

## **2.8. The Linear Differential Equation of the Second Order with Constant Coefficients and Special Right Part of the first type.**

Let us consider the inhomogeneous differential equation of the second order

$$y'' + p(x)y' + q(x)y = f(x). \quad (1)$$

**Theorem 3.** The general solution of INHLDE (1) is equal to the sum of general solution of corresponding homogeneous differential equation  $\bar{y}$  and some particular solution  $y^*$  of this equation .That's

$$y = \bar{y} + y^* \quad (1a)$$

**Proof of this theorem is presented in Lecture or textbook "L.V.Kurpa, O.S.Mazur, T.V.Shmatko. Differential equations and Series"**

**Conclusion.** *In order to find the general solution of the inhomogeneous linear differential equation (INHLDE) we should find the general solution of corresponding homogeneous differential equation and add some particular solution of the given INHLDE*

Let' s begin with solution of *the Linear Differential Equation of the Second Order with Constant Coefficients and Special Right Part*

Let us consider the differential equation

$$y'' + py' + qy = f(x), \quad (2)$$

where  $p = \text{const}$  and  $q = \text{const}$ . Since the general solution  $\bar{y}$  of the corresponding homogeneous equation we can find at once, then the integration of this equation is reduced to finding any particular

solution  $y^*(x)$  of this equation. It may be done enough simple for some types of the right part.

**I. The Right Side of the Differential Equation of the First Kind.** Let the right part have the following form:

$$f(x) = e^{\alpha x} P_n(x), \quad (3)$$

where  $P_n(x)$  is polynomial of  $n$ -th power. For example polynomial of the **zero power** is just constant  $P_0(x) = A$ ,

Polynomial of the **first power** is defined as

$$P_1(x) = Ax + B.$$

Polynomial of the **second power** is

$$P_2(x) = Ax^2 + Bx + C, \quad \text{and so on.}$$

There are possible three cases for function (3).

1) *Number  $\alpha$  is not root of the characteristic equation of the homogeneous linear differential equation*

$$\lambda^2 + p\lambda + q = 0.$$

In this case the particular solution should search for in the form

$$y^* = e^{\alpha x} Q_n(x), \quad (3a)$$

where  $Q_n(x)$  is unknown polynomial of the  $n$ -th power.

**Example.** Consider the equation

$$y'' + y = 4xe^x. \quad (4)$$

**Solution.** The general solution of the (4) is

$$y = \bar{y} + y^*$$

In order to find the general solution of the homogeneous equation  $\bar{y}$ , let us construct the characteristic equation

$$\lambda^2 + 1 = 0.$$

Its roots are  $\lambda_{1,2} = \pm i$ , therefore the general solution of the HLDE has the following form:

$$\bar{y} = C_1 \cos x + C_2 \sin x.$$

Let's find the partial solution of the initial differential equation  $y^*$ .

The right side of the given equation has the form

$$f(x) = e^{1 \cdot x} P_1(x).$$

In the given case  $\alpha = 1$ . Since  $\lambda_1 \neq 1$  and  $\lambda_2 \neq 1$ , then we will search for the solution  $y^*(x)$  in the form

$$y^* = e^{1 \cdot x} Q_1(x),$$

where  $Q_1(x)$  is polynomial of the first power, that's  $Q_1(x) = Ax + B$ .  
and

$$y^* = (Ax + B)e^x. \quad (5)$$

In order to find the unknown coefficients  $A$  and  $B$ , we should calculate derivatives of the first and second order of function (5), substitute into initial equation (4) and equate the coefficients at the same degree of the polynomials of the right and left sides.

Whence

$$(y^*)' = Ae^x + (Ax + B)e^x, \quad (6)$$

$$(y^*)'' = 2Ae^x + (Ax + B)e^x. \quad (7)$$

After substitution the obtained expressions (5),(6),(7) into equation (4) and reducing by  $e^x$  we obtain

$$2A + Ax + B + Ax + B = 4x.$$

$$2Ax + 2A + B = 4x$$

You can see that coefficient before  $x$  in the left hand side is  $2A$ , and in the right hand side is just  $2$ . Coefficient before zero power of the polynomial in the left hand side is  $2A+B$ , and in right hand side the free term is absent, this means that coefficient is equal to zero.

Thus, we go to the following system

$$\begin{cases} 2A = 4 \\ 2A + 2B = 0 \end{cases},$$

whence  $A = 2$ ,  $B = -2$ , it means that

$$y^* = (2x - 2)e^x.$$

Therefore the general solution has the following form:

$$y = C_1 \cos x + C_2 \sin x + 2(x - 1)e^x.$$

**2) Let number  $\alpha$  be simple root of the characteristic equation. This means that  $\alpha = \lambda_1, \alpha \neq \lambda_2$  or  $\alpha = \lambda_2, \alpha \neq \lambda_1$**

In the considering case we have to take as particular solution the following function:

$$y^* = e^{\alpha x} x Q_n(x). \quad (8)$$

**Example.** Find the general solution of the equation:

$$y'' + y' - 2y = e^x. \quad (9)$$

**Solution.** The general solution of the (9) is

$$y = \bar{y} + y^* \quad (10)$$

In order to find the general solution of the homogeneous equation  $\bar{y}$ , let us construct the characteristic equation of the corresponding homogeneous differential equation

$$\lambda^2 + \lambda - 2 = 0$$

This equations has the following roots  $\lambda_1 = 1$ ,  $\lambda_2 = -2$  and hence the general solution of an appropriate homogeneous equation can be written in the following form:

$$\bar{y} = C_1 e^x + C_2 e^{-2x}. \quad (11)$$

Let's find the partial solution of the initial differential equation  $y^*$ .

So, since here function

$$f(x) = e^{1 \cdot x} P_0(x), \quad (P_0 = 1)$$

has the special form of the first type and here  $\alpha = 1$   $\lambda_1 \neq \lambda_2$ , but  $\alpha = \lambda_1$  then we can put

$$y^* = e^{1 \cdot x} Q_0(x),$$

Where  $Q_0(x)$  is polynomial of the zero power and its general form is defined just as  $Q_0(x) = A$ . So partial solution we can search for as

$$y^* = Ae^x x. \quad (12)$$

Whence we can find derivative of the function (12)

$$(y^*)' = Ae^x x + Ae^x, \quad (y^*)'' = Ae^x x + 2Ae^x. \quad (13)$$

Substituting these expressions (12,13) into equation (9) and reducing by  $e^x$  we obtain

$$Ax + 2A + Ax + A - 2Ax = 1,$$

that is  $3A = 1$ , whence  $A = \frac{1}{3}$  and hence

$$y^* = \frac{1}{3} e^x x.$$

The general solution (10) is presented in form:

$$y = C_1 e^x + C_2 e^{-2x} + \frac{1}{3} e^x x.$$

**3) The number  $\alpha$  is multiple (double) root of the characteristic equation** ( $\lambda_1 = \lambda_2 = \alpha$ ), then partial solution  $y^*$  is defined as

$$y^* = e^{\alpha x} Q_n(x) x^2 \quad . \quad (14)$$

**Remark.** If the right side of the equation (1) is sum of some terms  $f(x) = f_1(x) + f_2(x)$  of the kind  $e^{\alpha x} P_n(x)$ , then  $y^* = y_1^* + y_2^*$ ,

Where  $y_1^*, y_2^*$  are relatively partial solutions of the equations

$$y'' + py' + qy = f_1(x), \quad (15)$$

$$y'' + py' + qy = f_2(x) \quad . \quad (16)$$

**Example.** Find the general solution of the equation

$$y'' - 4y' + 4y = e^{2x} - 4. \quad (17)$$

**Solution.** The general solution is

$$y = \bar{y} + y_1^* + y_2^*, \quad (18)$$

Because the right part of the equation (17) is sum, of two terms, each of them is function of the special type of the first kind.

First let's find the general solution of the homogeneous differential equation. Construct the characteristic equation

$$\lambda^2 - 4\lambda + 4 = 0,$$

whence  $\lambda_1 = \lambda_2 = 2$  and following general solution of HLDE:

$$\bar{y} = C_1 e^{2x} + C_2 e^{2x} x. \quad (19)$$

Now form the partial solution for differential equations

$$y'' - 4y' + 4y = e^{2x} \quad (20)$$

You can see that here  $\alpha = 2 = \lambda_1 = \lambda_2$ . The degree of the polynomial  $P(x)$  is equal to zero. So the particular solution for equation (20)  $y_1^*$  is defined as

$$y_1^* = Ae^{2x} x^2 \quad (21)$$

Particular solution for equation

$$y'' - 4y' + 4y = -4, \quad (22)$$

is defined as  $y_2^* = B$ , because in this case  $\alpha = 0, \alpha \neq \lambda_1 \neq \lambda_2$ . And we have polynomial  $P(x)$  of the zero degree in formula (22). That's we have the first case for construction of the partial solution. Therefore

$$y^* = Ae^{2x} x^2 + B.$$

Whence

$$\begin{aligned} (y^*)' &= 2Ae^{2x} x^2 + 2Ae^{2x} x, \\ (y^*)'' &= 4Ae^{2x} x^2 + 8Ae^{2x} x + 2Ae^{2x}. \end{aligned}$$

Let us substitute these expressions into initial equation (here we cannot reduce by  $e^{2x}$ ). As result we get

$$4Ae^{2x}x^2 + 8Ae^{2x}x + 2Ae^{2x} - 8Ae^{2x}x^2 - 8Ae^{2x}x + 4Ae^{2x}x^2 + 4B = e^{2x} - 4,$$

i.e.

$$2Ae^{2x} + 4B = e^{2x} - 4.$$

Whence

$$\begin{cases} 2A = 1 \\ 4B = -4 \end{cases},$$

that is  $A = \frac{1}{2}$ ,  $B = -1$ . It means that

$$y^* = \frac{1}{2}e^{2x}x^2 - 1.$$

Then the general solution we can write down in the following form:

$$y = C_1e^{2x} + C_2e^{-2x} + \frac{1}{2}e^{2x}x^2 - 1.$$

**Example .** Find the general solution of the given differential equation:

$$y'' - y = x^3e^{2x}.$$

**Solution.**  $P_n(x) = x^3 \Rightarrow n = 3; \alpha = 2$ .

Let us consider the corresponding homogeneous equation

$$y'' - y = 0.$$

The characteristic equation is  $\lambda^2 - 1 = 0$ , the roots of this equation are  $\lambda_1 = 1, \lambda_2 = -1$ .

Thus the value  $\alpha = 2$  is not a root of the characteristic equation.

The particular solution of LIDE is searched for as follows:

$$v(x) = (A_0 + A_1x + A_2x^2 + A_3x^3)e^{2x}.$$

Putting the function  $v(x)$  and its derivative  $v'(x)$  into the given differential equation we obtain the following identity:

$$2A_2 + 6A_3x + 4(A_1 + 2A_2x + 3A_3x^2) + 4(A_0 + A_1x + A_2x^2 + A_3x^3) - (A_0 + A_1x + A_2x^2 + A_3x^3) = x^3.$$

Equating coefficients at the same powers of  $x$ , we obtain the system of 4 equations:

$$\begin{array}{l|l} x^3 & 3A_3 = 1, \\ x^2 & 12A_3 + 3A_2 = 0, \\ x^1 & 6A_3 + 8A_2 + 3A_1 = 0, \\ x^0 & 2A_2 + 4A_1 + 3A_0 = 0. \end{array}$$

**Example 2.** Find the general solution of LIDE

$$y'' + 2y' - 3y = 8xe^x.$$

**Solution.**  $P_n(x) = 8x \Rightarrow n = 1; \alpha = 1$ .

Let us consider the corresponding LHDE

$$y'' + 2y' - 3y = 0.$$

The characteristic equation is  $\lambda^2 + 2\lambda - 3 = 0$ , the roots of the equation are  $\lambda_1 = 1, \lambda_2 = -3$ .

Thus the value  $\alpha = 1$  is a simple real root of the characteristic equation.

The particular solution is searched for as follows:

$$y^*(x) = x(A_1 + A_0x)e^x$$

Let us put the function  $v(x)$  and its derivatives  $y^{*'}(x)$  and  $y^{*''}(x)$  into the given differential equation and obtain the following identity:

$$2A_0 + 8A_0x + 4A_1 = 8x.$$

From here

$$\begin{array}{l|l} x^1 & 8A_0 = 8, \\ x^0 & 2A_0 + 4A_1 = 0. \end{array}$$

$$A_0 = 1, A_1 = -\frac{1}{2}.$$

Then  $y^*(x) = \left(x^2 - \frac{1}{2}x\right)e^x$ . The general solution of the given differential equation is:

$$y(x) = c_1e^x + c_2xe^{-3x} + \left(x^2 - \frac{1}{2}x\right)e^x.$$

**Examples.** Now you can train to construct the particular solution for equation

$y'' - 3y' + 2y = f(x)$ , if  $f(x)$  is presented by the different expressions. Let's find  $\bar{y}$ . Characteristic equation is

$$\lambda^2 - 3\lambda + 2 = 0, \Rightarrow \lambda_1 = 1; \lambda_2 = 2,$$

$$\bar{y} = C_1e^x + C_2e^{2x}$$

If

1)  $f(x) = 10e^{-x}$ ,  $y^* = Ae^{-x}$ ;

2)  $f(x) = 3e^{2x}$ ,  $y^* = Ax e^{2x}$

3)  $f(x) = 2x^3 - 30$ ,  $y^* = Ax^3 + Bx^2 + Cx + D$

4)  $f(x) = x + e^{-2x} + 1$ ,  $f_1(x) = x + 1$ ;  $f_2(x) = e^{-2x}$ ,  
 $y_1^* = Ax + B$ ,  $y_2^* = Ce^{-2x}$

5)  $f(x) = e^x(3 - 4x)$ ,  $y^* = x e^{2x}(Ax + B)$

### Self-Service Examples

<b>Task</b>	<b>Answers</b>
<i>Find the general solution of the equations:</i>	
<b>10.1.</b> $2y'' + y' - y = 2e^x$ .	$y = C_1 e^{-x} + C_2 e^{\frac{x}{2}} + e^x$ .
<b>10.3.</b> $y'' - 6y' + 9y = 2x^2 - x + 3$ .	$y = (C_1 + C_2 x)e^{3x} + \frac{2}{9}x^2 + \frac{5}{27}x + \frac{11}{27}$ .
<b>10.4.</b> $y'' + 4y' - 5y = 1$ .	$y = C_1 e^x + C_2 e^{-5x} - 0,2$ .
<i>Find the particular solution of the equations satisfying given initial conditions:</i>	
<b>10.6.</b> $4y'' + 16y' + 15y = 4e^{-\frac{3}{2}x}$ , $y(0) = 3$ , $y'(0) = -5,5$ .	$y = (1 + x)e^{-\frac{3}{2}x} + 2e^{-\frac{5}{2}x}$ .
<b>10.7.</b> $y'' - y' = 2(1 - x)$ , $y(0) = 1$ , $y'(0) = 1$ .	$y = e^x + x^2$ .
<b>10.11.</b> $y'' - 2y' + 2y = 2x$	$y = e^x(C_1 \cos x + C_2 \sin x) + x + 1$
<b>10.15.</b> $y'' - 2y' = e^x(x^2 + x - 3)$ , $y(0) = 2$ , $y'(0) = 2$ .	$y = e^x(e^x - x^2 - x + 1)$ .