2 = -1$ and hence the general solution has the form

$$y = C_1 + C_2 e^{-x}.$$

II. The roots λ_1 and λ_2 are real and equal each other $\lambda_1 = \lambda_2 = \lambda$. In this case we have the only particular solution: $y_1 = e^{\lambda x}$. Let us apply Ostrogradskiy-Liuvill’s formula to find the second solution y_2, which does not linearly depend on y_1. Then we get

$$y_2 = e^{\lambda x}\int \frac{1}{e^{2\lambda x}} e^{-\int p dx} dx = e^{\lambda x}\int \frac{1}{e^{2\lambda x}} e^{-px} dx.$$

But by virtue of Vieta's formulas in the given case we have $2\lambda = -p$ and hence

$$y_2 = e^{\lambda x}\int dx \Rightarrow y_2 = e^{\lambda x} x.$$

It means that the general solution may be presented in the following form:

$$y = C_1 e^{\lambda x} + C_2 x e^{\lambda x}.$$

III. The roots λ_1 and λ_2 are complex conjugate, that is $\lambda_{1,2} = \alpha \pm \beta i$. Since in the given case $\lambda_1 \neq \lambda_2$, then we can formally apply the same formula as in the case I :

$$y = \tilde{C}_1 e^{(\alpha+\beta i)x} + \tilde{C}_2 e^{(\alpha-\beta i)x} = \tilde{C}_1 e^{\alpha x}(\cos\beta x + i\sin\beta x) +$$
$$+ \tilde{C}_2 e^{\alpha x}(\cos\beta x - i\sin\beta x) == e^{\alpha x}\left[(\tilde{C}_1 + \tilde{C}_2)\cos\beta x + (\tilde{C}_1 - \tilde{C}_2)i\sin\beta x\right].$$

Let us denote

$$\tilde{C}_1 + \tilde{C}_2 = C_1,\ (\tilde{C}_1 - \tilde{C}_2)i = C_2. \tag{2.19}$$

Then we obtain

$$y = e^{\alpha x}(C_1\cos\beta x + C_2\sin\beta x). \tag{2.20}$$

Remark. From designation (2.19) it may be seemed that an arbitrary constant C_2 is imaginary number. Therefore let us show that if initial condition is real, then arbitrary constants C_1 and C_2 are also real. Indeed, in the given case the functions

$$y_1 = e^{\alpha x}\cos\beta x,\ y_2 = e^{\alpha x}\sin\beta x$$

are real ones. Therefore at real value of x_0 numbers $y_1(x_0)$ and $y_2(x_0)$ will be real. Due to initial conditions the numbers y_0 and y_0'

are real too. But then system (2.14), from which we find C_1 and C_2, cannot have complex solutions.

Example 1. Find the general solution of the equation

$$y'' + 4y' + 5y = 0.$$

Solution. Let us construct the characteristic equation

$$\lambda^2 + 4\lambda + 5 = 0,$$

whence

$$\lambda = -2 \pm i.$$

Consequently

$$y = e^{-2x}\left(C_1 \cos x + C_2 \sin x\right).$$

Example 2. For equation

$$y'' + 9y = 0$$

the characteristic equation will be

$$\lambda^2 + 9 = 0.$$

Its roots are $\lambda = \pm 3i$. You can see that these roots are just imaginary roots. Therefore due to (2.20) we get

$$y = C_1 \cos 3x + C_2 \sin 3x.$$