Lecture_01_04_20

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Lecture topic. 1.7. Linear Differential Equations and Solving Methods for Linear Equations of the First Order (second order). Equation's

The Second Approach (Method by Bernoulli). Let us search for unknown function $y(x)$ of the equation (1.13) in the following form

$$y(x) = u(x) \cdot v(x), \tag{1.14}$$

where $u(x)$ and $v(x)$ are another unknown functions, which have been selected so that to satisfy equation (1.13). For this reason let us substitute the expression (1.14) into (1.13). Then we obtain

$$u'(x)v(x) + u(x)v'(x) + p(x)u(x)v(x) = f(x).$$

Let us group the first and the third terms

$$v(x)(u'(x) + p(x)u(x)) + u(x)v'(x) = f(x).$$

If we require that the expression in the bracket is vanished

$$u'(x) + p(x)u(x) = 0, \tag{1.15}$$

we could find the one of the unknown functions $u(x)$. The another unknown function $v(x)$ we could find if we require that the expression

$$u(x)v'(x) = f(x) \tag{1.16}$$

holds. It should be noted that in the last equation the function $u(x)$ is any solution of the equation (1.15).

Example. Let us solve the example 1 by this way.

Solution. Let us seek for unknown function $y(x)$ in the form

$$y(x) = u(x)v(x). \tag{1.17}$$

After substitution it into initial differential equation we get

$$\left(x^2+1\right)\left(u'v+v'u\right)+xuv=x\left(x^2+1\right).$$

Let us group the terms, containing multiplier $v(x)$

$$v\left(\left(x^2+1\right)u'+xu\right)+v'u\left(x^2+1\right)=x\left(x^2+1\right)$$

and require that the following system of differential equation has place

$$\begin{cases}\left(x^2+1\right)u'+xu=0;\\ v'u\left(x^2+1\right)=x\left(x^2+1\right).\end{cases} \tag{1.18}$$

Whence we have

$$\left(x^2+1\right)u'=-xu \Rightarrow \frac{u'}{u}=-\frac{x}{\left(x^2+1\right)} \Rightarrow \ln u=-\frac{1}{2}\ln\left|x^2+1\right|+\ln C \Rightarrow$$

$$u=\frac{C}{\sqrt{x^2+1}}.$$

Assign $C=1$ and substitute obtained function $u=\frac{1}{\sqrt{x^2+1}}$ into the second equation of the system (1.18):

$$v'\frac{1}{\sqrt{x^2+1}}\left(x^2+1\right)=x\left(x^2+1\right)$$

or after transformation the last equation may be get as

$$v'=x\sqrt{x^2+1}.$$

Whence

$$v=\frac{\left(x^2+1\right)^{3/2}}{3}+C.$$

Then the general solution takes form

$$y=\frac{C}{\sqrt{1+x^2}}+\frac{x^2+1}{3}.$$

Conclusion. The typical form of the linear differential equation of the first order makes as

$$y'+p(x)y=q(x).$$

This equation may be solved by substitution $y(x)=u(x)v(x)$ or by method of variation of an arbitrary constant.

1.9. Bernoulli's Equation

If differential equation has the form

$$y'+p(x)y=q(x)y^{\alpha} \qquad (\alpha \neq 0,\ \alpha \neq 1),$$

then it is called **Bernoulli's equation**.

The Bernoulli's equation can be transformed to linear one by changing the variable. Let us divide this equation by y^{α} then we get

$$y^{-\alpha}y'+p(x)y^{1-\alpha}=q(x) \qquad (1.19)$$

Let us make the following substitution of variable

$$y^{1-\alpha}=z(x).$$

Then

$$z'=(1-\alpha)y^{-\alpha}y'.$$

Putting this expression into the equation (1.19) we get

$$\frac{z'}{1-\alpha}+p(x)z=q(x).$$

Linear inhomogeneous equation has been obtained.

Example. Solve the differential equation $xy'+y=y^2\ln x$.

Solution. Let us divide the equation by y^2 $(y^2 \neq 0)$:

$$xy^{-2}y'+y^{-1}=\ln x.$$

After changing the variable:

$$z(x)=y^{-1},\ z'=-y^{-2}y'$$

the equation is transformed as follows:

$$xz'-z=-\ln x.$$

1. Let us solve the corresponding homogeneous equation

$$xu'-u=0, \quad \frac{du}{u}=\frac{dx}{x}, \quad u=Cx.$$

2. Suppose

$$z(x)=C(x)x, \quad z'(x)=C'(x)x+C(x),$$

$$C'(x)x^2+xC(x)-C(x)x=-\ln x \quad \Rightarrow C'(x)=-\frac{\ln x}{x^2}.$$

$$C(x)=-\int\frac{\ln x}{x^2}dx=\frac{\ln x}{x}+\frac{1}{x}+C_1,\ z=\ln x+1+C_1x,$$

where $y=\frac{1}{z}$ and the general solution is

$$y=\frac{1}{C_1x+\ln x+1}.$$

The equation has also singular solution $y=0$.

The Bernoulli's equation can be also solved by **another method**. Let us explain this method on example of this Bernoulli's equation:

$$xy'+y=y^2\ln x.$$

Let us search for unknown function in the form of product of two another unknown functions

$$y=u\cdot v, \quad y'=u'v+v'u$$

(*u* and *v* are both functions of *x*). The equation is transformed as follows:

$$x(u'v+v'u)+u\cdot v=u^2v^2\ln x$$

or

$$(xu'+u)v+xv'u=u^2v^2\ln x.$$

Let us define the function *u* thus that the coefficient at *v* will vanish:

$$xu'+u=0 \quad \Rightarrow \quad \frac{du}{u}=-\frac{dx}{x} \quad \Rightarrow \quad u=\frac{1}{x}.$$

We accept particular solution of the equation at $C=1$ as *u*. Then for defining *v* we obtain the equation

$$xv'u = u^2 v^2 \ln x.$$

As $u = \dfrac{1}{x}$, then $v' = \dfrac{1}{x^2} v^2 \ln x$. This is equation with separable variables:

$$\frac{dv}{v^2} = \frac{1}{x^2} \ln x dx.$$

After integration we obtain:

$$-\frac{1}{v} = -\frac{1}{x} \ln x - \frac{1}{x} - C \quad \Rightarrow v = \frac{x}{Cx + \ln x + 1}.$$

It means that

$$y = u \cdot v = \frac{1}{Cx + \ln x + 1}$$

is general solution, $y \equiv 0$ is singular solution.