Lecture no 10: OPERATORS ON INNER PRODUCT SPACES. SPECTRAL THEOREM [1]

This chapter investigates the space $A(V)$ of linear operators F on an inner product space V. Thus, the base field K is either the real numbers **R** or the complex numbers **C**. In fact, different terminologies will be used for the real case and the complex case. We also use the fact that the inner products on real Euclidean space $\mathbf{R}^n$ and complex Euclidean space $\mathbf{C}^n$ may be defined, respectively, by

$$\langle u, v\rangle = u^T v \text{ and } \langle u, v\rangle = u^T \bar{v},$$

respectively, where u and v are column vectors.

It should be noted that earlier we mainly dealt with real inner product spaces, whereas here we assume that V is a complex inner product space unless otherwise stated or implied.

Lastly, we used A^H to denote the conjugate transpose of a complex matrix A; that is, $A^H = \bar{A}^T$ (*Hermitian*). This notation is not standard. Many texts, especially advanced texts, use A^* to denote such a matrix; we will use that notation in this chapter. That is, now $A^* = \bar{A}^T$.

1. Linear Functionals and Dual Spaces

In previous lectures, we have studied linear mappings (transformations) between vector spaces, for instance, linear transformations from the vector space V to the vector space W. Since the scalar field K is a vector space (the simplest of all nontrivial vector spaces), one class of linear transformations from a vector space V is into its field K of scalars. Naturally all the theorems and results for arbitrary mappings on V hold for this special case. However, these mappings should be considered separately because of their fundamental importance and because the special relationship of V to K gives rise to new notions and results that do not apply in the general case.

Definition: If V is a vector space with scalar field K, then a linear mapping (transformation):

$$\phi: V \rightarrow K$$

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is called a *linear functional* ϕ (or *linear form*) on a vector space *V*.

Note: These linear transformations take vectors from *V* as inputs and give *scalars* as outputs, herewith, for every $u, v \in V$ and every $a, b \in K$,

$\phi(au + bv) = a\phi(u) + b\phi(v)$

Example

1. Let $T: \mathbb{R}^2 \to \mathbb{R}$ be defined by $T(x, y) = 4x - 3y$.
2. Let $T: C[0,1] \to \mathbb{R}$ be defined

$$T(f) = \int_0^1 f(x)\,dx.$$

3. Let $\pi : K^n \to K$ be the ith projection mapping, that is, $\pi(a_1, a_2, \dots a_n) = a_i$. Then π is linear and so it is a linear functional on K^n.

4. Let *V* be the vector space of *n*-square matrices over *K*. Let $T: V \to K$ be the trace mapping $T(A) = a_{11} + a_{22} + \dots + a_{nn}$, where $A = [a_{ij}]$.

That is, *T* assigns to a matrix *A* the sum of its diagonal elements. This map is linear, and so it is a linear functional on *V*.

5. The set of linear functionals on a vector space *V* over a field *K* is also a vector space over *K*.

We denote the set of linear functionals on *V* by V^*. For v_1' and $v_2' \in V^*$, and $\alpha \in K$, we can define

$$v_1' + v_2' \in V^*, \text{ and } \alpha v_1' \in V^*$$

by

$$(v_1' + v_2')(\hat{x}) = v_1'(\hat{x}) + v_2'(\hat{x}) \quad \text{for } \hat{x} \in V$$
$$(\alpha v_1')(\hat{x}) = \alpha(v_1'(\hat{x})) \quad \text{for } \hat{x} \in V.$$

Definition: With these operations, V^* is a vector space, called the *dual of the vector space V*.

Example Let $V = K^n$, the vector space of *n*-tuples, which we write as column vectors. Then the *dual space* V^* can be identified with the space of row vectors. In particular, any linear functional $\phi = (a_1, \dots, a_n)$ in V^* has the representation

$\phi(x_1, x_2, \dots, x_n) = [a_1, a_2, \dots, a_n]\,[x_1, x_2, \dots, x_n]^T = a_1x_1 + a_2x_2 + \dots + a_nx_n$

Historically, the formal expression on the right was termed a *linear form*.

Suppose V is a vector space of dimension n over K. The dimension of the dual space V^* is also n (because K is of dimension 1 over itself). In fact, each basis of V determines a basis of V^* as follows:

Theorem: Suppose $\{v_1, \ldots, v_n\}$ is a basis of V over K. Let $\phi_1, \ldots, \phi_n \in V^*$ be the linear functionals ($\phi_i: V \to K$, $i = 1, \ldots, n$) as defined by

$$\phi_i(v_j) = \delta_{ij} = \begin{cases} 1 \; if \; i = j \\ 0 \; if \; i \neq j \end{cases}$$

The above basis $\{\phi_i\}$ is termed the *basis dual* to $\{v_i\}$ or the *dual basis*. The above formula, which uses the *Kronecker delta* δ_{ij}, is a short way of writing

$$\begin{array}{l} \phi_1(v_1) = 1, \quad \phi_1(v_2) = 0, \quad \phi_1(v_3) = 0, \quad \ldots, \quad \phi_1(v_n) = 0 \\ \phi_2(v_1) = 0, \quad \phi_2(v_2) = 1, \quad \phi_2(v_3) = 0, \quad \ldots, \quad \phi_2(v_n) = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \phi_n(v_1) = 0, \quad \phi_n(v_2) = 0, \quad \ldots, \phi_n(v_{n-1}) = 0, \quad \phi_n(v_n) = 1 \end{array}$$

Corollary: If the dimension of the vector space V is n, then the dimension of V^* is n.

Example Consider the basis $\{v_1 = (2; 1), v_2 = (3; 1)\}$ of $\mathbf{R}^2$. Find the dual basis $\{\phi_1, \phi_2\}$.

We seek linear functionals $\phi_1(x; y) = ax + by$ and $\phi_2(x; y) = cx + dy$ such that

$$\phi_1(v_1) = 1, \qquad \phi_1(v_2) = 0, \qquad \phi_2(v_2) = 0, \qquad \phi_2(v_2) = 1$$

These four conditions lead to the following two systems of linear equations:

$$\left.\begin{array}{l} \phi_1(v_1) = \phi_1(2,1) = 2a + b = 1 \\ \phi_1(v_2) = \phi_1(3,1) = 3a + b = 0 \end{array}\right\} \quad \text{and} \quad \left.\begin{array}{l} \phi_2(v_1) = \phi_2(2,1) = 2c + d = 0 \\ \phi_2(v_2) = \phi_2(3,1) = 3c + d = 1 \end{array}\right\}$$

The solutions yield $a = -1$, $b = 3$ and $c = 1$, $d = -2$. Hence, $\phi_1(x; y) = -x + 3y$ and $\phi 2(x; y) = x - 2y$ form the *dual basis*.

The next two theorems give relationships between bases and their duals.

***Theorem* 1:** Let $\{v, \ldots, v_n\}$ be a basis of V and let $\{\phi_1, \ldots, \phi_n\}$ be a dual basis in V^*. Then

(i) For any vector $u \in V$, $u = \phi_1(u)v_1 + \phi_2(u)v_2 + \ldots + \phi_n(u)v_n$

(ii) For any linear functional $\sigma \in V^*$, $\sigma = \sigma_1(v_1)\ \phi_1 + \sigma_2(v_2)\ \phi_2 + \ldots + \sigma_n(v_n)\ \phi_n$

***Theorem* 2: :** Let $\{v_1, \dots, v_n\}$ and $\{w_1, \dots, w_n\}$ be bases of V and let $\{\phi_1, \dots, \phi_n\}$ and $\{\sigma_1, \dots, \sigma_n\}$ be bases of V^* dual to $\{v_i\}$ and $\{w_i\}$, respectively. Suppose P is the *change-of-basis matrix* from $\{v_i\}$ to $\{w_i\}$. Then $(P^{-1})^T$ is the *change-of-basis matrix* from $\{\phi_i\}$ to $\{\sigma_i\}$.

2. Adjoint Operators

The adjoint of a linear operator has classically been defined for all vector spaces, but the most common applications occur with inner product spaces. It has become common in the literature to either differentiate between the adjugate of a linear transformation on a vector space and reserve the term adjoint for inner product spaces, or to only deal with inner product spaces. We will only consider inner product spaces.

Let U and V are finite dimensional inner product spaces and $T: U \to V$ is a linear transformation. We want to find a linear transformation:

$T^*: V \to U$ that satisfies $\langle T(u), v\rangle_U = \langle u, T^*(v)\rangle_V$

for any $u \in U$ and $v \in V$.

The notation $\langle\ ,\ \rangle_U$ and $\langle\ ,\ \rangle_V$ is used to emphasize that U and V may have different inner products.

Definition: A linear operator T on an inner product space V is said to have an *adjoint operator* T^* on V if, $\langle T(u), v\rangle = \langle u, T^*(v)\rangle$ for every $u, v \in V$

The following example shows that the *adjoint operator* has a simple description

Example

(a) Let A be a real n-square matrix viewed as a linear operator on $\mathbf{R}^n$. Then, for every $u, v \in \boldsymbol{R}^n$

$$\langle Au, v\rangle = (Au)^T v = u^T A^T v = \langle u, A^T v\rangle$$

Thus, the *transpose* A^T of A is the adjoint of A.

(b) Let B be a complex n-square matrix viewed as a linear operator on $\mathbf{C}^n$. Then for every $u, v \in \boldsymbol{C}^n$

$$\langle Bu, v\rangle = (Bu)^T \bar{v} = u^T B^T \bar{v} = u^T \overline{B^*} \bar{v} = \langle u, B^* v\rangle$$

Thus, the *conjugate transpose* B^* of B is the adjoint of B.

***Remark*:** B^* may mean either the adjoint of B as a linear operator or the conjugate transpose of B as a matrix. By Example (b), the ambiguity makes no difference, because they denote the same object.

The following theorem is the main result in this section.

Theorem 1: Let T be a linear operator on a finite-dimensional inner product space V over K. Then

(i) There exists a unique linear operator T^* on V such that $\langle T(u), v\rangle = \langle u, T^*(v)\rangle$ for every $u, v \in V$. (That is, T has an adjoint T^*.)

(ii) If A is the matrix representation T with respect to any orthonormal basis $S = \{u_i\}$ of V, then the matrix representation of T^* in the basis S is the conjugate transpose A^* of A (or the transpose A^{T} of A when K is real).

Remark: It should be emphasized that no such simple relationship exists between the matrices representing T and T^* if the basis is *not orthonormal*. Thus, it is seen one useful property of orthonormal bases. It is also emphasized that this theorem is not valid if V has infinite dimension.

The following theorem summarizes some of the properties of the adjoint.

Theorem 2: Let T, T_1, T_2 be linear operators on V and let $k \in K$. Then

(i) $(T_1 + T_2)^* = T_1^* + T_2^*$, (iii) $(T_1T_2)^* = T_2^*T_1^*$,

(ii) $(kT)^* = \bar{k}T^*$, (iv) $(T^*)^* = T$.

Observe the similarity between the above theorem and on properties of the transpose operation on matrices.

Linear Functionals and Inner Product Spaces

Recall that a linear functional f on a vector space V is a linear mapping ϕ: $V \rightarrow K$. Let V be an inner product space, each $u \in V$ determines a mapping $\hat{u}(v) = V \rightarrow K$ defined by

$$\hat{u}(v) = \langle v, u\rangle$$

and, with addition and scalar multiplication defined for any a, $b \in K$ and any v_1 and v_2 $\in V$

$$\hat{u}(av_1 + bv_2) = \langle av_1 + bv_2, \ u\rangle = a\langle v_1, u\rangle + b\langle v_2, u\rangle = a\hat{u}(v_1) + b\hat{u}(v_2)$$

that is, $\hat{u}$ is a *linear functional* on V.

The converse is also true for spaces of finite dimension and it is contained in the following important theorem:

Theorem 3: Let ϕ be a linear functional on a finite-dimensional inner product space V. Then there exists *a unique vector* $u \in V$ such that $\phi(v) = \langle v, u\rangle$ for every $v \in V$.

Analogy Between $A(V)$ and C, Special Linear Operators:

Let $A(V)$ denote the algebra of all linear operators on a finite-dimensional inner product space V. The adjoint mapping $T \mapsto T^*$ on $A(V)$ is quite analogous to the conjugation mapping $z \mapsto \bar{z}$ on the complex field **C**. To illustrate this analogy we identify in Table certain classes of operators $T \in A(V)$ whose behavior under the adjoint map imitates the behavior under conjugation of familiar classes of complex numbers.

Class of complex numbers	Behavior under conjugation	Class of operators in $A(V)$	Behavior under the adjoint map
Unit circle ($\lvert z\rvert = 1$)	$\bar{z} = 1/z$	Orthogonal operators (real case) Unitary operators (complex case)	$T^* = T^{-1}$
Real axis	$\bar{z} = z$	Self-adjoint operators Also called: symmetric (real case) Hermitian (complex case)	$T^* = T$
Imaginary axis	$\bar{z} = -z$	Skew-adjoint operators Also called: skew-symmetric (real case) skew-Hermitian (complex case)	$T^* = -T$
Positive real axis $(0, \infty)$	$z = \bar{w}w, w \neq 0$	Positive definite operators	$T = S^*S$ with S nonsingular

The analogy between these operators T and complex numbers z is reflected in the next theorem.

Theorem 4: Let λ be an *eigenvalue* of a linear operator T on V.

(i) If $T^* = T^{-1}$ (i.e., T is ***orthogonal*** or ***unitary***), then $\lvert\lambda\rvert = 1$.

(ii) If $T^* = T$ (i.e., T is ***self-adjoint***), then λ is real.

(iii) If $T^* = -T$ (i.e., T is ***skew-adjoint***), then λ is pure imaginary.

(iv) If $T = S^*S$ with S nonsingular (i.e., T is ***positive definite***), then λ is real and positive.

Self-Adjoint Operators Let T be a *self-adjoint operator* on an inner product space V, that is, suppose $T^* = T$ (If T is defined by a matrix A, then A is *symmetric* or *Hermitian* according as A is *real* or *complex*.)

By the Theorem 4 (ii), the eigenvalues of T are real. The following is another important property of T.

Theorem: Let T be a self-adjoint operator on V. Suppose u and v are eigenvectors of T belonging to distinct eigenvalues. Then u and v are orthogonal, that is, $\langle u, v \rangle = 0$.

Proof. Suppose $T(u) = \lambda_1 u$ and $T(v) = \lambda_2 v$, where $\lambda_1 \neq \lambda_2$. We show that $\lambda_1 \langle u, v\rangle = \lambda_2 \langle u, v\rangle$:

$$\begin{aligned}\lambda_1\langle u, v\rangle &= \langle \lambda_1 u, v\rangle = \langle T(u), v\rangle = \langle u, T^*(v)\rangle = \langle u, T(v)\rangle \\ &= \langle u, \lambda_2 v\rangle = \bar{\lambda}_2\langle u, v\rangle = \lambda_2\langle u, v\rangle\end{aligned}$$

Orthogonal and Unitary Operators Let U be a linear operator on a finite-dimensional inner product space V. Suppose $U^* = U^{-1}$ or equivalently $UU^* = U^*U = I$

Recall that U is said to be *orthogonal* or *unitary* according as the underlying field is real or complex. The next theorem gives alternative characterizations of these operators.

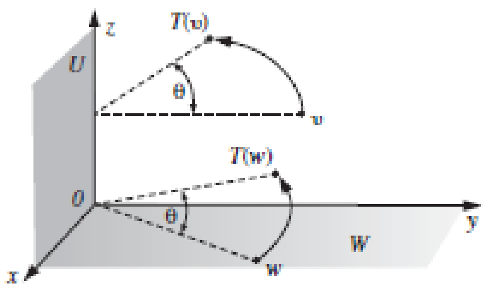
Theorem: The following conditions on an operator U are equivalent:

(i) $U^* = U^{-1}$, that is, $UU^* = U^*U = I$. [U is *unitary* (*orthogonal*).]

(ii) U preserves inner products, that is, for every v, $w \in V$, $\langle U(v), U(w)\rangle = \langle v, w\rangle$.

(iii) U preserves lengths, that is, for every $v \in V$, $\|U(v)\| = \|v\|$.

Example Let T: $\mathbf{R}^3 \to \mathbf{R}^3$ be the linear operator that rotates each vector v about the z-axis by a fixed angle θ as shown in Fig.



That is, T is defined by

$$T(x; y; z) = (x\cos\theta - y\sin\theta;\ x\sin\theta + y\cos\theta;\ z)$$

We note that lengths (distances from the origin) are preserved under T. Thus, T is an orthogonal operator.

Orthogonal and Unitary Matrices Let U be a linear operator on an inner product space V. By the Theorem 1, we obtain the following results.

Theorem **A**: A complex matrix A represents a unitary operator U (relative to an orthonormal basis) if and only if $A^* = A^{-1}$.
Theorem **B:** A real matrix A represents an orthogonal operator U (relative to an orthonormal basis) if and only if $A^T = A^{-1}$.

The above theorems motivate the following definitions:
Definition: A complex matrix A for which $A^* = A^{-1}$ is called a *unitary matrix*.
Definition: A real matrix A for which $A^T = A^{-1}$ is called an *orthogonal matrix*.

Theorem: The following conditions on a matrix A are equivalent:

(i) A is unitary (orthogonal).
(ii) The rows of A form an orthonormal set.
(iii) The columns of A form an orthonormal set.

Change of Orthonormal Basis. Orthonormal bases play a special role in the theory of inner product spaces V. Thus, we are naturally interested in the properties of the *change-of-basis matrix* from one such basis to another. The following theorem holds.
Theorem 5: Let $\{u_1, \dots, u_n\}$ be an orthonormal basis of an inner product space V. Then the *change-of-basis matrix* from $\{u_i\}$ into another orthonormal basis is *unitary* (*orthogonal*).

Conversely, if $P = [a_{ij}]$ is a unitary (orthogonal) matrix, then the following is an orthonormal basis:

$$\{u_i' = a_{1i}u_1 + a_{2i}u_2 + \cdots + a_{ni}u_n : i = 1, \dots, n\}$$

Recall that matrices A and B representing the same linear operator T are similar, that is, $B = P^{-1}A\,P$, where P is the (nonsingular) *change-of-basis matrix*. On the other hand, if V is an inner product space, we are usually interested in the case when P is *unitary* (or *orthogonal*) as suggested by the Theorem. Herewith,

P is *unitary* if the conjugate tranpose $P^* = P^{-1}$, and

P is *orthogonal* if the transpose $P^T = P^{-1}$.)

This leads to the following definition.

Definition: Complex matrices A and B are *unitarily equivalent* if there exists a unitary matrix P for which $B = P^*A\,P$.

Analogously, real matrices A and B are *orthogonally equivalent* if there exists an orthogonal matrix P for which $B = P^{\mathrm{T}}A\,P$.

Note that orthogonally equivalent matrices are necessarily *congruent*.

Positive Definite and Positive Operators Let P be a linear operator on an inner product space V. Then

(i) P is said to be *positive definite* if $P = S^*S$ for some nonsingular operators S;

(ii) P is said to be *positive* (or *nonnegative* or *semidefinite*) if $P = S^*S$ for some operator S.

The following theorems give alternative characterizations of these operators.

Theorem 6A: The following conditions on an operator P are equivalent:

(i) $P = T^2$ for some nonsingular self-adjoint operator T;

(ii) P is positive definite;

(iii) P is self-adjoint and $\langle P(u), u\rangle > 0$ for every $u \neq 0$ in V.

The corresponding theorem for positive operators follows.

Theorem 6B: The following conditions on an operator P are equivalent:

(i) $P = T^2$ for some self-adjoint operator T;

(ii) P is positive, that is, $P = S^*S$;

(iii) P is self-adjoint and $\langle P(u), u\rangle \geq 0$ for every $u \in V$.

Diagonalization and Canonical Forms in Inner Product Spaces

Let T be a linear operator on a finite-dimensional inner product space V over K. Representing T by a diagonal matrix depends upon the eigenvectors and eigenvalues of T, and hence, upon the roots of the characteristic polynomial $\Delta(t)$ of T. Now $\Delta(t)$ always factors into linear polynomials over the complex field **C** but may not have any linear polynomials over the real field **R**. Thus, the situation for real inner product spaces (sometimes called *Euclidean spaces*) is inherently different than the situation for complex inner product spaces (sometimes called *unitary spaces*). Thus, we treat them separately.

Real Inner Product Spaces, Symmetric and Orthogonal Operators The following theorem holds.

Theorem 7: Let T be a *symmetric* (*self-adjoint*) operator on a real finite-dimensional product space V. Then there exists an *orthonormal basis* of V consisting of eigenvectors of T, that is, T can be represented by a *diagonal matrix* relative to an orthonormal basis.

We give the corresponding statement for matrices.

Theorem 7': (*Alternative form*) Let A be a *real symmetric matrix*. Then there exists an *orthogonal matrix* P such that $B = P^{-1}AP = P^{\mathrm{T}}AP$ is *diagonal*.

Remark: We can choose the columns of the above matrix P to be *normalized orthogonal eigenvectors* of A, then the diagonal entries of B are the corresponding eigenvalues.

On the other hand, an orthogonal operator T need not be symmetric, and so it may not be represented by a diagonal matrix relative to an orthonormal matrix. However, such a matrix T does have a simple canonical representation, as described in the following theorem.

Theorem 8: Let T be an orthogonal operator on a real inner product space V. Then there exists an orthonormal basis of V in which T is represented by a *block diagonal matrix* M of the form:

$$M = \operatorname{diag}\left(I_s,\ -I_t,\ \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix},\ \ldots,\ \begin{bmatrix} \cos\theta_r & -\sin\theta_r \\ \sin\theta_r & \cos\theta_r \end{bmatrix} \right)$$

It may be recognized that each of the 2×2 diagonal blocks represents a rotation in the corresponding two-dimensional subspace, and each diagonal entry -1 represents a reflection in the corresponding one-dimensional subspace.

Complex Inner Product Spaces, Normal and Triangular Operators A linear operator T is said to be *normal* if it commutes with its adjoint — that is, if $TT^* = T^*T$. We note that normal operators include both self-adjoint and unitary operators.

Analogously, a *complex matrix A* is said to be *normal* if it commutes with its conjugate transpose — that is, if $AA^* = A^*A$.

Example Let $A = \begin{bmatrix} 1 & 1 \\ i & 3+2i \end{bmatrix}$, then $A^* = \begin{bmatrix} 1 & -i \\ 1 & 3-2i \end{bmatrix}$.

Also, $AA^* = \begin{bmatrix} 2 & 3-3i \\ 3+3i & 14 \end{bmatrix} = A^*A$. Thus, A is normal.

The following theorem holds.

Theorem 9: Let T be a *normal operator* on a complex finite-dimensional inner product space V. Then there exists an *orthonormal basis* of V consisting of eigenvectors of T, that is, T can be represented by a *diagonal matrix* relative to an orthonormal basis.

We give the corresponding statement for matrices.

Theorem 9': (*Alternative form*) Let A be a *normal matrix*. Then there exists a *unitary matrix P* such that $B = P^{-1}AP = P^*AP$ is *diagonal*.

The following theorem shows that even *non-normal operators* on unitary spaces have a relatively simple form.

Theorem 10: Let T be an *arbitrary operator* on a complex finite-dimensional inner product space V. Then T can be represented by a *triangular matrix* relative to an orthonormal basis of V.

Theorem 10': (*Alternative form*) Let A be an *arbitrary* complex matrix. Then there exists a *unitary matrix P* such that $B = P^{-1}AP = P^*AP$ is *triangular*.

Spectral Theorem

The Spectral Theorem is a reformulation of the diagonalization Theorems presented above.

Theorem 11: Let T be a *normal* (*symmetric*) operator on a complex (real) finite-dimensional inner product space V. Then there exists linear operators $E_1, \dots, E_r$ on V

and scalars $\lambda_1, \ldots, \lambda_r$ such that

$$\text{(i)} \quad T = \lambda_1 E_1 + \lambda_2 E_2 + \cdots + \lambda_r E_r, \qquad \text{(iii)} \quad E_1^2 = E_1, E_2^2 = E_2, \ldots, E_r^2 = E_r,$$
$$\text{(ii)} \quad E_1 + E_2 + \cdots + E_r = I, \qquad \text{(iv)} \quad E_i E_j = 0 \text{ for } i \neq j.$$

The above linear operators $E_1, \ldots, E_r$ are *projections* in the sense that $E_i^2 = E_i$. Moreover, they are said to be *orthogonal projections* because they have the additional property that $E_i E_j = 0$ for $i \neq j$.

The following example shows the relationship between a diagonal matrix representation and the corresponding orthogonal projections.

Example Consider the following diagonal matrices A, E_1, E_2, E_3:

$$A = \begin{bmatrix} 2 & & & \\ & 3 & & \\ & & 3 & \\ & & & 5 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{bmatrix},$$
$$E_2 = \begin{bmatrix} 0 & & & \\ & 1 & & \\ & & 1 & \\ & & & 0 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & 1 \end{bmatrix}$$

It could be verified that

$$\text{(i)} \quad A = 2E_1 + 3E_2 + 5E_3, \text{ (ii)} \quad E_1 + E_2 + E_3 = I, \text{ (iii)} \quad E_i^2 = E_i, \text{ (iv)} \quad E_i E_j = 0 \quad \text{for } i \neq j.$$