Lecture no 12: SYMMETRIC BILINEAR FORMS; LAW OF INERTIA [1]

This subsection treats symmetric bilinear forms and quadratic forms on vector spaces V over the real field **R**. The special nature of **R** permits an independent theory. The main result is as follows.

Theorem: Let f be a *symmetric form* on V over **R**. Then there exists a basis of V in which f is represented by a *diagonal matrix*. Every other diagonal matrix representation of f has the same number $\boldsymbol{p}$ of positive entries and the same number $\boldsymbol{n}$ of negative entries.

The above result is sometimes called the *Law of Inertia or Sylvester's Theorem*. The rank and signature of the symmetric bilinear form f are denoted and defined by

$$\operatorname{rank}(f) = \boldsymbol{p} + \boldsymbol{n} \text{ and } \operatorname{sig}(f) = \boldsymbol{p} - \boldsymbol{n}$$

These are uniquely defined by Theorem.

Definition: A real symmetric bilinear form f is said to be

(i) *positive definite* if $q(v) = f(v, v) > 0$ for every $v \neq 0$,

(ii) *nonnegative semidefinite* if $q(v) = f(v, v) \geq 0$ for every v.

Example: Let f be the dot product on $\mathbf{R}^n$. Recall that f is a *symmetric bilinear form* on $\mathbf{R}^n$. We note that f is also *positive definite*. That is, for any $u = (a_i) \neq 0$ in $\mathbf{R}^n$,

$$f(u, u) = a_1^2 + a_2^2 + \ldots + a_n^2 > 0$$

Corollary: Any real quadratic form q has a unique representation in the form

$$q(x_1, x_2, \ldots, x_n) = x_1 + \ldots + x_{\boldsymbol{p}} - x_{\boldsymbol{p}+1} - \ldots - x_r$$

where $r = \boldsymbol{p} + \boldsymbol{n}$ is the rank of the form.

Corollary: (Alternative Form) Any real symmetric matrix A is congruent to the *unique diagonal matrix*

$$D = \operatorname{diag}(I_{\boldsymbol{p}}; -I_{\boldsymbol{n}}; 0)$$

where $r = \boldsymbol{p} + \boldsymbol{n}$ is the rank of A.

Hermitian Forms

1

12	Л	2	Закон інерції квадратичних форм. Класифікація квадратичних форм. Критерій Сильвестра знаковизначеності квадратичної форми. Додаток теорії квадратичних форм до дослідження кривих і поверхонь другого порядку.
13	ПР	2	Приведення квадратичної форми до канонічного вигляду ортогональним перетворенням змінних.

Definition: Let V be a vector space of finite dimension over the complex field **C**. A *Hermitian form* on V is a mapping $f : V \times V \to \mathbf{C}$ such that, for all $a, b \in \mathbf{C}$ and all $u_i, v \in V$,

(i) $f(au_1 + bu_2, \ v) = af(u_1, v) + bf(u_2, v)$,

(ii) $f(u, v) = \overline{f(v, u)}$.

Using (i) and (ii), we get

$$f(u, \ av_1 + bv_2) = \overline{f(av_1 + bv_2, \ u)} = \overline{af(v_1, u) + bf(v_2, u)}$$
$$= \overline{af(v_1, u)} + \overline{bf(v_2, u)} = \bar{a}f(u, v_1) + \bar{b}f(u, v_2)$$

That is,

(iii) $f(u, \ av_1 + bv_2) = \bar{a}f(u, v_1) + \bar{b}f(u, v_2)$.

As before, we express condition (i) by saying f is linear in the first variable. On the other hand, we express condition (iii) by saying f is "*conjugate linear*" in the second variable. Moreover, condition (ii) tells us that $f(u, u) = \overline{f(u, u)}$ and hence, $f(u, u)$ is real for every $v \in V$.

Now suppose $S = \{u_1, ..., u_n\}$ is a basis of V. The matrix $H = [h_{ij}]$ where $h_{ij} = f(u_i, u_j)$ is called the *matrix representation* of f in the basis S. By (ii) $f(u_i, u_i) = \overline{f(u_i, u_i)}$, hence, H is *Hermitian* and, in particular, the diagonal entries of H are real. Thus, any diagonal representation of f contains only real entries.

The next theorem is the complex analog of theorem on real symmetric bilinear forms.

Theorem: Let f be a Hermitian form on V over **C**. Then there exists a basis of V in which f is represented by a *diagonal matrix*. Every other diagonal matrix representation of f has the same number $\boldsymbol{p}$ of positive entries and the same number $\boldsymbol{n}$ of negative entries.

Again the *rank* and *signature* of the Hermitian form f are denoted and defined by

$$\operatorname{rank}(f) = \boldsymbol{p} + \boldsymbol{n} \text{ and } \operatorname{sig}(f) = \boldsymbol{p} - \boldsymbol{n}$$

These are uniquely defined by Theorem.

Analogously, a Hermitian form f is said to be

(i) *positive definite* if $q(v) = f(v, v) > 0$ for every $v \neq 0$,

(ii) *nonnegative semidefinite* if $q(v) = f(v, v) \geq 0$ for every v.

Example: Let f be the dot product on $\mathbf{C}^n$, that is, for any $u = (z_i)$ and $v = (w_i)$ in $\mathbf{C}^n$,

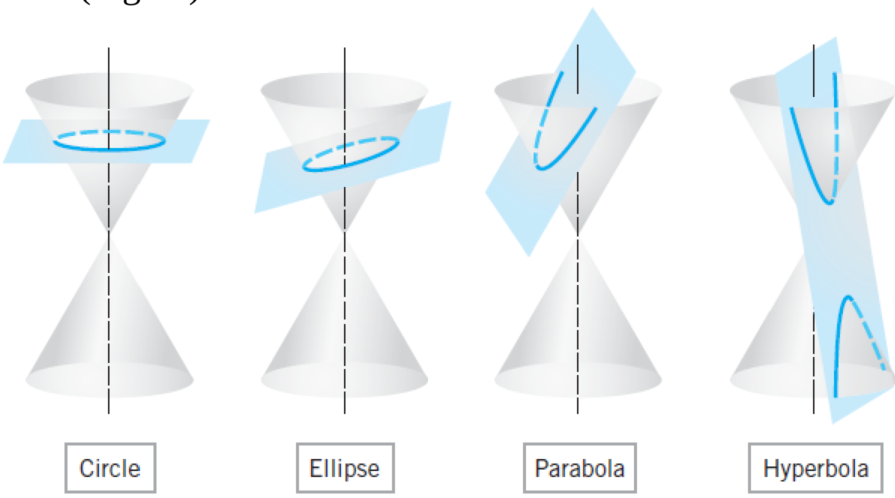
$$f(u, v) = u \cdot v = z_1\bar{w}_1 + z_2\bar{w}_2 + \cdots + z_n\bar{w}_n$$

Then f is a *Hermitian form* on $\mathbf{C}^n$. Moreover, f is also *positive definite*, because, for any $u = (z_i) \neq 0$ in $\mathbf{C}^n$,

$$f(u, u) = z_1\bar{z}_1 + z_2\bar{z}_2 + \cdots + z_n\bar{z}_n = |z_1|^2 + |z_2|^2 + \ldots + |z_n|^2 > 0$$

2. Quadratic Forms in Geometry

Recall that a *conic section* or *conic* is a curve that results by cutting a double-napped cone with a plane (Figure).



The most important conic sections are *ellipses*, *hyperbolas*, and *parabolas*, which result when the cutting plane does not pass through the vertex. Circles are special cases of ellipses that result when the cutting plane is perpendicular to the axis of symmetry of the cone. If the cutting plane passes through the vertex, then the resulting intersection is called a *degenerate conic*. The possibilities are a point, a pair of intersecting lines, or a single line.

Quadratic forms in $\mathbf{R}^2$ arise naturally in the study of conic sections. For example, it is shown in analytic geometry that an equation of the form:

$$ax^2 + 2bxy + cy^2 + dx + ey + f = 0$$

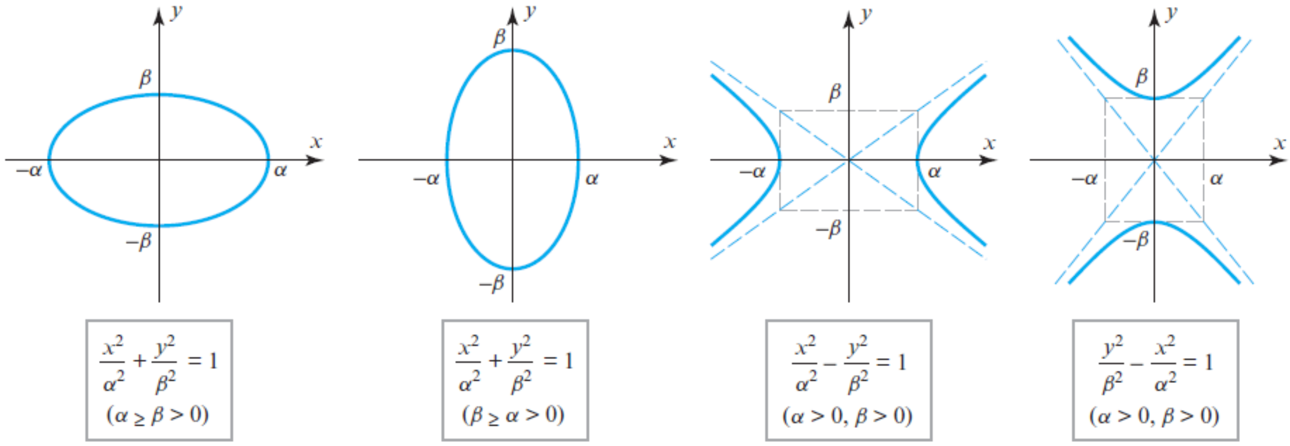
in which a, b, and c are not all zero, represents a conic section. If $d = e = 0$, then there are no linear terms, so the equation becomes

$$ax^2 + 2bxy + cy^2 + f = 0$$

and is said to represent a *central conic*. These include circles, ellipses, and hyperbolas, but not parabolas. Furthermore, if $b = 0$, then there is no cross product term (i.e., term involving xy), and the equation

$$ax^2 + cy^2 + f = 0$$

is said to represent a *central conic in standard position*. The most important conics of this type are shown in Figures:



If we take the constant f in Equations above to the right side and let $k = -f$, then we can rewrite these equations in matrix form as

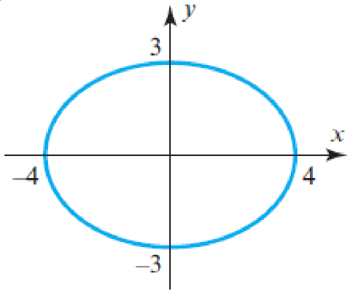
$$[x \quad y]\begin{bmatrix} a & b \\ b & c \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = k \quad \text{and} \quad [x \quad y]\begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = k$$

The *three-dimensional* analogs of these equations are

$$[x \quad y \quad z]\begin{bmatrix} a & d & e \\ d & b & f \\ e & f & c \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = k \quad \text{and} \quad [x \quad y \quad z]\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = k$$

If a, b, and c are not all zero, then the graphs in $\mathbf{R}^3$ of the equations are called *central quadrics*; the graph of the second of these equations, which is a special case of the first, is called a *central quadric in standard position*.

Example, the equation $9x^2 + 16y^2 - 144 = 0$ can be rewritten as $x^2/16 + y^2/9 = 1$, which is the ellipse shown in Figure



If a central conic is *rotated out of standard position*, then it can be identified by first rotating the coordinate axes to put it in standard position and then matching the resulting equation with one of the standard forms.

To find a rotation that eliminates the cross product term in the equation

$$ax^2 + 2bxy + cy^2 = k$$

it will be convenient to express the equation in the matrix form

$$\mathbf{x}^T A\mathbf{x} = [x \quad y]\begin{bmatrix} a & b \\ b & c \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = k$$

and look for a change of variable $\boxed{\mathbf{x} = P\mathbf{x}'}$ that *diagonalizes* A and for which det(P) = 1. The transition matrix

$$P = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

has the effect of rotating the xy-axes of a rectangular coordinate system through an angle θ, our problem reduces to finding θ that *diagonalizes* A, thereby eliminating the cross product term. If we make this change of variable, then in the $x'y'$-coordinate system, Equation above will become

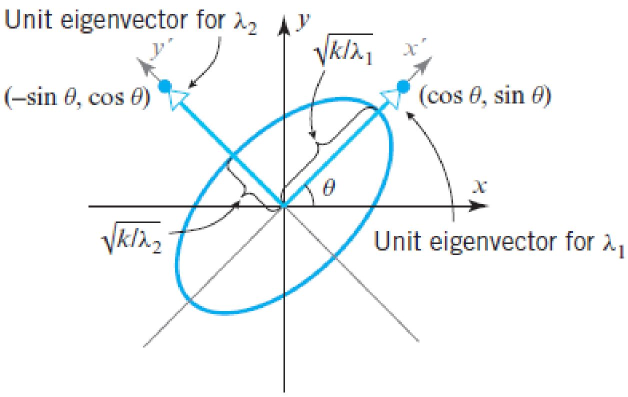
$$\boldsymbol{x}'^T D\boldsymbol{x}' = [x'y']\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix} = k$$

where λ_1 and λ_2 are the eigenvalues of A. The conic can now be identified by writing in the canonocal form

$$\lambda_1 x'^2 + \lambda_2 y'^2 = k$$

and performing the necessary algebra to match it with one of the standard forms.

The first column vector of P, which is a unit eigenvector corresponding to λ_1, is along the positive x'-axis; and the second column vector of P, which is a unit eigenvector corresponding to λ_2, is a unit vector along the y'-axis. These are called the *principal axes* of the ellipse.



Example: Identifying a conic by eliminating the cross product term.

(a) Identify the conic whose equation is $5x^2 - 4xy + 8y^2 - 36 = 0$ by rotating the xy-axes to put the conic in standard position.

(b) Find the angle θ through which you rotated the xy-axes in part (a).

Solution (a) The given equation can be written in the matrix form

$$\mathbf{x}^T A\mathbf{x} = 36$$

where

$$A = \begin{bmatrix} 5 & -2 \\ -2 & 8 \end{bmatrix}$$

The characteristic polynomial of A is

$$\begin{vmatrix} \lambda - 5 & 2 \\ 2 & \lambda - 8 \end{vmatrix} = (\lambda - 4)(\lambda - 9)$$

so the eigenvalues are $\lambda = 4$ and $\lambda = 9$. We leave it for you to show that orthonormal bases for the eigenspaces are

$$\lambda = 4: \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}, \quad \lambda = 9: \begin{bmatrix} -\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}$$

Thus, A is orthogonally diagonalized by

$$P = \begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}$$

Moreover, it happens by chance that $\det(P) = 1$, so we are assured that the substitution $\boldsymbol{x} = P\mathbf{x}'$ performs a rotation of axes. It follows from P that the equation of the conic in the $x'y'$-coordinate system is

$$[x' \quad y']\begin{bmatrix}4 & 0\\ 0 & 9\end{bmatrix}\begin{bmatrix}x'\\ y'\end{bmatrix} = 36$$

which we can write as

$$4x'^2 + 9y'^2 = 36 \quad \text{or} \quad \frac{x'^2}{9} + \frac{y'^2}{4} = 1$$

We can now see from Table 1 that the conic is an ellipse whose axis has length $2\alpha = 6$ in the x'-direction and length $2\beta = 4$ in the y'-direction.

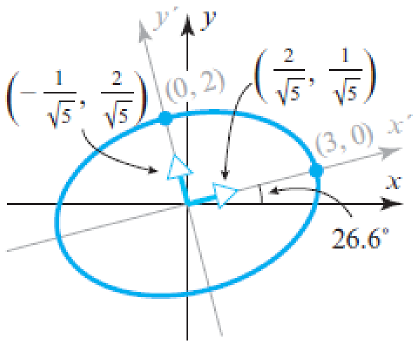
Solution (b) It follows from previous that

$$P = \begin{bmatrix}\frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}}\\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}}\end{bmatrix} = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$$

which implies that

$$\cos\theta = \frac{2}{\sqrt{5}}, \quad \sin\theta = \frac{1}{\sqrt{5}}, \quad \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{1}{2}$$

Thus, $\theta = \tan^{-1}\frac{1}{2} \approx 26.6°$



Positive Definite Quadratic Forms given in the matrix form

Definition: A quadratic form $\mathbf{x}^{\mathrm{T}}A\mathbf{x}$ is said to be:

positive definite if $\mathbf{x}^T A\mathbf{x} > 0$ for $\mathbf{x} \neq \mathbf{0}$;

negative definite if $\mathbf{x}^T A\mathbf{x} < 0$ for $\mathbf{x} \neq \mathbf{0}$;

indefinite if $\mathbf{x}^T A\mathbf{x}$ has both positive and negative values.

The following theorem, whose proof is deferred to the end of the section, provides a way of using eigenvalues to determine whether a matrix A and its associated quadratic form $\mathbf{x}^{\mathrm{T}}A\mathbf{x}$ are positive definite, negative definite, or indefinite.

Theorem: If A is a symmetric matrix, then:

(a) $\mathbf{x}^T A\mathbf{x}$ *is positive definite if and only if all eigenvalues of* A *are positive.*

(b) $\mathbf{x}^T A\mathbf{x}$ *is negative definite if and only if all eigenvalues of* A *are negative.*

(c) $\mathbf{x}^T A\mathbf{x}$ *is indefinite if and only if* A *has at least one positive eigenvalue and at least one negative eigenvalue.*

Example: Identifying whether positive definite or negative definite matrix is?

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$$

a) The matrix is indefinite since its eigenvalues are $\lambda = 1, 4, -2$.
b) Another way, let us write out the quadratic form as

$$\mathbf{x}^T A\mathbf{x} = [x_1 \quad x_2 \quad x_3] \begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 3x_1^2 + 2x_1x_2 + 2x_1x_3 + 4x_2x_3$$

We can now see, for example, that

$$\mathbf{x}^T A\mathbf{x} = 4 \quad \text{for} \quad x_1 = 0, \quad x_2 = 1, \quad x_3 = 1$$

and

$$\mathbf{x}^T A\mathbf{x} = -4 \quad \text{for} \quad x_1 = 0, \quad x_2 = 1, \quad x_3 = -1$$

If $\mathbf{x}^T A\mathbf{x} = k$ is the equation of a conic, and if $k \neq 0$, then we can divide through by k and rewrite the equation in the form $\mathbf{x}^T A\mathbf{x} = 1$. If A is a symmetric 2×2 matrix, then:

- $\mathbf{x}^T A\mathbf{x} = 1$ represents an ellipse if $\lambda_1 > 0$ and $\lambda_2 > 0$.
- $\mathbf{x}^T A\mathbf{x} = 1$ has no graph if $\lambda_1 < 0$ and $\lambda_2 < 0$.
- $\mathbf{x}^T A\mathbf{x} = 1$ represents a hyperbola if λ_1 and λ_2 have opposite signs.

In previous Example we performed a rotation to show that the equation

$$5x^2 - 4xy + 8y^2 - 36 = 0$$

represents an ellipse with a major axis of length 6 and a minor axis of length 4. This conclusion can also be obtained by rewriting the equation in the form

$$5/36x^2 - 1/9xy + 2/9\, y^2 = 1,$$

where the associated matrix is

$$A = \begin{bmatrix} \frac{5}{36} & -\frac{1}{18} \\ -\frac{1}{18} & \frac{2}{9} \end{bmatrix}$$

has eigenvalues $\lambda_1 = 1/9$ and $\lambda_2 = 1/4$. These eigenvalues are positive, so the matrix A is positive definite and the equation represents an ellipse. Moreover, the axes of the ellipse have lengths $2/\sqrt{\lambda_1} = 6$ and $2/\sqrt{\lambda_2} = 4$.

Identifying Positive Definite Matrices (Sylwester's Criterion)

We already know that a symmetric matrix is positive definite if and only if its eigenvalues are all positive; now we will give a criterion that can be used to determine whether a symmetric matrix is positive definite without finding the eigenvalues. For this purpose, we define the kth principal submatrix of an $n \times n$ matrix A to be the $k \times k$ submatrix consisting of the first k rows and columns of A. For example, here are the principal submatrices of a general 4×4 matrix:

$$\left[\begin{array}{c|ccc} a_{11} & a_{12} & a_{13} & a_{14} \\ \hline a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{array}\right] \quad \left[\begin{array}{cc|cc} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{array}\right] \quad \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ \hline a_{41} & a_{42} & a_{43} & a_{44} \end{array}\right] \quad \left[\begin{array}{cccc} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{array}\right]$$

First principal submatrix | Second principal submatrix | Third principal submatrix | Fourth principal submatrix $= A$

The following theorem provides a determinant test for ascertaining whether a symmetric matrix is positive definite.

Theorem: (*Sylwester's Criterion*) If A is a symmetric matrix, then:

(a) A is *positive definite* if and only if the determinant of every principal submatrix (*leading principal minors*) is *positive*.

(b) A is *negative definite* if and only if the determinants of the principal submatrices alternate between negative and positive values starting with a *negative value* for the determinant of the first principal submatrix.

(c) A is *indefinite* if and only if it is neither positive definite nor negative definite and at least one principal submatrix has a positive determinant and at least one has a negative determinant.

Example: Working with Principal Submatrices

The matrix

$$A = \begin{bmatrix} 2 & -1 & -3 \\ -1 & 2 & 4 \\ -3 & 4 & 9 \end{bmatrix}$$

is positive definite since the determinants

$$|2| = 2, \quad \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3, \quad \begin{vmatrix} 2 & -1 & -3 \\ -1 & 2 & 4 \\ -3 & 4 & 9 \end{vmatrix} = 1$$

are all positive. Thus, we are guaranteed that all eigenvalues of A are positive and $\mathbf{x}^T A\mathbf{x} > 0$ for $\mathbf{x} \neq \mathbf{0}$. ◀