Lecture no 9: COMPLEX INNER PRODUCT SPACES (supplement) [1]

1. Some Facts about Complex Numbers

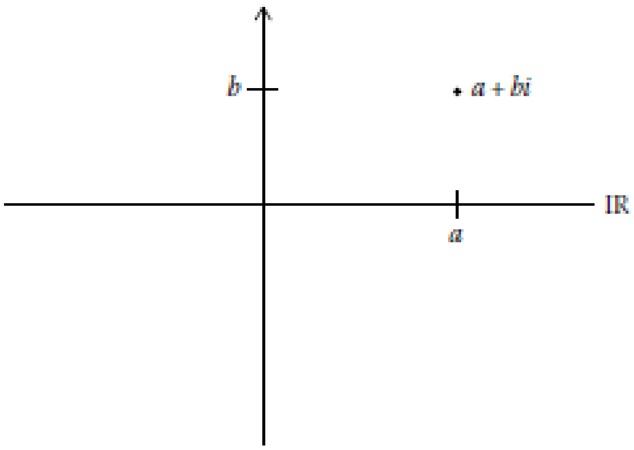
In this chapter, complex numbers come more to the forefront of our discussion. Here we give enough background in complex numbers to study vector spaces where the scalars are the complex numbers.

Complex numbers, denoted by $\mathbb{C}$, are defined by

$$\mathbb{C} = \{a+bi \mid a, b \in \mathbb{R}\},$$

where $i = \sqrt{-1}$.

To plot complex numbers, one uses a two-dimensional grid



Complex numbers are usually represented as z, where $z = a + i\,b$, with $a, b \in \mathbb{R}$. The arithmetic operations are what one would expect. For example, if

$$z_1 = a_1 + b_1 i \quad \text{and} \quad z_2 = a_2 + b_2 i,$$

then

$$z_1 + z_2 = (a_1 + b_1 i) + (a_2 + b_2 i) = (a_1 + a_2) + (b_1 + b_2) i$$

and

$$z_1 z_2 = (a_1 + b_1 i)(a_2 + b_2 i) = a_1 a_2 + (a_1 b_2 + b_1 a_2) i + b_1 b_2 i^2 = (a_1 a_2 - b_1 b_2) + (a_1 b_2 + b_1 a_2) i.$$

If a complex number $z = a + ib$, where $a, b \in \mathbf{R}$, then its complex conjugate is defined as $\bar{z} = a - ib$.

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The following relationships occur

$$z\bar{z} = a^2 + b^2, \qquad |z| = \sqrt{a^2 + b^2}, \qquad \overline{z_1 + z_2} = \overline{z_1} + \overline{z_2} \qquad \overline{z_1 z_2} = \bar{z}_1 \bar{z}_2, \qquad \bar{\bar{z}} = z$$

Also, z is real if and only if $\bar{z} = z$.

If the vector space is $\mathbb{C}^n$, the formula for the usual dot product breaks down in that it does not necessarily give the length of a vector. For example, with the definition of the *Euclidean dot product*, the length of the vector $(3i, 2i)$ would be $\sqrt{-13}$. In $\mathbb{C}^n$, the length of the vector $(a_1, \ldots, a_n)$ should be the distance from the origin to the terminal point of the vector, which is

$$\sqrt{a_1\overline{a_1} + \cdots + a_n\overline{a_n}} = \sqrt{|a_1|^2 + \cdots + |a_n|^2}.$$

The usual inner product in $\mathbb{C}^n$ is if $\hat{u} = (u_1, \ldots, u_n)$ and $\hat{v} = (v_1, \ldots, v_n)$, then

$$\langle \hat{u}, \hat{v} \rangle = \sum_{i=1}^{n} u_i \overline{v_i}.$$

2. Complex Inner Product Spaces

This section considers vector spaces over the complex field $\mathbb{C}$.

Note: The definition of inner product given in *Lecture 6* is not useful for complex vector spaces because no nonzero complex vector space has such an inner product. If it did, pick any vector $u \neq 0$ and then $\langle u, u \rangle > 0$. But also

$$0 < \langle i\mathbf{u}, i\mathbf{u} \rangle = i\langle \mathbf{u}, i\mathbf{u} \rangle = i^2\langle \mathbf{u}, \mathbf{u} \rangle = -\langle \mathbf{u}, \mathbf{u} \rangle < 0$$

which is a contradiction. Following is an altered definition which will work for complex vector spaces.

The following definition applies.

***Definition*:** Let V be a vector space over $\mathbb{C}$. Suppose to each pair of vectors, $u, v \in V$ there is assigned a complex number, denoted by $\langle u, v \rangle$. This function is called a *complex inner product* (*Hermitian inner product*) on V if it satisfies the following axioms:

$[I_1^*]$ (***Linear Property***) $\langle au_1 + bu_2,\ v\rangle = a\langle u_1, v\rangle + b\langle u_2, v\rangle$
$[I_2^*]$ (***Conjugate Symmetric Property***) $\langle u, v\rangle = \overline{\langle v, u\rangle}$
$[I_3^*]$ (***Positive Definite Property***) $\langle u, u\rangle \geq 0$; and $\langle u, u\rangle = 0$ if and only if $u = 0$.

The vector space V over $\mathbb{C}$ with an inner product is called a (*complex*) *inner product space*. Observe that a complex inner product differs from the real case only in the second axiom $[I_2^*]$. Axiom $[I_1^*]$ (*Linear Property*) is equivalent to the two conditions:

(a) $\langle u_1 + u_2, v\rangle = \langle u_1, v\rangle + \langle u_2, v\rangle$
(b) $\langle ku, v\rangle = k\langle u, v\rangle$

On the other hand, applying $[I_1^*]$ and $[I_2^*]$, we obtain

$$\langle u, kv\rangle = \overline{\langle kv, u\rangle} = \overline{k\langle v, u\rangle} = \bar{k}\overline{\langle v, u\rangle} = \bar{k}\langle u, v\rangle$$

That is, we must take the conjugate of a complex number when it is taken out of the second position of a complex inner product. In fact, the inner product is *conjugate linear* in the second position, that is,

$$\langle u, av_1 + bv_2\rangle = \bar{a}\langle u, v_1\rangle + \bar{b}\langle u, v_2\rangle$$

Combining linear in the first position and conjugate linear in the second position, we obtain, by induction,

$$\left\langle \sum_i a_i u_i,\ \sum_j b_j v_j \right\rangle = \sum_{i,j} a_i \overline{b_j} \langle u_i, v_j\rangle$$

The following remarks are in order.

Remark 1: Axiom $[I_1^*]$ by itself implies that $\langle 0,0\rangle = \langle 0u, 0\rangle = 0\langle u, 0\rangle$. Accordingly, $[I_1^*]$, $[I_2^*]$ and $[I_3^*]$ are equivalent to $[I_1^*]$, $[I_2^*]$ and the following axiom:

$[I_3^*]$’: If $u \neq 0$, then $\langle u, u\rangle > 0$

That is, a function satisfying $[I_1^*]$, $[I_2^*]$ and $[I_3^*]$’ is a (complex) inner product on V.

Remark 2: By $[I_2^*]$ $\langle u, u\rangle = \overline{\langle u, u\rangle}$. Thus, $\langle u, u\rangle$ must *be real*. By $[I_3^*]$ $\langle u, u\rangle$ must *be nonnegative*, and hence, its positive real square root exists. As with real inner product spaces, we define $\|u\| = \sqrt{\langle u, u\rangle}$ to be the norm or length of u.

Remark 3: In addition to the norm, we define the notions of orthogonality, orthogonal complement, and orthogonal and orthonormal sets as before. In fact, the definitions of distance and Fourier coefficient and projections are the same as in the real case.

Example (Complex Euclidean Space $\mathbf{C}^n$). Let $V = \mathbf{C}^n$, and let $u = (z_i)$ and $v = (w_i)$ be vectors in $\mathbf{C}^n$. Then

$$\langle u, v\rangle = \sum_k z_k\overline{w_k} = z_1\overline{w_1} + z_2\overline{w_2} + \cdots + z_n\overline{w_n}$$

is an inner product on V, called the *usual* or *standard inner product* on $\mathbf{C}^n$. V with this inner product is called **Complex Euclidean Space**. We assume this inner product on $\mathbf{C}^n$ unless otherwise stated or implied. Assuming u and v are column vectors, the above inner product may be defined by

$$\langle u, v\rangle = u^T\bar{v}$$

where, as with matrices, $\bar{v}$ means the conjugate of each element of v. If u and v are real, we have $\overline{w}_i = w_i$. In this case, the inner product reduced to the analogous one on $\mathbf{R}^n$.

Example The vector space $\mathbf{C}^n$ has a standard inner product, $\langle u, v\rangle = \bar{u}^T v$. So,

$$\langle [1+i, 2-i], [3-2i, 1+i]\rangle = (1-i)(3-2i) + (2+i)(1+i) = 1-5i+1+3i = 2-2i.$$

Example Another example is the complex vector space $C_C[a, b]$ of complex valued continuous functions with domain [a, b]. So any vector in $C_C[a, b]$ is of the form $f(t) + ig(t)$ where f and g are in C[a, b]. Examples are

(a) $t^2 + it^3$ and

(b) $e^{i5t} = \cos(5t) + i\sin(5t)$.

We can define a Hermitian inner product on $C_C[a, b]$ by

$$\langle u, v\rangle = \int_a^b \overline{u(t)}v(t)dt$$

For example in $C_C[0, 2\pi]$, if $l \neq k$

$$\langle e^{kti}, e^{\ell ti}\rangle = \int_0^{2\pi} \overline{e^{kti}}e^{\ell ti}\,dt = \int_0^{2\pi} e^{(\ell-k)ti}\,dt = \left.\frac{e^{(\ell-k)ti}}{(\ell-k)i}\right]_0^{2\pi} = 0$$

Note: $\overline{e^{ia}} = e^{-ia}$

Just as for inner products, the length of a vector u is defined $\|u\| = \sqrt{\langle u, v\rangle}$. The angle θ between two vectors v and u is defined by $\|u\|\|v\|\cos\theta = \Re\langle u, v\rangle$, where $\Re\langle u, v\rangle$ denotes the real part of the Hermitian inner product.

Example to find the angle between $[3\ i]^T$ and $[2+i\ 1-i]^T$ in $\mathbb{C}^2$ we have,

$$||[3\ i]^T|| = \sqrt{3\cdot 3 + (-i)\cdot i} = \sqrt{10}$$

$$||[2+i\ 1-i]^T|| = \sqrt{(2-i)\cdot(2+i)+(1+i)\cdot(1-i)} = \sqrt{2^2+1^2+1^2+1^2} = \sqrt{7}$$

$$\langle [3\ i]^T, [2+i\ 1-i]^T\rangle = 3\cdot(2+i)+(-i)\cdot(1-i) = 6+3i-i-1 = 5+2i$$

$$\theta = \cos^{-1}(5/(\sqrt{10}\sqrt{7})) \approx .93 \text{ radians}$$

Example

a) Let V be the vector space of complex continuous functions on the (real) interval $a \le t \le b$. Then the following is the usual inner product on V:

$$\langle f, g\rangle = \int_a^b f(t)\overline{g(t)}\,dt$$

(b) Let U be the vector space of $m \times n$ matrices over **C**. Suppose $A = (z_{ij})$ and $B = (w_{ij})$ are elements of U. Then the following is the usual inner product on U:

$$\langle A, B\rangle = \operatorname{tr}(B^H A) = \sum_{i=1}^{m}\sum_{j=1}^{n} \bar{w}_{ij} z_{ij}$$

Here, $B^H = \bar{B}^T$; that is, B^H is the conjugate transpose of B.

Remark: A matrix $A = A^H = \bar{A}^T$ a *Hermitian matrix* that plays the same role that a symmetric matrix A (i.e., one where $A^T = A$) plays in the real case.

The following is a list of theorems for complex inner product spaces that are analogous to those for the real case.

Theorem 1: (Cauchy–Schwarz) Let V be a complex inner product space. Then

$$|\langle u, v\rangle| \le \|u\|\|v\|$$

Theorem 2: Let W be a subspace of a complex inner product space V. Then $V = W \oplus W^{\perp}$.

Theorem 3: Suppose $\{u_1, u_2, \dots, u_n\}$ is a basis for a complex inner product space *V*. Then, for any $v \in V$:

$$v = \frac{\langle v, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 + \frac{\langle v, u_2 \rangle}{\langle u_2, u_2 \rangle} u_2 + \cdots + \frac{\langle v, u_n \rangle}{\langle u_n, u_n \rangle} u_n$$

Theorem 4: Suppose $\{u_1, u_2, \dots, u_n\}$ is a basis for a complex inner product space *V*. Let $A = [a_{ij}]$ be the complex matrix defined by $a_{ij} = \langle u_i, u_j \rangle$. Then, for any

$$\langle u, v \rangle = [u]^T A \overline{[v]}$$

where $[u]$ and $[v]$ are the coordinate column vectors in the given basis fuig. (***Remark***: This matrix *A* is said to represent the inner product on *V*.)

Theorem 5: Let *A* be a Hermitian matrix (i.e. $A = A^H = \bar{A}^T$) such that $X^T A \bar{X}$ is real and positive for every nonzero vector $X \in \mathbf{C}^n$. Then $\langle u, v \rangle = u^T A \bar{v}$ is an inner product on $\mathbf{C}^n$.

Theorem 6: Let *A* be the matrix that represents an inner product on *V*. Then *A* is *Hermitian*, and $X^T A \bar{X}$ is real and positive for any nonzero vector in $\mathbf{C}^n$.

Complex numbers

1. Perform the arithmetic operations:

1. $(2+4i)-3(-1+2i)$
2. $2(5-3i)(-6+i)$
3. $6(2+i)\overline{(3-i)}$
4. $|-2-3i|$
5. Show that if

$$z = a + bi$$

then

$$z^{-1} = \frac{1}{z} = \frac{1}{a^2+b^2}(a-bi)$$

if a and b are not both 0.

2. Find the inner product $\langle \hat{u}, \hat{v} \rangle$ for the following pairs of vectors:

(a) $\hat{u} = (1-2i, 3+6i, 2), \hat{v} = (i, -2i, 3-2i)$

(b) $\hat{u} = (i, 3, 2-i)$, $\hat{v} = (3, 4+i, 7i)$

(c) $\hat{u} = (2i, 5i, -3i), \hat{v} = (1, 5, 3)$

(d) $\hat{u} = (1, 5, 3)$, $\hat{v} = (2i, 5i, -3i)$

3. Here a connection between complex numbers and 2 × 2 matrices is demonstrated. We associate

$$1 \text{ with} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \; i \text{ with} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

(a) Show that

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

analogous to $i^2 = -1$.

Identify

$$a + bi \sim \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

(b) Identify $z_1 \sim \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ and $z_2 \sim \begin{pmatrix} c & -d \\ d & c \end{pmatrix}$.

Show that

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} c & -d \\ d & c \end{pmatrix}=\begin{pmatrix} c & -d \\ d & c \end{pmatrix}\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

as $z_1z_2 = z_2z_1$ even though matrices do not typically commute in multiplication.

(c) Show that if $a+bi \sim \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, then $\overline{a+bi} \sim \begin{pmatrix} a & -b \\ b & a \end{pmatrix}^T$.

(d) Show that if $z=a+bi \sim \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, then

$$det\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = a^2+b^2=|z|^2.$$

(e) In complex numbers, Euler's theorem states that

$$e^{i\theta}=\cos\theta+i\sin\theta.$$

In the matrix analogy, this says

$$\exp\begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix}=\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Recall that $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ is the rotation matrix in two dimensions.

Taking derivatives gives $\frac{d}{d\theta}e^{i\theta}=ie^{i\theta}=i\cos\theta-\sin\theta$ whose matrix representation is

$$\begin{pmatrix} -\sin\theta & -\cos\theta \\ \cos\theta & -\sin\theta \end{pmatrix}=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}=i\exp\begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix}.$$

Complex Inner products

1 Suppose $\langle u, v\rangle = 3 + 2i$ in a complex inner product space V. Find
(a) $\langle (2-4i)u, v\rangle$; (b) $\langle u, (4+3i)v\rangle$; (c) $\langle (3-6i)u, (5-2i)v\rangle$.

(a) $\langle (2-4i)u, v\rangle = (2-4i)\langle u, v\rangle = (2-4i)(3+2i) = 14-8i$
(b) $\langle u, (4+3i)v\rangle = \overline{(4+3i)}\langle u, v\rangle = (4-3i)(3+2i) = 18-i$
(c) $\langle (3-6i)u, (5-2i)v\rangle = (3-6i)\overline{(5-2i)}\langle u, v\rangle = (3-6i)(5+2i)(3+2i) = 129-18i$

2. Find the Fourier coefficient (component) c and the projection cw of $v = (3 + 4i; 2 - 3i)$ along $w = (5 + i; 2i)$ in $\mathbf{C}^2$.

Recall that $c = \langle v, w\rangle / \langle w, w\rangle$. Compute

$$\begin{aligned}\langle v, w\rangle &= (3+4i)\overline{(5+i)} + (2-3i)\overline{(2i)} = (3+4i)(5-i) + (2-3i)(-2i)\\ &= 19 + 17i - 6 - 4i = 13 + 13i\\ \langle w, w\rangle &= 25 + 1 + 4 = 30\end{aligned}$$

Thus, $c = (13+13i)/30 = \frac{13}{30} + \frac{13}{30}i$. Accordingly, $\operatorname{proj}(v, w) = cw = (\frac{26}{15} + \frac{39}{15}i, \ -\frac{13}{15} + \frac{1}{15}i)$

3. Find an orthogonal basis for $u^{\perp}$ in $\mathbf{C}^3$ where $u = (1; i; 1 + i)$.

Here $u^{\perp}$ consists of all vectors $s = (x, y, z)$ such that

$$\langle w, u\rangle = x - iy + (1-i)z = 0$$

Find one solution, say $w_1 = (0, \ 1-i, \ i)$. Then find a solution of the system

$$x - iy + (1-i)z = 0, \qquad (1+i)y - iz = 0$$

Here z is a free variable. Set $z = 1$ to obtain $y = i/(1+i) = (1+i)/2$ and $x = (3i-3)2$. Multiplying by 2 yields the solution $w_2 = (3i-3, \ \ 1+i, \ \ 2)$. The vectors w_1 and w_2 form an orthogonal basis for $u^{\perp}$.

4. Find an orthonormal basis of the subspace W of $\mathbf{C}^3$ spanned by

$$v_1 = (1, i, 0) \qquad \text{and} \qquad v_2 = (1, \ 2, \ 1-i).$$

Apply the Gram–Schmidt algorithm. Set $w_1 = v_1 = (1, i, 0)$. Compute

$$v_2 - \frac{\langle v_2, w_1\rangle}{\langle w_1, w_1\rangle} w_1 = (1, \ 2, \ 1-i) - \frac{1-2i}{2}(1, i, 0) = (\tfrac{1}{2} + i, \ \ 1 - \tfrac{1}{2}i, \ \ 1-i)$$

Multiply by 2 to clear fractions, obtaining $w_2 = (1+2i, \ 2-i, \ 2-2i)$. Next find $\|w_1\| = \sqrt{2}$ and then $\|w_2\| = \sqrt{18}$. Normalizing $\{w_1, w_2\}$, we obtain the following orthonormal basis of W:

$$\left\{u_1 = \left(\frac{1}{\sqrt{2}}, \ \frac{i}{\sqrt{2}}, \ 0\right), \ u_2 = \left(\frac{1+2i}{\sqrt{18}}, \ \frac{2-i}{\sqrt{18}}, \ \frac{2-2i}{\sqrt{18}}\right)\right\}$$

5. Find the matrix P that represents the usual inner product on $\mathbf{C}^3$ relative to the basis $\{1; i; 1-i\}$.

Compute the following six inner products:

$$\begin{array}{lll}\langle 1, 1\rangle = 1, & \langle 1, i\rangle = \bar{i} = -i, & \langle 1, 1-i\rangle = \overline{1-i} = 1+i\\ \langle i, i\rangle = i\bar{i} = 1, & \langle i, 1-i\rangle = i(\overline{1-i}) = -1+i, & \langle 1-i, 1-i\rangle = 2\end{array}$$

Then, using $(u, v) = \overline{\langle v, u\rangle}$, we obtain

$$P = \begin{bmatrix} 1 & -i & 1+i\\ i & 1 & -1+i\\ 1-i & -1-i & 2\end{bmatrix}$$

(As expected, P is Hermitian; that is, $P^H = P$.)