GROUP I-219ia, KH-219 ae,219iae,219,iбe,219iвe

The Linear Differential Equation of the Second Order with Constant Coefficients and Special Right Part of the second type.

II. The Right Side of the Differential Equation of the Second Kind. Let the right side of the differential equation

$$y'' + p(x)y' + q(x)y = f(x) \qquad (1)$$

have the form

$$f(x) = e^{\alpha x}[P_n(x)\cos\beta x + Q_m(x)\sin\beta x], \qquad (2)$$

where $P_n(x)$ and $Q_m(x)$ are polynomials *n-th* and *m-th* degree relatively. It may be shown that in this case the form of the particular solution is determined as follows:

1) *If the number* $\alpha + \beta i$ ***is not a root of the characteristic equation***, then the particular solution of the equation (1) should be sought for in the form

$$y^* = e^{\alpha x}(U_s(x)\cos\beta x + V_s(x)\sin\beta x), \qquad (3)$$

where $U_s(x)$ and $V_s(x)$ are polynomials of the same degree equal to the highest degree of the polynomials $P_n(x)$ and $Q_m(x)$, that is $s = \max(n, m)$.

2) *If the number* $\alpha + \beta i$ *is a root of the characteristic equation*, then we write the particular solution in the form

$$y^* = x\, e^{\alpha x}(U_s(x)\cos\beta x + V_s(x)\sin\beta x). \qquad (4)$$

Note. Here in order to avoid mistakes we must remember that expressions (3) and (4) are retained when one of the polynomials

$P_n(x)$ or $Q_m(x)$ on the right side of equation (3) is identically zero; that is when the right side has the following form

$$f(x) = P_n(x)e^{\alpha x}\cos\beta x$$

or

$$f(x) = Q_m(x)e^{\alpha x}\sin\beta x.$$

Consider now an important *special case*. Let the right side of the second order LDE has the form:

$$f(x) = M\cos\beta x + N\sin\beta x, \tag{5}$$

where M and N are constants:

1) *if βi is not a root of the characteristic equation,* then the particular solution should be sought for in the form

$$y^* = A\cos\beta x + B\sin\beta x. \tag{6}$$

2) *if βi is a root of the characteristic equation,* then the particular solution should be sought for in the form

$$y^* = x(A\cos\beta x + B\sin\beta x). \tag{7}$$

We remark that the function (5) is a special case of the right side of the second kind $(P(x) = M,\ Q(x) = N,\ \alpha = 0)$, the functions (6) and (7) are special cases of the function (3) and (4).

Example. Find the general solution of the INHLDE.

$$y'' + 2y' + 5y = 2\cos x. \tag{8}$$

Solution. The general solution of the (8) is

$$y = \bar{y} + y^*$$

In order to find the general solution of the *homogeneous equation* $\bar{y}$, let us construct the characteristic equation

$$\lambda^2 + 2\lambda + 5 = 0$$

and find roots of the square equation

$$\lambda_1 = -1 \pm 2i.$$

The general solution of *homogenous equation* has the following form

$$\bar{y} = e^{-x}\left(c_1 \cos 2x + c_2 \sin 2x\right).$$

Let's analyze the right part of the given equation (8) and compare with function (2). There is only polynomial *P(x)* of the zero power $P(x) = 2$, polynomial $Q(x) = 0$. So *max(m,n)= 0* . Parameter $\alpha = 0$, and parameter $\beta = 1$. Construct number $\alpha + \beta i = 0 + i = i$. Number i is not a root of the characteristic equation, therefore we will seek for the particular solution in the form:

$$y^* = A\cos x + B\sin x,$$

where A and B are unknown constant coefficients, which should be determined. Putting y^* into the given equation we get

$$-A\cos x - B\sin x + 2(-A\sin x + B\cos x) +$$
$$+5(A\cos x + B\sin x) = 2\cos x.$$

Equating the coefficients of $\cos x$ and $\sin x$ we obtain two equations for determining A and B:

$$-A + 2B + 5A = 2,$$
$$-B - 2A + 5B = 0.$$

Whence $A = \frac{2}{5}$; $B = \frac{1}{5}$. The general solution of the given equation is $y = \bar{y} + y^*$, that is,

$$y = e^{-x}\left(c_1 \cos 2x + c_2 \sin 2x\right) + \frac{2}{5}\cos x + \frac{1}{5}\sin x.$$

Example 2. Find the general solution of the given INHLIDE

$$y'' + y = x\cos x + \sin x. \qquad (9)$$

Solution. Let us consider the corresponding homogeneous differential equation

$$y'' + y = 0.$$

Characteristic equation is

$$\lambda^2 + 1 = 0,\ \lambda_1 = i,\ \lambda_2 = -i. \qquad (9a)$$

The general solution of the homogeneous differential equation $\bar{y}$ is defined as:

$$\bar{y} = C_1 \cos x + C_2 \sin x, \qquad (10)$$

The value $\alpha + i\beta = i$ is a root of the characteristic equation. Let 's analyse the right part of differential equation (9)

$$f(x) = x\cos x + \sin x,$$

$$\Rightarrow P_1(x) = x, \quad Q_0(x) = 1; \Rightarrow \max(n, m) = 1; \quad \alpha = 0, \beta = 1.$$

Form the expression $\alpha \pm \beta i = i$ and compare with roots of the characteristic equation. You can see that this expression coincides with roots of characteristic equation (9a). So the particular solution we will search, applying formula (4), that's as follows

$$y^*(x) = x\big((Ax + B)\cos x + (Cx + D)\sin x\big).$$

Putting the function $y^*(x)$ and its derivatives $y^{*/}(x)$ and $y^{*//}(x)$ into the given differential equation, we obtain the following identity

$$2C\sin x + 2A\cos x + (4Cx + 2D)\cos x - (4Ax + 2B)\sin x = x\cos x + \sin x.$$

From here
$$\begin{array}{r|l} \cos x & 2A + 2D = 0, \\ x\cos x & 4C = 1, \\ \sin x & 2C - 2B = 1, \\ x\sin x & -4A = 0. \end{array}$$

$$A = 0, \; D = 0, C = \frac{1}{4}, \; B = -\frac{1}{4}.$$

Then, $y^*(x) = \frac{1}{4}x^2 \sin x - \frac{1}{4}x\cos x$. The general solution of the differential equation is:

$$y(x) = c_1 \cos x + c_2 \sin x + \frac{1}{4}x^2 \sin x - \frac{1}{4}x\cos x.$$

<u>Self-Service Examples</u>

10.2. $y'' - 7y' + 6y = \sin x$.	$y = C_1 e^{6x} + C_2 e^x + \dfrac{5\sin x + 7\cos x}{74}$.
10.10. $y'' + 2y' + 5y = -\dfrac{7}{12}\cos 2x$	$y = e^{-x}(C_1 \cos 2x + C_2 \sin 2x) - \dfrac{1}{2}\cos 2x - 2\sin 2x$.
$y'' + 9y = \cos 3x$	

$y'' + y = \cos x \cos 2x$	