Methodical instructions to IDZ

" Differential calculus for functions with several variables"

GROUP I-219ia, КН-219 ae,219iae,219,iбe,219iвe

Task 1.

Example 1.

Show that the function $z = x^2 - y^2$ satisfies the following equation

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0 \qquad (1)$$

Solution.

In order to solve the task you have to calculate the second derivatives of the given function and substitute them in the given equation

$$\frac{\partial z}{\partial x} = 2x; \frac{\partial^2 z}{\partial x^2} = 2;$$

$$\frac{\partial z}{\partial y} = -2y; \frac{\partial^2 z}{\partial y^2} = -2;$$

After substitution we get

$$2 - 2 = 0.$$

So indeed the given function satisfies the equation (1).

Example 2.

Prove that the function

$$z = y\cos(x - y) \qquad (2)$$

satisfies the following equation

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = \frac{z}{y} \qquad (3)$$

Solution.

$$\frac{\partial z}{\partial x} = -y\sin(x - y);$$

$$\frac{\partial z}{\partial y} = \cos(x-y) + y\sin(x-y).$$

Let's substitute these partial derivatives into Eq. (3). We have

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = -y\sin(x-y) + \cos(x-y) + y\sin(x-y) = \cos(x-y).$$

From (2) we can find

$$\cos(x-y) = \frac{z}{y};$$

so, we have identity

$$\frac{z}{y} \equiv \frac{z}{y}.$$

Example 3.

Find $\frac{du}{dt}$ if $u = xyz$,where $x = t^2 + 1,\ y = \ln t,\ z = \tan t.$

Solution.

In order to solve this task you should know the differentiation rule of the complex function. In the given case we have the following situation: $u = u(x, y, z),$ where $x = x(t),\ y = y(t),\ z = z(t).$ That is function $u(x, y, z)$ depends on only variable t by complex way. So $\frac{du}{dt}$ is total derivative, which is calculated by the following formulas:

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt}. \qquad (4)$$

Now calculate all derivatives:

$$\frac{\partial u}{\partial x} = yz;\ \frac{\partial u}{\partial y} = xz;\ \frac{\partial u}{\partial z} = xy; \qquad (5)$$

$$\frac{dx}{dt} = 2t;\ \frac{dy}{dt} = \frac{1}{t};\ \frac{dz}{dt} = \frac{1}{\cos^2 t}. \qquad (6)$$

After substitution (5) and (6) into (4) we obtain:

$$\frac{du}{dt}=2yzt+\frac{xz}{t}+\frac{xy}{\cos^2 t}.$$

Task 2.

Example. Investigate the function for an extreme

$$z=3x^2-x^3+3y^2+4y.$$

Solution.

Necessary conditions to be extreme at the point M_0 is equality to zero of all partial derivatives at this point. Function may also have extreme at the point where its partial derivatives don’t exist. So, first of all you must find critical (stationary) points. In the given case we have:

$$\frac{\partial z}{\partial x}=6x-3x^2,\ \frac{\partial z}{\partial y}=6y+4.$$

Generate system

$$\begin{cases}\frac{\partial z}{\partial x}=0\\ \frac{\partial z}{\partial y}=0\end{cases}\Rightarrow\begin{cases}6x-3x^2=0\\ 6y+4=0\end{cases}\Rightarrow\begin{cases}2x-x^2=0\\ 3y=-2\end{cases}\Rightarrow\begin{cases}x(2-x)=0\\ y=-\frac{2}{3}\end{cases}\Rightarrow\begin{cases}x_1=0\\ y_1=-\frac{2}{3}\end{cases} and \begin{cases}x_2=2\\ y_2=-\frac{2}{3}\end{cases}$$

Next step. You should find the second derivatives of the given function and calculate their values at the critical points.

$$\frac{\partial^2 z}{\partial x^2}=6-6x,\ \frac{\partial^2 z}{\partial x\partial y}=0,\ \frac{\partial^2 z}{\partial y^2}=6.$$

Calculate values of the second derivatives at the point $M_1\left(0;\frac{2}{3}\right)$:

$$a_{11}=\left.\frac{\partial^2 z}{\partial x^2}\right|_{M_1}=6,\ a_{12}=\left.\frac{\partial^2 z}{\partial x\partial y}\right|_{M_1}=0,\ a_{22}=\left.\frac{\partial^2 z}{\partial y^2}\right|_{M_1}=6.$$

Sufficient condition for existence of the extremum is:

$$\Delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix} > 0,$$

if $a_{11} > 0,$ than M_1 is point of minimum;

if $a_{11} < 0,$ than M_1 is point of maximum.

In the given case

$\Delta = \begin{vmatrix} 6 & 0 \\ 0 & 6 \end{vmatrix} = 36 > 0,\ a_{11} = 6 > 0 \Rightarrow$ point $M_1\left(0; -\frac{2}{3}\right)$ is point of minimum.

$$z_{\min}\Big|_{\substack{x=0 \\ y=-\frac{2}{3}}} = 3x^2 - x^3 + 3y^2 + 4y\Big|_{\substack{x=0 \\ y=-\frac{2}{3}}} = \frac{3\cdot 4}{9} - \frac{4\cdot 2}{3} = \frac{4}{3}(1-2) = -\frac{4}{3}.$$

Investigate the point $M_2\left(2; -\frac{2}{3}\right)$:

$$a_{11} = \frac{\partial^2 z}{\partial x^2}\bigg|_{M_2} = 6 - 6x\big|_{M_2} = 6(1-2) = -6,$$

$$a_{12} = \frac{\partial^2 z}{\partial x \partial y}\bigg|_{M_2} = 0,\ a_{22} = \frac{\partial^2 z}{\partial y^2}\bigg|_{M_2} = 6.$$

$\Delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix} = \begin{vmatrix} -6 & 0 \\ 0 & 6 \end{vmatrix} = -36 < 0.$ If $\Delta < 0$ then function doesn't have extreme at the point.

<u>Task 3.</u>

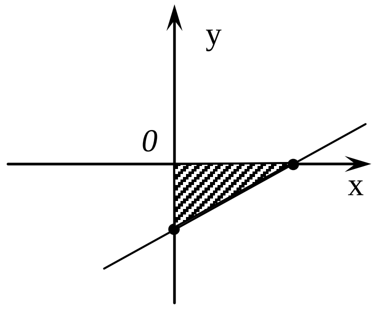
Example. Find the greatest and the smallest values of the function

$$z = \cos^2 x + \cos^2 y \qquad (7)$$

in the domain D, bounded by the given curves $x - y = \frac{\pi}{4}; x \geq 0, y \leq 0.$

<u>Solution.</u>

Let's draw the domain. First of all find critical points that belong to given domain. These points satisfy the system



$$\begin{cases} \frac{\partial z}{\partial x} = 0 \\ \frac{\partial z}{\partial y} = 0 \end{cases}$$

$$\frac{\partial z}{\partial x} = -2\cos x \sin x = -\sin 2x; \qquad \frac{\partial z}{\partial y} = -2\cos y \sin y = -\sin 2y;$$

$$\begin{cases} \sin 2x = 0 \\ \sin 2y = 0 \end{cases} \Rightarrow \begin{cases} 2x = \pi n \\ 2y = \pi k \end{cases} n, k \in Z \Rightarrow \begin{cases} x = \frac{\pi n}{2} \\ y = \frac{\pi k}{2} \end{cases} n, k \in Z \Rightarrow$$

we have only one point, if $n = 0, k = 0,\ x = 0, y = 0, \Rightarrow M_1(0,0)$.

Second step is reduced to solution of the given problem on the domain boundary.

Consider the following part of the domain

$$\begin{cases} x = 0 \\ -\frac{\pi}{4} \le y \le 0 \end{cases}$$

Then the initial function $z = \cos^2 x + \cos^2 y$ becomes function of one variable:

$$z = 1 + \cos^2 y. \tag{8}$$

We should find the greatest and the smallest of the function (8) on segment

$-\frac{\pi}{4} \le y \le 0.$ So let’s find derivative of (8)

$$\frac{dz}{dy} = -2\cos y \sin y = -\sin 2y;$$

$\frac{dz}{dy} = 0 \Rightarrow -\sin 2y = 0 \Rightarrow 2y = \pi k \Rightarrow y = \frac{\pi k}{2}$, if $k = 0 \Rightarrow y = 0.$ We again obtain

point $M_0(0,0)$.

We should include the end points of this segment $M_2\left(0, -\frac{\pi}{4}\right)$.

Similarly we investigate the given function on the segment of straight line

$$\begin{cases} y = 0 \\ 0 \le x \le \frac{\pi}{4} \end{cases}$$

On this segment the given functions depend on only x and take the form

$z=\cos^2 x.$

$$\frac{dz}{dx}=-2\cos x\sin x=-\sin 2x;$$

$\frac{dz}{dx}=0\Rightarrow -\sin 2x=0\Rightarrow 2x=\pi k\Rightarrow x=\frac{\pi n}{2}$, if $n=0\Rightarrow x=0.$ We have again only end point $M_3\left(\frac{\pi}{4},0\right)$.

Let's investigate the third border of the domain: $x-y=\frac{\pi}{4}$. Taking into account that $y=x-\frac{\pi}{4}$ on this straight line, we can again obtain function

$z=\cos^2 x+\cos^2\left(x-\frac{\pi}{4}\right)$, depending only of x, where $0\le x\le\frac{\pi}{4}$.

Let's find the point of which the function $z=\cos^2 x+\cos^2\left(x-\frac{\pi}{4}\right)$ can take the greatest of the smallest values:

$$\frac{dz}{dx}=-2\cos x\sin x-2\cos\left(x-\frac{\pi}{4}\right)\sin\left(x-\frac{\pi}{4}\right)=-\sin 2x-\sin 2\left(x-\frac{\pi}{4}\right)=$$
$$=-\sin 2x+\cos 2x;$$

$$\frac{dz}{dx}=0\Rightarrow \cos 2x-\sin 2x=0\Rightarrow 1-\tan 2x=0\Rightarrow \tan 2x=1\Rightarrow$$
$$2x=\frac{\pi}{4}+k\pi\Rightarrow x=\frac{\pi}{8}+k\frac{\pi}{2},$$

if $k=0$ then point $\begin{cases} x=\dfrac{\pi}{8} \\ y=\dfrac{\pi}{8}-\dfrac{\pi}{4}=-\dfrac{\pi}{8}\end{cases}\Rightarrow M_4\left(\frac{\pi}{8},-\frac{\pi}{8}\right)$ belong to investigated segment.

Now let's calculate the values of the function z of all points M_1,M_2,M_3,M_4.

$$z\big|_{M_1(0,0)}=\cos^2 x+\cos^2 y\big|_{M_1(0,0)}=2;$$

$$z\big|_{M_2\left(0,-\frac{\pi}{4}\right)} = 1+\frac{2}{4} = \frac{3}{2};$$

$$z\big|_{M_3\left(\frac{\pi}{4},0\right)} = \frac{3}{2};$$

$$z\big|_{M_4\left(\frac{\pi}{8},-\frac{\pi}{8}\right)} = \cos^2\frac{\pi}{8} + \cos^2\left(-\frac{\pi}{8}\right) = 2\cos^2\frac{\pi}{8} = 1.8477 < 2;$$

$$z_{great} = z\big|_{M_1(0,0)} = 2;$$

$$z_{\min} = z\big|_{M_2,M_3} = \frac{3}{2}.$$

Task 4.

Example. On the curve $x=\frac{t^4}{4}, y=\frac{t^3}{3}, z=\frac{t^2}{2}$ find the points at which the tangent to this curve is parallel to the plane $x+3y+2z-10=0$

Solution. Equation of the tangent line to the curve is:

$$\frac{x-x_0}{x'_{t_0}} = \frac{y-y_0}{y'_{t_0}} = \frac{z-z_0}{z'_{t_0}},$$

where $M_0(x_0, y_0, z_0)$ is tangency point, value of the parameter t_0 corresponds to position of the tangency point, x'_{t_0}, y'_{t_0}, z'_{t_0} are coordinates of the directional vector of the tangent line.

In the given case, we don't know tangency point. In order to find that point let's use the additional condition: tangent line is parallel to the plane $x+3y+2z-10=0$. So directional vector of the unknown tangent line $\vec{a}\left(x'_{t_0},\ y'_{t_0},\ z'_{t_0}\right)$ is perpendicular to normal vector $\vec{n}(1,\ 3,\ 2)$ of the plane. Condition of the perpendicularity is $(\vec{n},\vec{a})=0.$

Find the coordinate of the directional vector $\vec{a}\left(x'_{t_0},\ y'_{t_0},\ z'_{t_0}\right)$:

$$x'_{t_0} = t_0^3;\ y'_{t_0} = t_0^2;\ z'_{t_0} = t_0$$

So we have

$$1\cdot t_0^3+3\cdot t_0^2+2t_0=0\Rightarrow t_0\cdot\left(t_0^2+3\cdot t_0+2\right)=0\Rightarrow$$
$$t_0^{(1)}=0;\quad t_0^{(2)}=-1;\quad t_0^{(3)}=-2.$$

Now calculate coordinates of the points, corresponding to parameters $t_0^{(1)}, t_0^{(2)}, t_0^{(3)}$.

$$x_0^{(1)}=\frac{t_0^4}{4}=0,\ y_0^{(1)}=0,\ z_0^{(1)}=0;$$

That is $M_0(0,0,0)$ and derivatives vanish as well $x_{t_0^{(1)}}=0,\ y_{t_0^{(1)}}=0,\ z_{t_0^{(1)}}=0.$

Tangent line doesn't exist at this point, because ${x'_{t_0}}^2+{y'_{t_0}}^2+{z'_{t_0}}^2=0.$ It's singular point.

Investigate points corresponding to $t_0^{(2)}=-1$ and $t_0^{(3)}=-2$.

$$t_0^{(2)}=-1\Rightarrow x_0^{(2)}=\left.\frac{t_0^4}{4}\right|_{t=-1}=\frac{1}{4},\ y_0^{(2)}=\left.\frac{t_0^3}{3}\right|_{t=-1}=-\frac{1}{3},\ z_0^{(2)}=\left.\frac{t_0^2}{2}\right|_{t=-1}=\frac{1}{2}.$$

$${x'_{t_0}}^{(2)}=\left.t^3\right|_{t=-1}=-1;\ {y'_{t_0}}^{(2)}=\left.t^2\right|_{t=-1}=1;\ {z'_{t_0}}^{(2)}=\left.t\right|_{t=-1}=-1.$$

Equation of the tangent line is

$$\frac{x-\frac{1}{4}}{-1}=\frac{y+\frac{1}{3}}{1}=\frac{z-\frac{1}{2}}{-1}.$$

Analogously we can investigate the last point

$$t_0^{(3)}=-2\Rightarrow x_0^{(3)}=\left.\frac{t_0^4}{4}\right|_{t=-2}=4,\ y_0^{(3)}=\left.\frac{t_0^3}{3}\right|_{t=-2}=-\frac{8}{3},\ z_0^{(3)}=\left.\frac{t_0^2}{2}\right|_{t=-2}=2.$$

$${x'_{t_0}}^{(3)}=\left.t^3\right|_{t=-2}=-8;\ {y'_{t_0}}^{(3)}=\left.t^2\right|_{t=-2}=4;\ {z'_{t_0}}^{(3)}=\left.t\right|_{t=-2}=-2.$$

Equation of the tangent line is

$$\frac{x-4}{-8}=\frac{y+\frac{8}{3}}{4}=\frac{z-2}{-2}\ ,$$

or

$$\frac{x-4}{4}=\frac{y+\frac{8}{3}}{-2}=\frac{z-2}{1}.$$

Task 5

Find the gradient of the function $\underline{u = F(x,y,z)}$ at the point $\underline{A(x_0, y_0, z_0)}$

Notes. You should remember that gradient $\vec{\nabla}$ of the function is vector, coordinates of which coincide with partial derivatives of the given functions, that is,

$$\boxed{\vec{\nabla}u = \frac{\partial u}{\partial x}\cdot\vec{i} + \frac{\partial u}{\partial y}\cdot\vec{j} + \frac{\partial u}{\partial z}\cdot\vec{k}}$$

Example. Find gradient of the function

$u = x^2 + y^2 - z^2 - 2y,$ at the point $A(0,3,1)$

Solution. $\vec{\nabla}u = (2x)_A \cdot\vec{i} + (2y-2)_A\cdot\vec{j} - 6z_A\cdot\vec{k} = 0\cdot\vec{i} + 4\cdot\vec{j} - 6\cdot\vec{k}.$

So, $\vec{\nabla}u = (0;4;-6)$

Task 6

Find the derivative of the function $\varphi(x,y,z)$ with respect to the given direction.

Note. Directional derivative is defined by the following formula

$$\boxed{\frac{\partial\varphi}{\partial l} = \frac{\partial\varphi}{\partial x}\cdot\cos\alpha + \frac{\partial\varphi}{\partial y}\cdot\cos\beta + \frac{\partial\varphi}{\partial z}\cdot\cos\gamma}$$

or $\boxed{\frac{\partial\varphi}{\partial l} = \left(\vec{\nabla}\varphi, \vec{l}^0\right)}$. It`s scalar product of the $\vec{\nabla}\varphi$ and ort $\vec{l}^{\,0}$ of the directional vector $\vec{l}$. Ia you calculate directional derivative at the fixed point, then you have to calculate partial derivatives $\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z}$ at this point $A(x_0, y_0, z_0)$.

The main problem is finding of the directional vector ort $\vec{l}^{\,0}$.

If vector $\vec{l}$ is known, then $\vec{l}^0 = \frac{\vec{l}}{|\vec{l}|}$, or $\vec{l}^0 = \left(\frac{l_x}{|l|}, \frac{l_y}{|l|}, \frac{l_z}{|l|}\right) = (\cos\alpha, \cos\beta, \cos\gamma)$

Example 1. Find directional derivative $\frac{\partial\varphi}{\partial l}$, of the function

$\varphi(x,y,z) = y^2 + art\tan(x-z)$ at the point A(0;-2;-1) with respect to the direction $\vec{l} = \{2;-6;3\}$

<u>Solution.</u> *Step 1*. Let's find partial derivatives of the function $\varphi(x,y,z)$ and calculate these values at the point $A(0,-2,-1)$

$$\left.\frac{\partial\varphi}{\partial x}\right|_A=\left.\frac{1}{1+(x-z)^2}\right|_A=\frac{1}{1+1}=\frac{1}{2}$$

$$\left.\frac{\partial\varphi}{\partial y}\right|_A=2y\big|_A=-4;\quad \left.\frac{\partial\varphi}{\partial z}\right|_A=\left.\frac{(-1)}{1+(x-1)^2}\right|_A=-\frac{1}{1+1}=-\frac{1}{2}$$

Step 2. Determination of the ort $\vec{l}^0=(\cos\alpha,\cos\beta,\cos\gamma)$

$$\left|\vec{l}\right|=\sqrt{2^2+(-6)^2+(-3)^2}=\sqrt{4+36+9}=7$$

$$\cos\alpha=\frac{l_x}{|l|}=\frac{2}{7};\quad \cos\beta=\frac{l_y}{|l|}=\frac{-6}{7}=-\frac{6}{7};\quad \cos\gamma=\frac{l_z}{|l|}=\frac{-3}{7}=-\frac{3}{7}$$

So $\vec{\nabla}\varphi=\left\{\frac{1}{2};-4;-\frac{1}{2}\right\};\ \vec{l}^0=\left\{\frac{2}{7},-\frac{6}{7},-\frac{3}{7}\right\}$

Step 3. Calculation of the directional derivative

$$\frac{\partial\varphi}{\partial l}=\left(\vec{\nabla}\varphi,\vec{l}^0\right)=\frac{1}{2}\cdot\frac{2}{7}-4\cdot\left(-\frac{6}{7}\right)-\frac{1}{2}\cdot\left(-\frac{3}{7}\right)=\frac{1}{7}+\frac{24}{7}+\frac{3}{14}=\frac{53}{14}=3\frac{11}{14}$$

Answer: $\left.\frac{\partial\varphi}{\partial l}\right|_A=3\frac{11}{14}$

If it`s needed to calculate directional derivative with respect to the direction of the <u>some curve</u> (circle, parabola and etc.), **you should know** that *direction of the curve is defined by direction of the tangent line to this curve at the given point. Counterclockwise movement along the curve is considered to be the positive direction of the curve*

<u>Example</u>. Find direction of the circle $x^2+y^2=4x$ at the point $A(1;-\sqrt{3})$.

Solution. Calculate derivative of the function $x^2+y^2=4x$**,** considering *y* as function of *x*, given in implicit way

$$2x+2y\cdot y'=4\Rightarrow x+y\cdot y'=2\Rightarrow y'=\frac{2-x}{y}$$

By geometric sense of the derivative it coincides with slope of the tangent line to the given curve. So $y' = \tan\alpha$, where α is angle between tangent line and positive direction of *OX*-axis.

So in the given case $y'\Big|_A = \frac{2-1}{-\sqrt{3}} = -\frac{1}{\sqrt{3}}$, So $\tan\alpha = -\frac{1}{\sqrt{3}} \Rightarrow \alpha = -\frac{\pi}{6}$;

Now we can calculate $\cos\alpha = \cos\left(-\frac{\pi}{6}\right) = +\frac{\sqrt{3}}{2}$; and $\cos\beta = \sin\alpha = \sin\left(-\frac{\pi}{6}\right)1 = -\frac{1}{2}$.

Answer: $\vec{l}^0 = \left(\frac{\sqrt{3}}{2}; -\frac{1}{2}\right)$

Example 2. Find directional derivative of the function $\varphi = y^2 + 2yx$ at the point $A(0;-2)$ of the ellipse $x^2 + \frac{y^2}{4} = 1$ with respect to the direction of this curve.

Solution. Let`s find $\vec{l}^0(\cos\alpha, \cos\beta) = \vec{l}^0(\cos\alpha, \sin\alpha)$. So we should find derivative $y' = \frac{dy}{dx}$ of the function $x^2 + \frac{y^2}{4} = 1$, in implicit way.

$$2x + \frac{2y \cdot y'}{4} = 0 \Rightarrow 2x + \frac{y \cdot y'}{2} = 0 \Rightarrow y \cdot y' = -4x, \quad y' = -\frac{4x}{y}\Big|_A = 0$$

So $\tan\alpha = 0 \Rightarrow \quad \alpha = 0, \Rightarrow \cos\alpha = 1; \quad \cos\beta = \sin\alpha = 0$.

Now calculate $\vec{\nabla}\varphi = \left\{\frac{\partial\varphi}{\partial x}; \frac{\partial\varphi}{\partial y}\right\}\Big|_A = \{2y; 2y + 2x\}\Big|_A = \{-4; -4\}$.

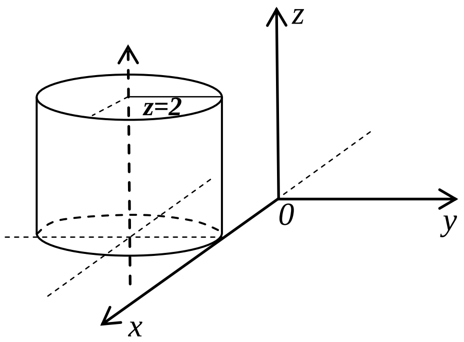
So $\frac{\partial\varphi}{\partial l} = \left(\vec{\nabla}\varphi, \vec{l}^0\right) = -4 \times 1 + 0 \times (-4) = -4$

<u>Answer</u> $\frac{\partial\varphi}{\partial l}\Big|_A = -4$

Example 3.

Find directional derivative of the function $\varphi(x, y, z) = xz^2 + 2yz$ at the point $A(1; 0, 2)$ with respect to direction of the space curve (circle), given by equations:

$$\begin{cases} x^2 + y^2 - 2x + 2y + 1 = 0, \\ z = 2 \end{cases}$$



Solution. We deal with circle in space, obtained as intersection of the cylinder $x^2+y^2-2x+2y+1=0$ and plane $z=2$. Find center of this circle, allocating the full square

$$x^2+y^2-2x+2y+1=0 \Rightarrow (x-1)^2+(y+1)^2=1$$

Let`s find the direction of this circle;

$$2x+2y\cdot y'-2+2y'=0$$

$$x+y\cdot y'+y'=0 \Rightarrow y'(y+1)=1-x\,,\;\; y'=\left.\frac{1-x}{1+y}\right|_A=0$$

$$\mathrm{tg}\alpha=-0 \Rightarrow \alpha=0, \cos\alpha=1;$$

$$\tan\alpha=0, \;\Rightarrow\; \alpha=\pi, \;\; (\textit{we take positive direction of the curve}), \;\; \cos\alpha=-1;$$

$\cos\beta=\sin\alpha=0;\;\; \cos\gamma=0$ (because tangent line is perpendicular to *OZ*-axis)

Calculate partial derivative of the function $\varphi(x,y,z)=xz^2+2yz$.

$$\frac{\partial\varphi}{\partial x}=z^2;\;\frac{\partial\varphi}{\partial y}=2z;\;\frac{\partial\varphi}{\partial z}=2zx+2y$$

$$\vec{\nabla}\varphi_A=\left.\left\{\frac{\partial\varphi}{\partial x};\frac{\partial\varphi}{\partial y};\frac{\partial\varphi}{\partial z}\right\}\right._A=\{4;4;4\}$$

$$\frac{\partial\varphi}{\partial l}=\left(\vec{\nabla}\varphi,\vec{l}^{\,0}\right)=-4\times1+4\times0+4\times0=-4$$