Distance learning materials for students

E-118іа.е, E-318іб.е, E-118іл.е, E-618іб.е, МІТ-203.8і

Lecture 3 (17.04.2020)

4.1. Triple Integral in Cylindrical and Spherical Coordinate Systems

There are plenty of problems when calculation of triple integrals is more convenient in cylindrical, spherical and other coordinate systems.

The question of change of variables in a triple integral is solved similarly to the case of a double integral, i.e. if the function $f(x, y, z)$ is continuous in some domain V and the formulas

$$x = \varphi(u, v, w); \quad y = \psi(u, v, w), \quad z = \chi(u, v, w) \tag{4.2}$$

establish one-to-one correspondence between the points $M(x, y, z)$ of the domain V and the points $M'(u, v, w)$ of some domain V', then

$$\iiint\limits_V f(x, y, z)\,dxdydz = \iiint\limits_{V'} f(\varphi(u, v, w), \psi(u, v, w), \chi(u, v, w))\,|J|\,dudvdw,$$

where $|J|$ is absolute value of Jacobian. For imagination (4.2) of functional determinant by Jacobian is

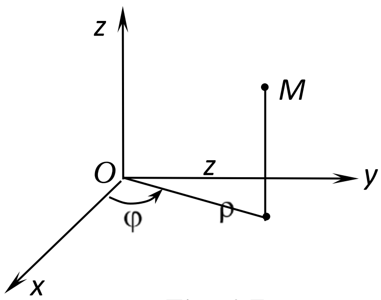

Fig. 4.7*a*

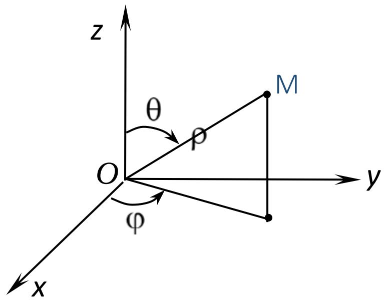

Fig. 4.7*b*

$$J = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}. \tag{4.3}$$

In **cylindrical** coordinate system location of a point is defined by the polar coordinates φ, ρ and z-coordinate (Fig. 4.7*a*) (formulas connecting rectangular and cylindrical coordinates look as follows $x=\rho\cos\varphi;\ y=\rho\sin\varphi;\ z=z$), absolute value of Jacobian make $\left|J\right|=\rho$ (Prove it by yourself).

$$\iiint\limits_V f(x,y,z)dxdydz=\iiint\limits_{V'} f(\rho\cos\varphi,\rho\sin\varphi,z)\rho d\rho d\varphi dz.$$

The coordinate surfaces $\rho=const,\ \varphi=const,\ z=const$ of the of cylindrical system coordinates are present relatively: circle cylinders with the axis *Oz*, half-planes coming from the axis *Oz* and planes parallel to the plane *xOy*.

Therefore if an integrating domain is a circle cylinder with the axis *Oz,* then the corresponding triple integral over this domain in the cylindrical coordinates will have constant limits of an integration for all variables, i.e.

$$\iiint\limits_V f(x,y,z)dV=\int\limits_0^{2\pi}d\varphi\int\limits_0^R\rho d\rho\int\limits_0^H f(\rho\cos\varphi,\rho\sin\varphi,z)dz.$$

Spherical coordinates of a point *M* of the area *V* are defined through ρ,φ,θ, where ρ is distance between the origin and the point *M* $\rho^2=x^2+y^2+z^2$, φ is angle between the axis *Ox* and projection of the radius-vector *OM* on the plane *xOy*, and θ is angle between positive directions of the axis *Oz* and the radius-vector *OM* (Fig. 4.7*b*). It is obviously that $(\rho\ge 0,\ 0\le\theta\le\pi,\ 0\le\varphi\le 2\pi)$. Here are the following coordinate surfaces: $\rho=\mathrm{const}$ is spheres with center at the origin, $\varphi=\mathrm{const}$ is semi-planes coming out the axis *Oz*, $\theta=\mathrm{const}$ is a circle cones with the axis *Oz*. The spherical coordinates ρ,φ,θ are connected with rectangular one by the following relations:

$$x=\rho\sin\theta\cos\varphi,\ y=\rho\sin\theta\sin\varphi,\ z=\rho\cos\theta.$$

Applying the formula (4.3) it is possible to show that

$$\left|J\right|=\rho^2\sin\theta.$$

Passing to the spherical coordinates in a triple integral is carried out according to next formula:

$$\iiint\limits_V f(x,y,z)dxdydz=$$

$$=\iiint\limits_V f(\rho\sin\theta\cos\varphi,\rho\sin\theta\sin\varphi,\rho\cos\theta)\rho^2\sin\theta d\rho d\varphi d\theta.$$

It is obviously that if the domain of integration is a sphere with center at the origin and radius *R*, then the triple integral over the domain will have constant integrating limits on all variables in the spherical system coordinates, i.e.

$$\iiint\limits_V f(x,y,z)dV = \int\limits_0^{\pi}\sin\theta d\theta\int\limits_0^{2\pi}d\varphi\int\limits_0^R \rho^2 f(\rho\sin\theta\cos\varphi, \rho\sin\theta\sin\varphi, \rho\cos\theta)d\rho .$$

4.2. Some Rules of Finding Integration Limits in the Cylindrical and Spherical Coordinates. Geometrical and Physical Applications of the Triple Integrals

Below we will consider some examples in order to illustrate the applications of the triple integrals.

Example 1. Calculate $\iiint\limits_V xyzdxdydz$, where V is part of the domain, bounded by the sphere $x^2+y^2+z^2=4$ and paraboloid $x^2+y^2=3z$, located in the 1st octant (Fig. 4.9).

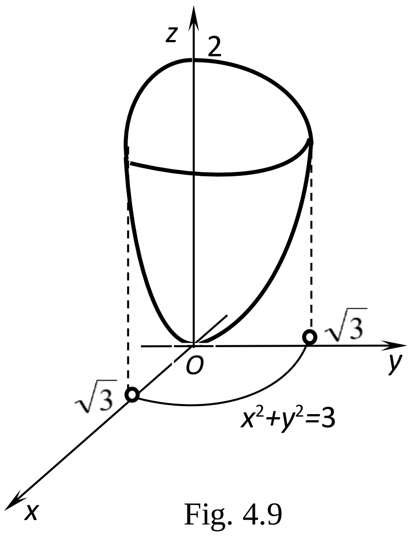

Fig. 4.9

Solution. The 1st way. Calculation of an integral in the Cartesian coordinates.

Before projecting the domain V on the plane xOy we should define the line of intersection of the sphere and paraboloid. For this let us solve these two equations jointly:

$$\left\{\begin{array}{c} x^2+y^2+z^2=4 \\ x^2+y^2=3z \end{array}\right\} \Rightarrow z^2+3z-4=0 \Rightarrow$$

$$z=1 \Rightarrow x^2+y^2=3.$$

i.e. the surfaces are crossed on a circle with radius $R=\sqrt{3}$, belonging to the plane $z=1$. The domain V is projected on the plane xOy in a quarter of a circle with the same diameter lying at the first quarter.

Thus we obtain:

$$\iiint\limits_V xyzdxdydz = \int\limits_0^{\sqrt{3}} xdx \int\limits_0^{\sqrt{3-x^2}} ydy \int\limits_{\frac{x^2+y^2}{3}}^{\sqrt{4-x^2-y^2}} zdz =$$

$$=\iint_D xyzdxdydz=\int_0^{\sqrt{3}}xdx\int_0^{\sqrt{3-x^2}}ydy\int_{\frac{x^2+y^2}{3}}^{\sqrt{4-x^2-y^2}}zdz=$$

$$=\frac{1}{2}\int_0^{\sqrt{3}}xdx\int_0^{\sqrt{3-x^2}}yz^2\Bigg|_{\frac{x^2+y^2}{3}}^{\sqrt{4-x^2-y^2}}dy=$$

$$=\frac{1}{2}\int_0^{\sqrt{3}}xdx\int_0^{\sqrt{3-x^2}}yz^2\Bigg|_{\frac{x^2+y^2}{3}}^{\sqrt{4-x^2-y^2}}dy=$$

$$=\frac{1}{2}\int_0^{\sqrt{3}}xdx\int_0^{\sqrt{3-x}}y\left(4-x^2-y^2-\frac{\left(x^2+y^2\right)^2}{9}\right)dy=$$

$$=\frac{1}{2}\int_0^{\sqrt{3}}x\left(2y^2-\frac{x^2y^2}{2}-\frac{y^4}{4}-\frac{x^4y^2}{18}-\frac{x^2y^4}{18}-\frac{y^6}{54}\right)\Bigg|_0^{\sqrt{3-x^2}}dx=$$

$$=\frac{1}{2}\int_0^{\sqrt{3}}x\left(\frac{13}{2}x-2x^3+\frac{x^5}{4}+\frac{x^7}{54}\right)dx=\frac{27}{32}.$$

The 2nd way. Calculation of an integral in the cylindrical coordinates:

$$I=\iiint_V xyzdxdydz=\iiint_{V'}\rho^3\sin\varphi\cos\varphi zd\rho d\varphi dz\,,$$

where V' is image of the solid *V* in cylindrical system coordinates of the points belonging to the area *V*.

After finding integrating limits we obtain the following:

$$I=\int_0^{\frac{\pi}{2}}\sin\varphi\cos\varphi d\varphi\int_0^{\sqrt{3}}\rho^3d\rho\int_{\frac{\rho^2}{3}}^{\sqrt{4-\rho^2}}zdz=$$

$$=\int_{\frac{\rho^2}{3}}^{\sqrt{4-\rho^2}}zdz=\frac{1}{2}\frac{\sin^2\varphi}{2}\Bigg|_0^{\frac{\pi}{2}}\cdot\int_0^{\sqrt{3}}\rho^3\left(4-\rho^2-\frac{\rho^4}{9}\right)d\rho=\frac{27}{32}.$$

Example 2. Calculate volume of part of the sphere $x^2+y^2+z^2=R^2$, located inside the cylinder

$$\left(x^2+y^2\right)^2=R^2\left(x^2-y^2\right)\quad(z\geq 0)$$

(Fig. 4.10).

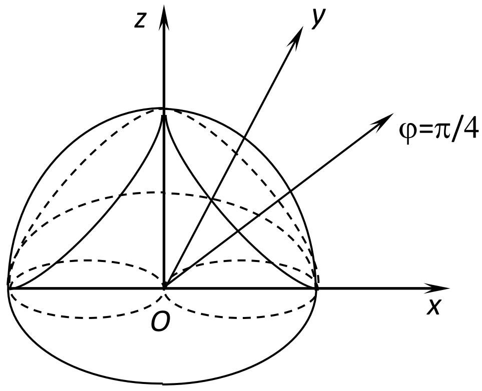

Fig. 4.10

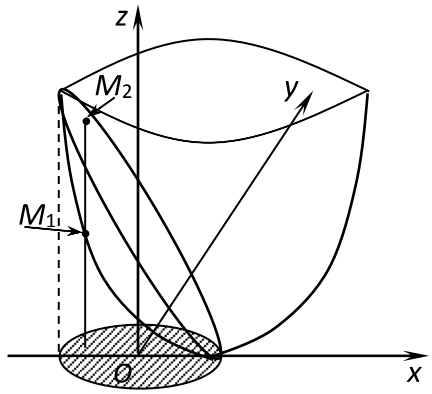

Fig. 4.11

Solution. Let us form the generatrix of the cylinder, bounded by a lemniscates.

Pass to the polar coordinates $x = \rho\cos\varphi,\ \ y = \rho\sin\varphi$. Polar equation of this curve is $\rho = R\sqrt{\cos 2\varphi}$. The curve is symmetrical about the axes *Ox* and *Oy* and while changing φ from 0 to
$\pi/4$ the current point (ρ,φ) will cover one quarter of the domain.

The required volume in the cylindrical coordinates is presented as follows:

$$V = \iiint\limits_{V_1} \rho d\rho d\varphi dz = 4\int_0^{\pi/4} d\varphi \int_0^{R\sqrt{\cos 2\varphi}} \rho d\rho \int_0^{\sqrt{R^2-\rho^2}} dz =$$

$$= 4\int_0^{\pi/4} d\varphi \int_0^{R\sqrt{\cos 2\varphi}} \rho\sqrt{R^2-\rho^2}\,d\rho = 4\int_0^{\pi/4} -\frac{1}{2}\cdot\frac{2}{3}\left(R^2-\rho^2\right)^{3/2}\Big|_0^{R\sqrt{\cos 2\varphi}} d\varphi =$$

$$= \frac{4}{3}\int_0^{\pi/4} R^3\left(1-\left(1-\cos 2\varphi\right)^{3/2}\right)d\varphi = \frac{4}{3}R^3\left(\frac{\pi}{4}+\frac{5-4\sqrt{2}}{3}\right).$$

Example 3. Calculate volume of the solid limited by the following surfaces: paraboloid $(x-1)^2 + y^2 = z$ and the plane $2x + z = 2$ (Fig. 4.11).

Solution. Let us find equation of projection of intersection line of the surface $(x-1)^2 + y^2 = z$ with the plane $2x + z = 2$ on the plane *xOy*, *z*-coordinates coincide on the line of intersection and then we obtain

$$(x-1)^2 + y^2 = 2-2x, \Rightarrow\ x^2 + y^2 = 1.$$

As the domain *V* is projected in the circle $D_{xy} : x^2 + y^2 \le 1$, it is expediently to pass to the cylindrical coordinates. Equation of the border D_{xy} in the cylindrical coordinates

$$\rho^2\cos^2\varphi+\rho^2\sin^2\varphi=1\Rightarrow\rho=1.$$

Equation of the plane is $z=2(1-\rho\cos\varphi)$; equation of the paraboloid is

$$z=x^2-2x+1+y^2\Rightarrow z=\rho^2-2\rho\cos\varphi+1.$$

At each current value of (ρ,φ) $(0\le\rho\le1,\ 0\le\varphi\le2\pi)$ the variable z is changing from $z_1=\rho^2-2\rho\cos\varphi+1$ (at the point M_1 – point of entrance into the domain V) to $z_2=2(1-\rho\cos\varphi)$ (at the point M_2 – point of exit from the domain V). Volume of the solid in the cylindrical coordinates makes:

$$V=\iiint_v\rho d\rho d\varphi dz=\int_0^{2\pi}d\varphi\int_0^1\rho d\rho\int_{\rho^2-2\rho\cos\varphi+1}^{2(1-\rho\cos\varphi)}dz=\int_0^{2\pi}d\varphi\int_0^1\rho\cdot z\Big|_{\rho^2-2\rho\cos\varphi+1}^{2(1-\rho\cos\varphi)}d\rho=$$

$$=\int_0^{2\pi}d\varphi\int_0^1\rho\left(2-2\rho\cos\varphi-\rho^2+2\rho\cos\varphi-1\right)d\rho=\int_0^{2\pi}d\varphi\int_0^1\left(\rho-\rho^3\right)d\rho=$$

$$=\varphi\Big|_0^{2\pi}\cdot\left(\frac{\rho^2}{2}-\frac{\rho^4}{4}\right)\Bigg|_0^1=2\pi\left(\frac{1}{2}-\frac{1}{4}\right)=\frac{\pi}{2}.$$

Example 4. Calculate volume of the solid bounded by the sphere $x^2+y^2+z^2=4Rz-3R^2$ and the cone $z^2=4\left(x^2+y^2\right)$ (solid is considered to be a part of the sphere lying inside the cone) Fig. 4.12.

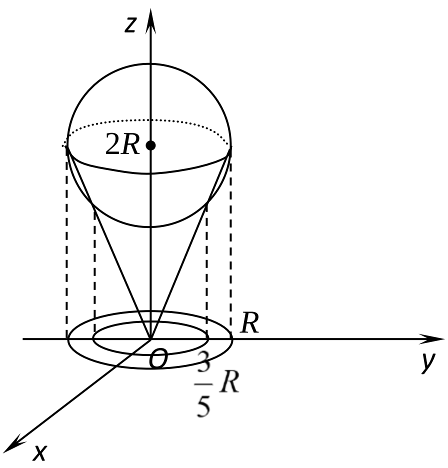

Fig. 4.12

Solution. Let us transform equation of the sphere:

$$x^2+y^2+(z-2R)^2=R^2.$$

The center of the sphere is the point (0,0,2R); radius makes R. Let us find projections of intersection lines of the cone and the sphere on the plane xOy. For this let us present equations of the surfaces in the cylindrical coordinates and then equate their z-coordinates.

$$z^2=4\left(x^2+y^2\right)\Rightarrow z^2=4\rho^2,\quad z=2\rho$$

$$x^2+y^2+(z-2R)^2\Rightarrow z=2R\pm\sqrt{R^2-\rho^2}\qquad\Rightarrow$$

$$2R\pm\sqrt{R^2-\rho^2}=2\rho$$

$$\Rightarrow R^2-\rho^2=4\rho^2-8R\rho+4R^2\Rightarrow5\rho^2-8R\rho+3R^2=0;\rho_1=\frac{3}{5}R;\ \rho_2=R.$$

At the first let us calculate volume of a part of the sphere lying outside the cone:

$$V_1 = \int_0^{2\pi} d\varphi \int_{\frac{3}{5}R}^{R} \rho d\rho \int_{2R-\sqrt{R^2-\rho^2}}^{2\rho} dz = 2\pi \int_{\frac{3}{5}R}^{R} \rho\left(2\rho - 2R + \sqrt{R^2-\rho^2}\right)d\rho =$$

$$= 2\pi \int_{\frac{3}{5}R}^{R} \left(2\rho^2 - 2R\rho + \rho\sqrt{R^2-\rho^2}\right)d\rho = 2\pi\left(2\frac{\rho^3}{3} - 2R\frac{\rho^2}{2}\right)\Bigg|_{\frac{3}{5}R}^{R} -$$

$$-2\pi\frac{1}{2}\int_{\frac{3}{5}R}^{R}\left(R^2-\rho^2\right)^{1/2} d\left(R^2-\rho^2\right) = 2\pi\left(\frac{2}{3}\left(R^3 - \frac{27}{125}R^3\right) - R^3 + \frac{9}{25}R^3\right) -$$

$$-2\pi\frac{1}{2}\frac{\left(R^2-\rho^2\right)^{3/2}}{3/2}\Bigg|_{\frac{3}{5}R}^{R} = \frac{8}{75}\pi R^3.$$

Then volume of the sphere lying inside the cone is the following:

$$V = \frac{4}{3}\pi R^3 - V_1 = \frac{4}{3}\pi R^3 - \frac{8}{75}\pi R^3 = \frac{92}{75}\pi R^3.$$

Example 5. Find inertia moment of a homogeneous solid ($\delta(x,y,z)\equiv 1$), limited by the sphere $x^2+y^2+z^2=2z$ and the cone $x^2+y^2=z^2$ about the axis *Oz* (Fig. 4.13).

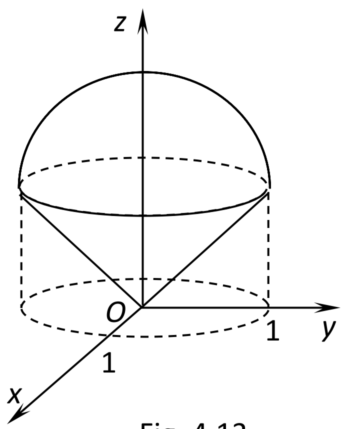

Fig. 4.13

Solution. Let us form the mentioned solid. For this let us find the interception line of the surfaces:

$$\begin{cases} x^2+y^2+z^2=2z \\ x^2+y^2=z^2 \end{cases} \Rightarrow 2z^2 = 2z \Rightarrow z=1,$$ i.e. this line is a circle with radius $R=1$ lying on the plane $z=1$. Projection of the solid on the plane *xOy* is the circle $x^2+y^2\le 1$.

Inertia moment is calculated according to the formula:

$$I_z = \iiint_V \left(x^2+y^2\right)dxdydz.$$

Let us pass to the spherical coordinates, then all integrating limits will be constant. Besides this, we can define limits for θ by means of the equation

$x^2+y^2=z^2$ considering $x=0$ (or $y=0$) we obtain that $y=z \Rightarrow \theta=\frac{\pi}{4}$, i.e. $0 \le \theta \le \frac{\pi}{4}$.

$$I_z=\iiint\limits_{V'} r^2 \sin^2\theta r^2 \sin\theta dr d\varphi d\theta = \int\limits_0^{2\pi} d\varphi \int\limits_0^{\pi/4} \sin^3\theta d\theta \int\limits_0^{2\cos\theta} r^4 dr = \frac{11\pi}{30}.$$

Example 6. Find gravity center of common part of the spheres $x^2+y^2+z^2 \le R^2$ and $x^2+y^2+z^2 \le 2Rz$, if density at each point of the given solid is equal to the distance between this point and the plane *xOy*.

Solution. Coordinates of gravity centre of a solid is calculated as follows:

$$x_c=\frac{M_{yz}}{M}=\frac{\iiint\limits_D x\delta(x,y,z)dxdydz}{\iiint\limits_D \delta(x,y,z)dxdydz};$$

$$y_c=\frac{M_{xz}}{M}=\frac{\iiint\limits_D y\delta(x,y,z)dxdydz}{\iiint\limits_D \delta(x,y,z)dxdydz};$$

$$z_c=\frac{M_{xy}}{M}=\frac{\iiint\limits_D z\delta(x,y,z)dxdydz}{\iiint\limits_D \delta(x,y,z)dxdydz},$$

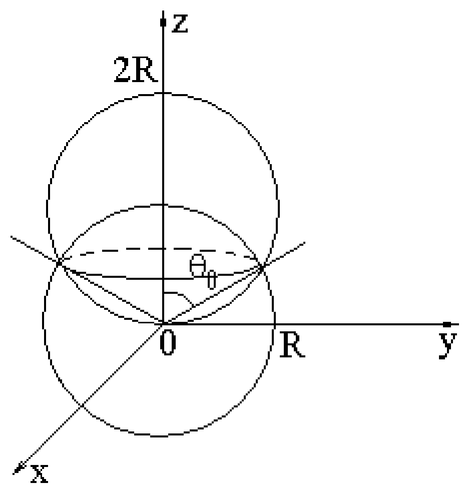

Fig. 4.14

where M – weight of the solid, M_{yz}, M_{xz}, M_{xy} – static moments. In this example $\delta(x,y,z)=z$. Due to the symmetry of the solid (Fig. 4.14) about the axis *Oz* we obtain that $x_c=y_c=0$. Let us define the angle θ_0 solving jointly equations of the spheres (in the spherical coordinates):

$$\begin{cases}\rho=R;\\ \rho=2R\cos\theta;\end{cases} \Rightarrow \cos\theta_0=\frac{1}{2}; \Rightarrow \theta_0=\frac{\pi}{3}.$$

Then

$$M_x=\iiint\limits_V z^2 dV=\iiint\limits_V \rho^4\cos^2\theta\sin\theta\, d\rho\, d\varphi\, d\theta=$$

$$= \int_0^{\frac{\pi}{3}} \sin\theta\cos^2\theta\, d\theta \int_0^{2\pi} d\varphi \int_0^R \rho^4 d\rho + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin\theta\cos^2\theta\, d\theta \int_0^{2\pi} d\varphi \int_0^{2R\cos\theta} \rho^4 d\rho =$$

$$= \frac{7\pi R^5}{60} + \frac{\pi R^5}{160} = \frac{59\pi R^5}{480}.$$

$$M = \iiint_V z\,dV = \iiint_V \rho^3 \cos\theta\sin\theta\, d\rho\, d\varphi\, d\theta = \int_0^{\frac{\pi}{3}} \sin\theta\cos\theta\, d\theta \int_0^{2\pi} d\varphi \int_0^R \rho^3 d\rho +$$

$$+ \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin\theta\cos\theta\, d\theta \int_0^{2\pi} d\varphi \int_0^{2R\cos\theta} \rho^3 d\rho = \frac{19\pi R^4}{80}.$$

So

$$z_c = \frac{59\pi R^5}{480} \Bigg/ \frac{19\pi R^4}{80} = \frac{59R}{114}.$$