Lecture no 7: EUCLIDEAN SPACE. ORTHOGONAL SETS AND BASES. GRAM–SCHMIDT ORTHOGONALIZATION [1]

1. Orthogonal Sets and Bases

Consider a set $S = (u_1, u_2, ..., u_r)$ of nonzero vectors in an inner product space V. S is called *orthogonal* if each pair of vectors in S are orthogonal, and S is called *orthonormal* if S is orthogonal and each vector in S has unit length. That is,

$$\text{(i) } \textbf{Orthogonal: } \langle u_i, u_j \rangle = 0 \quad \text{for } i \neq j$$

$$\text{(ii) } \textbf{Orthonormal: } \langle u_i, u_j \rangle = \begin{cases} 0 & \text{for } i \neq j \\ 1 & \text{for } i = j \end{cases}$$

Normalizing an orthogonal set S refers to the process of multiplying each vector in S by the reciprocal of its length in order to transform S into an orthonormal set of vectors.

The following theorems apply.

Theorem: Suppose S is an orthogonal set of nonzero vectors. Then S is *linearly independent.*

Theorem: (*Pythagoras*) Suppose $(u_1, u_2, ..., u_r)$ is an orthogonal set of vectors. Then

$$\|u_1 + u_2 + \cdots + u_r\|^2 = \|u_1\|^2 + \|u_2\|^2 + \cdots + \|u_r\|^2$$

Here we prove the Pythagorean theorem in the special and familiar case for two vectors. Specifically, suppose $\langle u, v\rangle=0$. Then

$$\|u + v\|^2 = \langle u + v, \ \ u + v\rangle = \langle u, u\rangle + 2\langle u, v\rangle + \langle v, v\rangle = \langle u, u\rangle + \langle v, v\rangle = \|u\|^2 + \|v\|^2$$

which gives our result.

Example

[1]

9	Л	2	Евклідов простір. Аксіоми скалярного добутку. Існування ортонормованого базису евклідового простору. Ортонормовані системи векторів. Процес ортогоналізації Грама-Шмідта.
10	ПР	2	Побудова ортонормованого базису евклідового простору. Ортогональне доповнення. Ортогональне проектування

(a) Let $E = \{e_1, e_2, e_3\} = \{(1,0,0),\ (0,1,0),\ (0,0,1)\}$ be the usual basis of Euclidean space $\mathbf{R}^3$. It is clear that

$$\langle e_1, e_2\rangle = \langle e_1, e_3\rangle = \langle e_2, e_3\rangle = 0 \quad \text{and} \quad \langle e_1, e_1\rangle = \langle e_2, e_2\rangle = \langle e_3, e_3\rangle = 1$$

Namely, E is an orthonormal basis of $\mathbf{R}^3$. More generally, the usual basis of $\mathbf{R}^n$ is orthonormal for every n.

(b) Let $V = C[-\pi, \pi]$ be the vector space of continuous functions on the interval $-\pi \le t \le \pi$ with inner product defined by $\langle f, g\rangle = \int_{-\pi}^{\pi} f(t)g(t)\,dt$. Then the following is a classical example of an orthogonal set in V:

$$\{1, \cos t, \cos 2t, \cos 3t, \ldots, \sin t, \sin 2t, \sin 3t, \ldots\}$$

This orthogonal set plays a fundamental role in the theory of Fourier series.

Orthogonal Basis and Linear Combinations; Fourier Coefficients

Let S consist of the following three vectors in $\mathbf{R}^3$:

$u_1=(1,2,1)$, $u_2=(2,1,-4)$, $u_3=(3,-2,1)$,

The reader can verify that the vectors are orthogonal; hence, they are linearly independent. Thus, S is an orthogonal basis of $\mathbf{R}^3$.

Suppose we want to write $v = (7; 1; 9)$ as a linear combination of u_1, u_2, u_3. First we set v as a linear combination of u_1, u_2, u_3. using unknowns x_1, x_2, x_3. as follows:

$$v = x_1u_1 + x_2u_2 + x_3u_3 \quad \text{or} \quad (7,1,9) = x_1(1,2,1) + x_2(2,1,-4) + x_3(3,-2,1)$$

We can proceed in two ways.

Method 1: Expand the expression to obtain the system

$$x_1 + 2x_2 + 3x_3 = 7, \qquad 2x_1 + x_2 - 2x_3 = 1, \qquad x_1 - 4x_2 + x_3 = 9$$

Solve the system by Gaussian elimination to obtain $x_1 = 3$, $x_2 = -1$, $x_3 = 2$. Thus, $v = 3\,u_1 - u_2 + 2\,u_3$.

Method 2: (This method uses the fact that the basis vectors are orthogonal, and the arithmetic is much simpler.) If we take the inner product of each side of the expression with respect to u_i, we get

$$\langle v, u_i\rangle = \langle x_1u_2 + x_2u_2 + x_3u_3,\ u_i\rangle \quad \text{or} \quad \langle v, u_i\rangle = x_i\langle u_i, u_i\rangle \quad \text{or} \quad x_i = \frac{\langle v, u_i\rangle}{\langle u_i, u_i\rangle}$$

Here two terms drop out, because u_1, u_2, u_3 are orthogonal. Accordingly,

$$x_1 = \frac{\langle v, u_1\rangle}{\langle u_1, u_1\rangle} = \frac{7+2+9}{1+4+1} = \frac{18}{6} = 3, \qquad x_2 = \frac{\langle v, u_2\rangle}{\langle u_2, u_2\rangle} = \frac{14+1-36}{4+1+16} = \frac{-21}{21} = -1$$

$$x_3 = \frac{\langle v, u_3\rangle}{\langle u_3, u_3\rangle} = \frac{21-2+9}{9+4+1} = \frac{28}{14} = 2$$

Thus, again, we get $v = 3\,u_1 - u_2 + 2\,u_3$.

The procedure in Method 2 is true in general. Namely, we have the following theorem:

Theorem: Let $\{u_1, u_2, \ldots, u_n\}$ be an orthogonal basis of V. Then, for any $v \in V$,

$$v = \frac{\langle v, u_1\rangle}{\langle u_1, u_1\rangle}u_1 + \frac{\langle v, u_2\rangle}{\langle u_2, u_2\rangle}u_2 + \cdots + \frac{\langle v, u_n\rangle}{\langle u_n, u_n\rangle}u_n$$

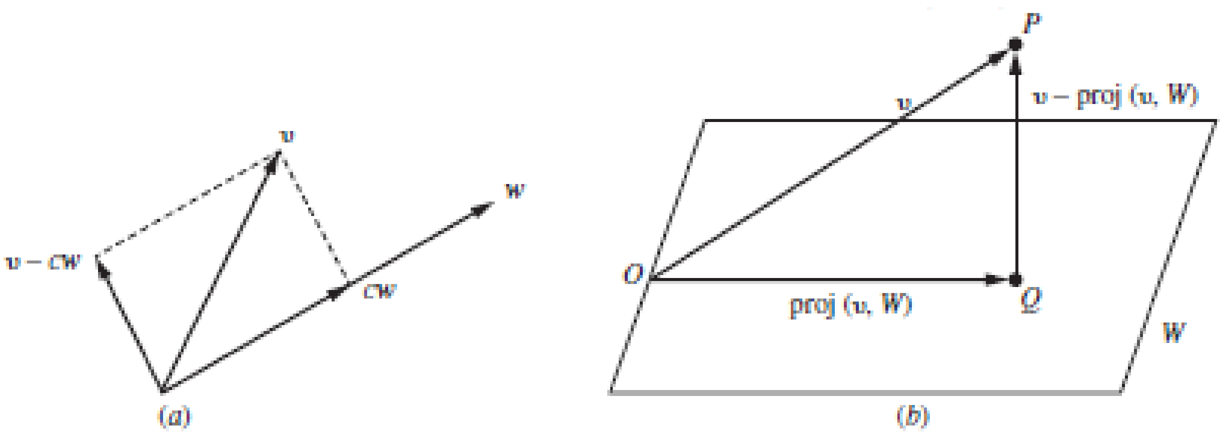
Remark: The scalar $k_i = \frac{\langle v,u_i\rangle}{\langle u_i,u_i\rangle}$ is called *the Fourier coefficient* of v with respect to u_i, because it is analogous to a coefficient in the Fourier series of a function. This scalar also has a geometric interpretation, which is discussed below.

Projections

Let V be an inner product space. Suppose w is a given nonzero vector in V, and suppose v is another vector.

We seek the "*projection of v along w*," which, as indicated in Fig., will be the multiple cw of w such that $v' = v - cw$ is orthogonal to w. This means

$$\langle v - cw,\ w\rangle = 0 \quad \text{or} \quad \langle v, w\rangle - c\langle w, w\rangle = 0 \quad \text{or} \quad c = \frac{\langle v, w\rangle}{\langle w, w\rangle}$$



Accordingly, the projection of v along w is denoted and defined by

$$\text{proj}(v, w) = cw = \frac{\langle v, w\rangle}{\langle w, w\rangle}w$$

Such a scalar c is unique, and it is called the *Fourier coefficient* of v with respect to w or the component of v along w.

The above notion is generalized as follows:

Theorem: Suppose $w_1, w_2, \ldots, w_r$ form an orthogonal set of nonzero vectors in V. Let v be any vector in V. Define

$$v' = v - (c_1w_1 + c_2w_2 + \cdots + c_rw_r),$$

where

$$c_1 = \frac{\langle v, w_1 \rangle}{\langle w_1, w_1 \rangle}, \qquad c_2 = \frac{\langle v, w_2 \rangle}{\langle w_2, w_2 \rangle}, \qquad \ldots, \qquad c_r = \frac{\langle v, w_r \rangle}{\langle w_r, w_r \rangle}$$

Then v' is orthogonal to $w_1, w_2, \ldots, w_r$.

Note that each ci in the above theorem is the *component* (*Fourier coefficient*) of v along the given w_i.

Remark: The notion of the projection of a vector $v \in V$ along a subspace W of V is defined as follows. By Theorem $V = W \oplus W^{\perp}$. Hence, v may be expressed uniquely in the form:

$v = w + w'$, where $w \in W$ and $w' \in W^{\perp}$

We define w to be the projection of v along W, and denote it by **proj**(v, W), as pictured in Figure above.

In particular, if $W = span(w_1, w_2, \ldots, w_r)$, where the w_i form an orthogonal set, then

proj$(v, W) = c_1 w_1 + c_2 w_2 + \cdots + c_r w_r$

Here c_i is the component of v along w_i, as above.

2. Gram–Schmidt Orthogonalization Process

Suppose $\{v_1, v_2, \ldots, v_n\}$ is a basis of an inner product space V. One can use this basis to construct an orthogonal basis $\{w_1, w_2, \ldots, w_n\}$ of V as follows. Set

$$w_1 = v_1$$

$$w_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1$$

$$w_3 = v_3 - \frac{\langle v_3, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle v_3, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2$$

....

$$w_n = v_n - \frac{\langle v_n, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle v_n, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2 - \cdots - \frac{\langle v_n, w_{n-1} \rangle}{\langle w_{n-1}, w_{n-1} \rangle} w_{n-1}$$

In other words, for $k = 2, 3, \ldots$ n, we define

$$w_k = v_k - \left(c_{k1} w_1 + c_{k2} w_2 + \cdots + c_{k,k-1} w_{k-1}\right),$$

where $c_{ki} = \frac{\langle v_k, w_i \rangle}{\langle w_i, w_i \rangle}$ is the component of v_k along w_i. By Theore, each w_k is orthogonal to the preceeding w's. Thus, $w_1, w_2, \ldots, w_n$ form an orthogonal basis for V as claimed. Normalizing each w_i will then yield an orthonormal basis for V.

The above construction is known as the *Gram–Schmidt orthogonalization process*. The following remarks are in order.

Remark 1: Each vector w_k is a linear combination of v_k and the preceding w's. Hence, one can easily show, by induction, that each w_k is a linear combination of $v_1, v_2, \ldots, v_n$.

Remark 2: Because taking multiples of vectors does not affect orthogonality, it may be simpler in hand calculations to clear fractions in any new *wk*, by multiplying w_k by an appropriate scalar, before obtaining the next w_{k+1}.

Remark 3: Suppose $u_1, u_2, \ldots, u_r$ are linearly independent, and so they form a basis for $U = span(u_i)$. Applying the *Gram–Schmidt orthogonalization process* to the u's yields an orthogonal basis for U.

The following theorems use the above algorithm and remarks.

Theorem 1. Let $\{ v_1, v_2, \ldots, v_n \}$ be any basis of an inner product space V. Then there exists an orthonormal basis $\{u_1, u_2, \ldots, u_n \}$ of V such that the change-of-basis matrix from $\{v_i\}$ to $\{u_i\}$ is triangular; that is, for $k = 1, 2, \ldots n$,

$$u_k = a_{k1}v_1 + a_{k2}v_2 + \cdots + a_{kk}v_k$$

Theorem 2. Suppose $S = \{ w_1, w_2, \ldots, w_r \}$ is an orthogonal basis for a subspace W of a vector space V. Then one may extend S to an orthogonal basis for V; that is, one may find vectors $w_{r+1}, \ldots, w_n$ such that $\{w_1, w_2, \ldots, w_n\}$ is an orthogonal basis for V.

Example 1: Apply the Gram–Schmidt orthogonalization process to find an orthogonal basis and then an orthonormal basis for the subspace U of $\mathbf{R}^4$ spanned by

$$v_1 = (1,1,1,1), \qquad v_2 = (1,2,4,5), \qquad v_3 = (1,-3,-4,-2)$$

(1) First set $w_1 = v_1 = (1,1,1,1)$.

(2) Compute

$$v_2 - \frac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 = v_2 - \frac{12}{4} w_1 = (-2,-1,1,2)$$

Set $w_2 = (-2,-1,1,2)$.

(3) Compute

$$v_3 - \frac{\langle v_3, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle v_3, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2 = v_3 - \frac{(-8)}{4} w_1 - \frac{(-7)}{10} w_2 = \left(\tfrac{8}{5}, -\tfrac{17}{10}, -\tfrac{13}{10}, \tfrac{7}{5}\right)$$

Clear fractions to obtain $w_3 = (-6, -17, -13, 14)$.

Thus, w_1, w_2, w_3 form an orthogonal basis for U. Normalize these vectors to obtain an orthonormal basis $\{u_1, u_2, u_3\}$ of U. We have $\|w_1\|^2 = 4$, $\|w_2\|^2 = 10$, $\|w_3\|^2 = 910$, so

$$u_1 = \frac{1}{2}(1, 1, 1, 1), \qquad u_2 = \frac{1}{\sqrt{10}}(-2, -1, 1, 2), \qquad u_3 = \frac{1}{\sqrt{910}}(16, -17, -13, 14)$$

Example 2: Let V be the vector space of polynomials $f(t)$ with inner product $\langle f, g \rangle = \int_{-1}^{1} f(t)g(t)dt$. Apply the Gram–Schmidt orthogonalization process to $\{1, t, t^2, t^3)$ to find an orthogonal basis $\{f_0; f_1; f_2; f_3\}$ with integer coefficients for $\mathbf{P}_3(t)$.

Here we use the fact that, for $r + s = n$,

$$\langle t^r, t^s \rangle = \int_{-1}^{1} t^n \, dt = \left. \frac{t^{n+1}}{n+1} \right|_{-1}^{1} = \begin{cases} 2/(n+1) & \text{when } n \text{ is even} \\ 0 & \text{when } n \text{ is odd} \end{cases}$$

(1) First set $f_0 = 1$.

(2) Compute $t - \frac{\langle t, 1 \rangle}{\langle 1, 1 \rangle}(1) = t - 0 = t$. Set $f_1 = t$.

(3) Compute

$$t^2 - \frac{\langle t^2, 1 \rangle}{\langle 1, 1 \rangle}(1) - \frac{\langle t^2, t \rangle}{\langle t, t \rangle}(t) = t^2 - \frac{\frac{2}{3}}{2}(1) + \frac{0}{2}(t) = t^2 - \tfrac{1}{3}$$

Multiply by 3 to obtain $f_2 = 3t^2 = 1$.

(4) Compute

$$t^3 - \frac{\langle t^3, 1 \rangle}{\langle 1, 1 \rangle}(1) - \frac{\langle t^3, t \rangle}{\langle t, t \rangle}(t) - \frac{\langle t^3, \; 3t^2 - 1 \rangle}{\langle 3t^2 - 1, \; 3t^2 - 1 \rangle}(3t^2 - 1)$$

$$= t^3 - 0(1) - \frac{\frac{2}{5}}{\frac{2}{3}}(t) - 0(3t^2 - 1) = t^3 - \tfrac{3}{5}t$$

Multiply by 5 to obtain $f_3 = 5t^3 - 3t$.

Thus, $\{1, \; t, \; 3t^2 - 1, \; 5t^3 - 3t\}$ is the required orthogonal basis.

Remark: Normalizing the polynomials in the last Example so that $p(1) = 1$ yields the polynomials $1, \; t, \; \frac{1}{2}(3t^2 - 1), \; \frac{1}{2}(5t^3 - 3t)$.

These are the first four *Legendre polynomials*, which appear in the study of differential equations.

3. Orthogonal and Positive Definite Matrices

This section discusses two types of matrices that are closely related to real inner product spaces V. Here vectors in $\mathbf{R}^n$ will be represented by column vectors. Thus, $\langle u, v\rangle = u^T v$ denotes the inner product in Euclidean space $\mathbf{R}^n$.

Orthogonal Matrices

Definition A real matrix P is *orthogonal* if P is nonsingular and $P^{-1} = P^T$, or, in other words, if $P P^T = P^T P = I$.

First we recall an important characterization of such matrices.

Theorem 1: Let P be a real matrix. Then the following are equivalent:

(a) P is orthogonal;

(b) the rows of P form an orthonormal set;

(c) the columns of P form an orthonormal set.

(This theorem is true only using the usual inner product on $\mathbf{R}^n$. It is not true if $\mathbf{R}^n$ is given any other inner product.)

Example 7.12

(a) Let $P = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 2/\sqrt{6} & -1/\sqrt{6} & -1/\sqrt{6} \end{bmatrix}$. The rows of P are orthogonal to each other and are unit vectors. Thus P is an orthogonal matrix.

(b) Let P be a 2×2 orthogonal matrix. Then, for some real number θ, we have

$$P = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \quad \text{or} \quad P = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$

The following two theorems show important relationships between orthogonal matrices and orthonormal bases of a real inner product space V.

***Theorem* 1**: Suppose $E = \{e_i\}$ and $E' = \{e_i'\}$ are orthonormal bases of V. Let P be the change-of-basis matrix from the basis E to the basis E'. Then P is *orthogonal*.

***Theorem* 2**: Let $\{ e_1, e_2, \ldots, e_n \}$ be an orthonormal basis of an inner product space V. Let $P = [a_{ij}]$ be an orthogonal matrix. Then the following n vectors form an orthonormal basis for V:

$e_i' = a_{i1}e_1 + a_{i2}e_2 + \cdots + a_{in}e_n$, $i = 1, 2, \ldots, n$.

Positive Definite Matrices

Let A be a real symmetric matrix, that is, $A^T = A$. Then A is said to be positive definite if, for every nonzero vector u in $\mathbf{R}^n$,

$$\langle u, Au\rangle = u^T A u > 0$$

Algorithms to decide whether or not a matrix A is positive definite will be given later. However, for 2×2 matrices, we have simple criteria that we state formally in the following theorem:

Theorem : A 2×2 real symmetric matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$ is *positive definite* if and only if the diagonal entries a and d are positive and the determinant $\|A\| = a\,d - b\,c = a\,d - b^2$ is *positive*.

Example : Consider the following symmetric matrices

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & -2 \\ -2 & -3 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}$$

A is not positive definite, because $\|A\| = 4 - 9 = -5$ is negative.

B is not positive definite, because the diagonal entry -3 is negative.

However, C is positive definite, because the diagonal entries 1 and 5 are positive, and the determinant $\|C\| = 5 - 4 = 1$ is also positive.

The following theorem holds.

Theorem : Let A be a real positive definite matrix. Then the function $\langle u, Av\rangle = u^T A v$ is an inner product on $\mathbf{R}^n$.

Matrix Representation of an Inner Product

The Theorem presented above says that every positive definite matrix A determines an inner product on $\mathbf{R}^n$. This subsection may be viewed as giving the converse of this result.

Let V be a real inner product space with basis $S = \{ u_1, u_2, \ldots, u_n \}$. The matrix $A = [a_{ij}]$, where $a_{ij} = \langle u_i, u_j\rangle$

is called the *matrix representation of the inner product* on V relative to the basis S.

Observe that A is *symmetric*, because the inner product is symmetric, that is, $\langle u_i, u_j \rangle = \langle u_j, u_i \rangle$.

Also, A depends on both the inner product on V and the basis S for V. Moreover, if S is an orthogonal basis, then A is *diagonal*, and if S is an orthonormal basis, then A is the *identity matrix*.

Example The vectors $u_1 = (1, 1, 0)$, $u_2 = (1, 2, 3)$, $u_3 = (1, 3, 5)$ form a basis S for Euclidean space $\mathbf{R}^3$. Find the matrix A that represents the inner product in $\mathbf{R}^3$ relative to this basis S.

First compute each $\langle u_i, u_j \rangle$ to obtain

$$\langle u_1, u_1 \rangle = 1 + 1 + 0 = 2, \qquad \langle u_1, u_2 \rangle = 1 + 2 + 0 = 3, \qquad \langle u_1, u_3 \rangle = 1 + 3 + 0 = 4$$
$$\langle u_2, u_2 \rangle = 1 + 4 + 9 = 14, \qquad \langle u_2, u_3 \rangle = 1 + 6 + 15 = 22, \qquad \langle u_3, u_3 \rangle = 1 + 9 + 25 = 35$$

Then $A = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 14 & 22 \\ 4 & 22 & 35 \end{bmatrix}$. As expected, A is symmetric.

The following theorems hold.

***Theorem* 1**: Let A be the matrix representation of an inner product relative to basis S for V. Then, for any vectors $u, v \in V$, we have

$$\langle u, v \rangle = [u]^T A [v],$$

where $[u]$ and $[v]$ denote the (column) coordinate vectors relative to the basis S.

***Theorem* 2**: Let A be the matrix representation of any inner product on V. Then A is a *positive definite matrix*.

4. Normed Vector Spaces

Definition: Let V be a real or complex vector space. Suppose to each $v \in V$ there is assigned a real number, denoted by $\|v\|$. This function $\|\cdot\|$ is called *a norm* on V if it satisfies the following axioms:

$[N_1]$ $\|v\| \geq 0$; and $\|v\| = 0$ if and only if $v = 0$.

$[N_2]$ $\|kv\| = |k| \|v\|$.

$[N_3]$ $\|u + v\| \leq \|u\| + \|v\|$.

A vector space V with a norm is called a *normed vector space*.

Suppose V is a normed vector space. The distance between two vectors u and v in V is denoted and defined by

$$d(u, v) = \|u - v\|$$

The following theorem is the main reason why $d(u, v)$ is called the *distance* between u and v.

Theorem : Let V be a normed vector space. Then the function $d(u, v) = \|u - v\|$ satisfies the following three axioms of a metric space:

$[M_1]$ $d(u, v) \geq 0$; and $d(u, v) = 0$ if and only if $u = v$.

$[M_2]$ $d(u, v) = d(v, u)$.

$[M_3]$ $d(u, v) \leq d(u, w) + d(w, v)$.

Normed Vector Spaces and Inner Product Spaces

Suppose V is an inner product space. Recall that the norm of a vector v in V is defined by

$$\|v\| = \sqrt{\langle v, v\rangle}$$

One can prove that this norm satisfies $[N_1]$, $[N_2]$, and $[N_3]$. Thus, every inner product space V is a *normed vector space*. On the other hand, there may be norms on a vector space V that do not come from an inner product on V, as shown below.

Norms on $\mathbf{R}^n$ and $\mathbf{C}^n$

The following define three important norms on $\mathbf{R}^n$ and $\mathbf{C}^n$:

$$\|(a_1, \ldots, a_n)\|_\infty = \max(|a_i|)$$
$$\|(a_1, \ldots, a_n)\|_1 = |a_1| + |a_2| + \cdots + |a_n|$$
$$\|(a_1, \ldots, a_n)\|_2 = \sqrt{|a_1|^2 + |a_2|^2 + \cdots + |a_n|^2}$$

(Note that subscripts are used to distinguish between the three norms.) The norms $\|\cdot\|_\infty$, $\|\cdot\|_1$, and $\|\cdot\|_2$ are called *the infinity-norm*, *one-norm*, and *two-norm*, respectively. Observe that $\|\cdot\|_2$ is the norm on $\mathbf{R}^n$ (respectively, $\mathbf{C}^n$) induced by the usual inner product on $\mathbf{R}^n$ (respectively, $\mathbf{C}^n$). We will let d_∞, d_1, d_2 denote the corresponding *distance functions*.

***Example* 1**: Consider vectors $u = (1;-5; 3)$ and $v = (4; 2;-3)$ in $\mathbf{R}^3$.

(a) The infinity norm chooses the maximum of the absolute values of the components. Hence,

$$\|u\|_\infty = 5, \text{ and } \|v\|_\infty = 4$$

(b) The one-norm adds the absolute values of the components. Thus,

$$\|u\|_1 = 1 + 5 + 3 = 9, \text{ and } \|v\|_1 = 4 + 2 + 3 = 9$$

(c) The two-norm is equal to the square root of the sum of the squares of the components (i.e., the norm induced by the usual inner product on $\mathbf{R}^3$). Thus,

$$\|u\|_2 = \sqrt{1+25+9} = \sqrt{35} \text{ and } \|v\|_2 = \sqrt{16+4+9} = \sqrt{29}$$

(d) Because $u - v = (1 - 4, -5 - 2, 3 + 3) = (-3,-7, 6)$, we have

$$d_\infty(u,v) = 7,\ d_1(u,v) = 3 + 7 + 6 = 16, \text{ and } d_2(u,v) = \sqrt{9+49+36} = \sqrt{94}$$

Norms on C[a, b]

Consider the vector space $V = \mathbf{C}[a, b]$ of real continuous functions on the interval $a \leq t \leq b$. Recall that the following defines an inner product on V:

$$\langle f, g\rangle = \int_a^b f(t)g(t)dt$$

Accordingly, the above inner product defines the following norm on $V = \mathbf{C}[a, b]$ (which is analogous to the $\|\cdot\|$ norm on $\mathbf{R}^n$)

$$\|f\|_2 = \sqrt{\int_a^b [f(t)]^2 dt}$$

The following define the other norms on $V = \mathbf{C}[a, b]$:

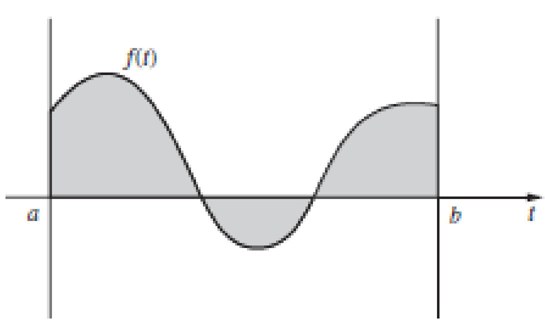
$$\|f\|_1 = \int_a^b |f(t)|dt \text{ and } \|f\|_\infty = max(|f(t)|)$$

There are geometrical descriptions of these two norms and their corresponding distance functions, which are described below.

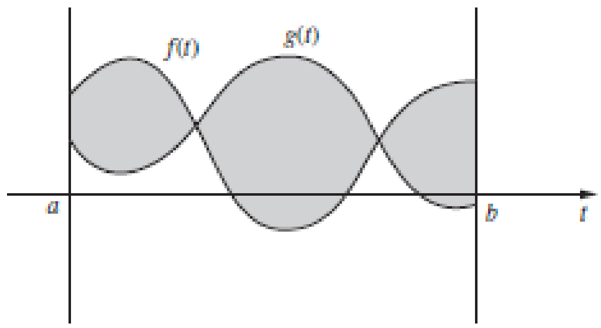
The first norm is pictured in Figure. Here

$\|f\|_1$ = area between the function $|f|$ and the t-axis

$d_1(f, g)$ = area between the functions f and g



(a) $\|f\|_1$ is shaded



(b) $d_1(f, g)$ is shaded