1.a) $z = \ln(y^2 - 4x + 8)$

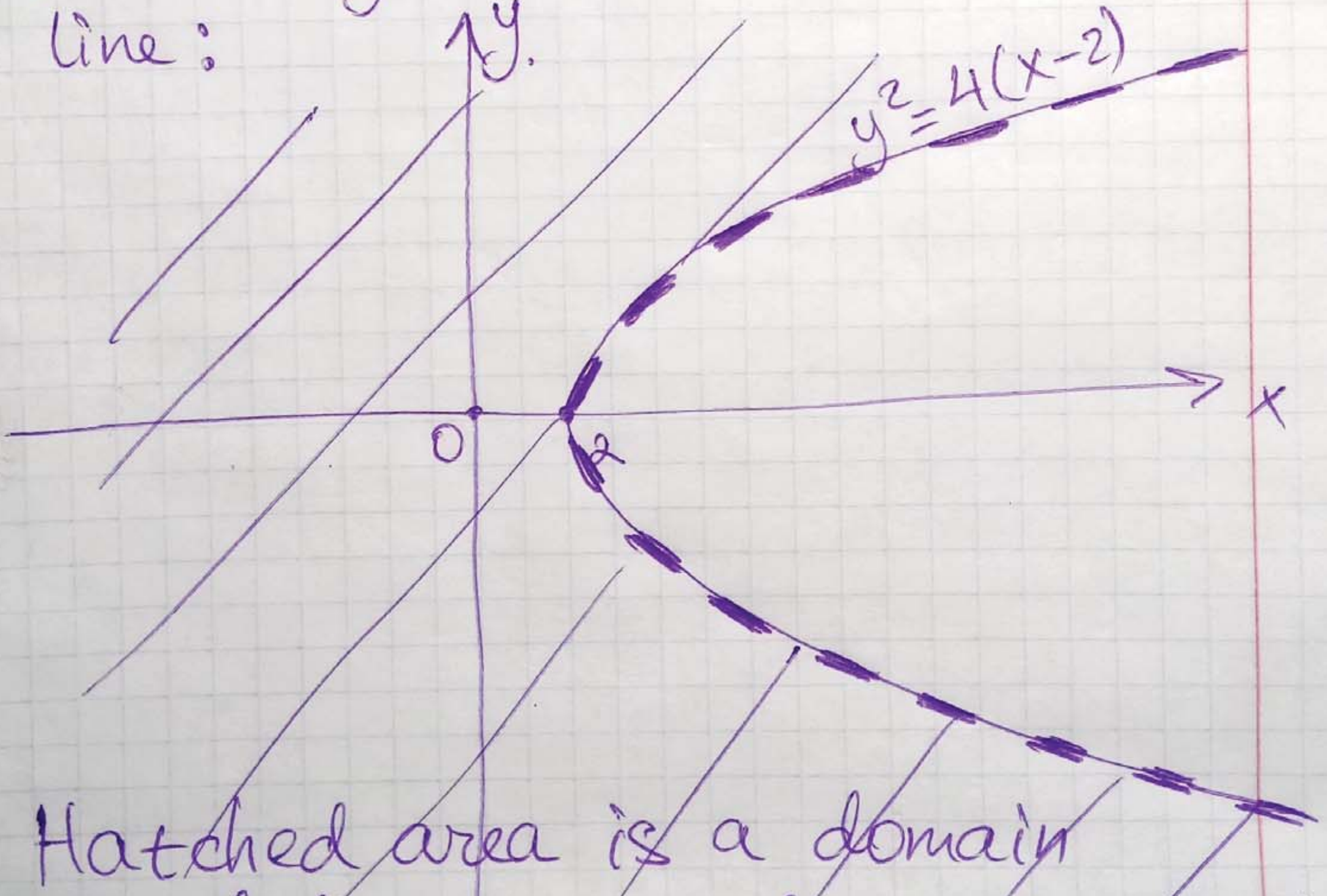
$D: \; y^2 - 4x + 8 > 0$

$y^2 > 4x - 8$

$y^2 > 4(x-2)$

Boundary: $y^2 = 4(x-2)$ parabola

Sinc equality „=" is not included in the studied inequality, the boundary is drawn with a dashed line:



Hatched area is a domain of difinition of function $z(x,y)$.

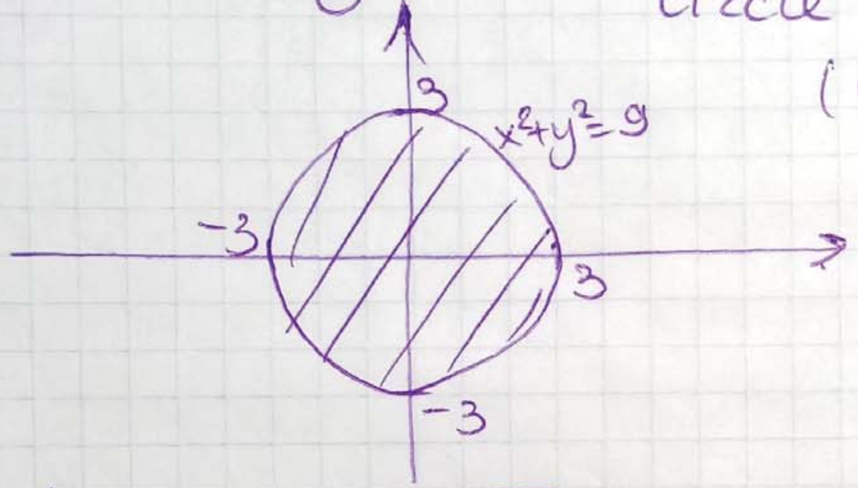
1.b) $z = \sqrt{9 - x^2 - y^2}$

$D: 9 - x^2 - y^2 \geq 0.$

$9 \geq x^2 + y^2$

Boundary: $9 = x^2 + y^2$ (included)

circle with $R = 3$.



$(0,0)$: $9 \geq 0^2 + 0^2$

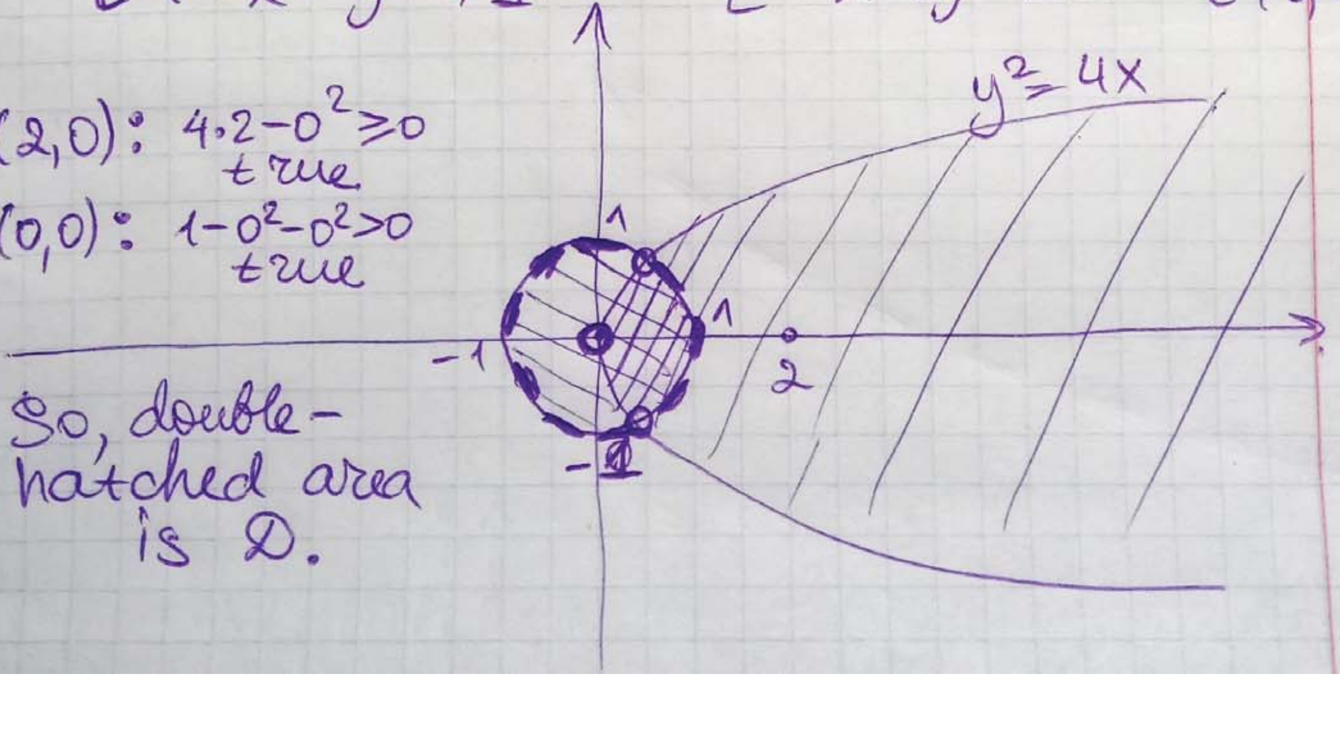
$9 \geq 0$

true

So, $(0,0)$ belongs to the domain of definition.

1.c) $z = \dfrac{\sqrt{4x - y^2}}{\ln(1 - x^2 - y^2)}$

$$D: \begin{cases} 4x - y^2 \geq 0 \\ 1 - x^2 - y^2 > 0 \\ 1 - x^2 - y^2 \neq 1 \end{cases} \Leftrightarrow \begin{cases} 4x \geq y^2 \\ 1 > x^2 + y^2 \\ x^2 + y^2 \neq 0 \end{cases} \Leftrightarrow \begin{cases} y^2 \leq 4x \\ x^2 + y^2 < 1 \\ (0,0) \notin D. \end{cases}$$



$(2,0)$: $4 \cdot 2 - 0^2 \geq 0$

true

$(0,0)$: $1 - 0^2 - 0^2 > 0$

true

So, double-hatched area is D.

1.d) $z = \sqrt{x \cdot \sin y}$

$$D: x \cdot \sin y \geq 0 \iff \left[\begin{array}{l} \begin{cases} x \geq 0 \\ \sin y \geq 0 \end{cases} \\ \begin{cases} x \leq 0 \\ \sin y \leq 0 \end{cases} \end{array} \right. \iff \left[\begin{array}{l} \begin{cases} x \geq 0 \\ 0 + 2\pi k \leq y \leq \pi + 2\pi k \end{cases} \quad k \in \mathbb{Z} \\ \begin{cases} x \leq 0 \\ \pi + 2\pi k \leq y \leq 2\pi + 2\pi k \end{cases} \quad k \in \mathbb{Z} \end{array} \right.$$

2.a) Partial derivatives:

$$z = x \cdot \sqrt{y} + y / \sqrt{x} = x \cdot y^{1/2} + y \cdot x^{-\frac{1}{2}}$$

$$\frac{\partial z}{\partial x} = 1 \cdot y^{1/2} + y \cdot \left(-\tfrac{1}{2}\right) x^{-3/2}$$

$$\frac{\partial z}{\partial y} = x \cdot \tfrac{1}{2} \cdot y^{-1/2} + 1 \cdot x^{-\frac{1}{2}}$$

2.b) $z = x^y$

$$\frac{\partial z}{\partial x} = y \cdot x^{y-1}$$

$$\frac{\partial z}{\partial y} = x^y \cdot \ln x.$$

2.c) $z = x^2y + y^2x + \ln y - e^x + 5$

$$\frac{\partial z}{\partial x} = 2x \cdot y + y^2 \cdot 1 + 0 - e^x + 0$$

$$\frac{\partial z}{\partial y} = x^2 \cdot 1 + 2y \cdot x + \frac{1}{y} - 0 + 0$$

2.d) $u = (x)^{y \cdot z}$

$$\frac{\partial u}{\partial x} = y \cdot z \cdot x^{yz-1}$$

$$\frac{\partial u}{\partial y} = \left((x^z)^y\right)'_y = x^{z \cdot y} \cdot \ln(x^z)$$

$$\frac{\partial u}{\partial z} = \left((x^y)^z\right)'_z = x^{y \cdot z} \cdot \ln(x^y)$$

2.e) $e^z - xyz = 0.$

So, $z = z(x, y)$. Let us differentiate both parts of this equation with respect to x, considering y as a constant.

$$(e^z - xyz)'_x = 0'_x$$

$$e^z \cdot z'_x - 1 \cdot yz - x \cdot y \cdot z'_x = 0.$$

$$z'_x\,(e^z - xy) = yz$$

$z'_x = \dfrac{yz}{e^z - xy}$. In the same way: $z'_y = \dfrac{xz}{e^z - xy}$

3) $z = \frac{y}{f(x^2-y^2)}$ $\frac{1}{x}\cdot z'_x + \frac{1}{y}\cdot z'_y = \frac{z}{y^2}$

Let $x^2-y^2=u$. $u'_x=2x$, $u'_y=-2y$.

$$z'_x = \left(\frac{y}{f(u)}\right)'_x = \frac{0-y\cdot f'(u)\cdot u'_x}{f^2(u)} = -\frac{2xy\cdot f'}{f^2}$$

$$z'_y = \left(\frac{y}{f(u)}\right)'_y = \frac{1\cdot f(u) - y\cdot f'(u)\cdot u'_y}{f^2(u)} =$$

$$= \frac{f+2y^2f'}{f^2}$$

$$\frac{z}{y^2} = \frac{y}{f\cdot y^2} = \frac{1}{y\cdot f}$$

$$\frac{1}{x}z'_x + \frac{1}{y}z'_y = \frac{1}{x}\cdot\frac{-2xyf'}{f^2} + \frac{1}{y}\frac{f+2y^2f'}{f^2} =$$

$$= -\cancel{\frac{2yf'}{f^2}} + \frac{f}{yf^2} + \cancel{\frac{2yf'}{f^2}} = \frac{f}{yf^2} = \frac{1}{yf^2}$$

So, both parts of equation are equal. $\Rightarrow$ function satisfies this equation.

4) $z = x^3 + xy^2 - 5xy^3 + y^5$

$z'_x = 3x^2 + 1 \cdot y^2 - 5 \cdot y^3 + 0$

$z'_y = 0 + x \cdot 2y - 5x \cdot 3y^2 + 5y^4$

$z''_{xx} = 6x + 0 - 0 + 0 = 6x$

$z''_{xy} = 0 + 2y - 15y^2 = 2y - 15y^2$

$z''_{yx} = 1 \cdot 2y - 5 \cdot 3y^2 + 0 = 2y - 15y^2$

$z''_{yy} = x \cdot 2 - 5x \cdot 6y + 20y^3 = 2x - 30xy +$
$+ 20y^3.$

5) $u = x^2y + 3\sin(2x+5y) + x^3$

$du = \frac{\partial u}{\partial x} \cdot dx + \frac{\partial u}{\partial y} \cdot dy =$

$= (2xy + 3\cos(2x+5y) \cdot 2 + 3x^2)\, dx +$

$+ (x^2 \cdot 1 + 3\cos(2x+5y) \cdot 5 + 0)\, dy.$

6) $u = \ln(x + 2y + 3z)$, $M_0(1, 2, 0)$

$du(1,2,0) = \frac{\partial u}{\partial x}(M_0) \cdot dx + \frac{\partial u}{\partial y}(M_0)\, dy + \frac{\partial u}{\partial z}(M_0)\, dz$

$\frac{\partial u}{\partial x} = \frac{1}{x+2y+3z} \cdot 1$ $\quad \frac{\partial u}{\partial y} = \frac{1}{x+2y+3z} \cdot 2$ $\quad \frac{\partial u}{\partial z} = \frac{1}{x+2y+3z} \cdot 3$

$du(M_0) = \frac{1}{5}dx + \frac{2}{5}dy + \frac{3}{5}dz$

7) Calculate approximatly.

a) $\ln\left(\sqrt[3]{1{,}03} + \sqrt[4]{0{,}98} - 1\right)$

Let $x = 1{,}03$, $y = 0{,}98$.

Thus $u = \ln\left(\sqrt[3]{x} + \sqrt[4]{y} - 1\right) = f(x,y)$

$x_0 = 1$, $y_0 = 1$.

$$f(x,y) \approx f(x_0,y_0) + f'_x(x_0,y_0)(x-x_0) + f'_y(x_0,y_0)(y-y_0)$$

$$f(x_0,y_0) = \ln\left(\sqrt[3]{1} + \sqrt[4]{1} - 1\right) = \ln 1 = 0$$

$$f'_x = \frac{1}{\sqrt[3]{x} + \sqrt[4]{y} - 1} \cdot \frac{1}{3} x^{-2/3} \qquad f'_x(x_0,y_0) = \frac{1}{3}$$

$$f'_y = \frac{1}{\sqrt[3]{x} + \sqrt[4]{y} - 1} \cdot \frac{1}{4} y^{-3/4} \qquad f'_y(x_0,y_0) = \frac{1}{4}$$

$$\ln\left(\sqrt[3]{1{,}03} + \sqrt[4]{0{,}98}\right) \approx 0 + \frac{1}{3}(1{,}03-1) + \frac{1}{4}(0{,}98-1) =$$

$$= 0{,}01 - 0{,}005 = 0{,}005.$$

7.b) $(1,04)^{2,02}$

$x = 1,04 \qquad y = 2,02$

$x_0 = 1 \qquad y_0 = 2$

$f(x,y) = x^y$.

$f(x_0,y_0) = 1^2 = 1$

$f'_x = y \cdot x^{y-1} \Rightarrow f'_x(x_0,y_0) = 2 \cdot 1^{2-1} = 2$

$f'_y = x^y \ln x \Rightarrow f'_y(x_0,y_0) = 1^2 \ln 1 = 0.$

$(1,04)^{2,02} \approx 1 + 2 \cdot (1,04-1) + 0 \cdot (2,02-2) =$

$= 1 + 0,08 = 1,08.$