Three Particular Cases for Plane Equations

Case 1 Suppose we know one point $M_0(x_0, y_0, z_0)$ of the plane and any two uncollinear vectors $\vec{a}, \vec{b}$ parallel to this plane (Fig.32*a*).

In this case we have a point and to get equation of the plane we should just find the normal vector. But

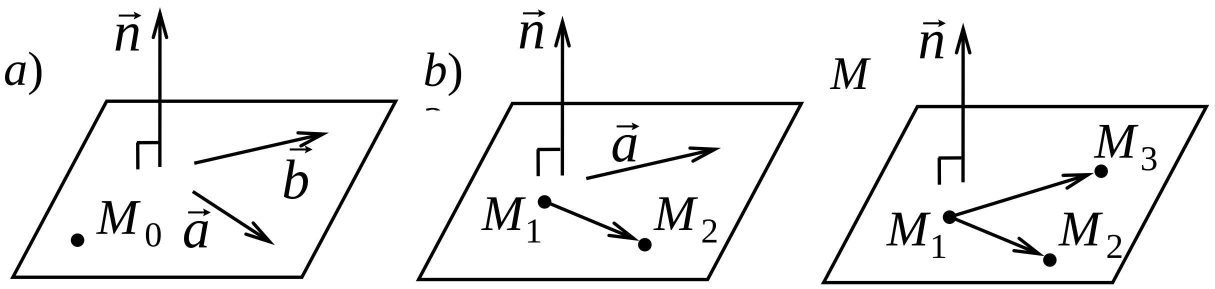

Figure 32

$$\begin{cases} \vec{n} \perp \vec{a} \\ \vec{n} \perp \vec{b} \end{cases} \Rightarrow \vec{n} \parallel \vec{a} \times \vec{b}.$$

Therefore as normal vector we can choose the vector

$$\vec{n} = \lambda \vec{a} \times \vec{b},$$

where $\lambda \in R$, $\lambda \neq 0$.

Case 2 Suppose we know two points $M_1(x_1, y_1, z_1)$ and $M_2(x_2, y_2, z_2)$ of the plane and a vector $\vec{a}$ parallel to this plane and uncollinear to the vector $\overrightarrow{M_1M_2}$ (Fig.32*b*).

In this case we have a point but do not have a normal vector. Since

$$\begin{cases} \vec{n} \perp \vec{a} \\ \vec{n} \perp \overrightarrow{M_1M_2} \end{cases} \Rightarrow \vec{n} \parallel \vec{a} \times \overrightarrow{M_1M_2}.$$

Therefore as normal vector we can choose the vector

$$\vec{n} = \lambda \vec{a} \times \overrightarrow{M_1M_2},$$

where $\lambda \in R$, $\lambda \neq 0$.

Case 3 Suppose we know three points $M_1(x_1, y_1, z_1)$, $M_2(x_2, y_2, z_2)$ and $M_2(x_2, y_2, z_2)$ of the plane such that vectors $\overrightarrow{M_1M_2}$ and $\overrightarrow{M_1M_3}$ are uncollinear (Fig.32*c*).

Since

$$\begin{cases} \vec{n} \perp \overrightarrow{M_1M_2} \\ \vec{n} \perp \overrightarrow{M_1M_3} \end{cases} \Rightarrow \vec{n} \parallel \overrightarrow{M_1M_2} \times \overrightarrow{M_1M_3}.$$

Therefore as normal vector we can choose the vector

$$\vec{n} = \lambda \overrightarrow{M_1M_2} \times \overrightarrow{M_1M_3},$$

where $\lambda \in R$, $\lambda \neq 0$. Then

$$\vec{n} = \lambda \overrightarrow{M_1M_2} \times \overrightarrow{M_1M_3} = \lambda \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = (A, B, C).$$

Thus, the equation of the plane passing through three given points has the form

$$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0 \text{ or}$$

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0.$$

Note. The last equation of the plane can be obtained directly from the condition of complanarity for vectors $\overrightarrow{M_1M_2}$, $\overrightarrow{M_1M_3}$, $\overrightarrow{M_1M}$, where point $M(x,y,z)$ is an arbitrary point of this plane.

Example. Find equation of the plane passing through the points $M_1(1,2,-1)$, $M_2(0,3,0)$ and $M_2(2,-1,1)$.

From the last formula we have

$$\begin{vmatrix} x-1 & y-2 & z+1 \\ 0-1 & 3-2 & 0+1 \\ 2-1 & -1-2 & 1+1 \end{vmatrix} = \begin{vmatrix} x-1 & y-2 & z+1 \\ -1 & 1 & 1 \\ 1 & -3 & 2 \end{vmatrix} =$$

$$= 5(x-1) + 3(y-2) + 2(z+1) = 5x + 3y + 2z - 9 = 0.$$

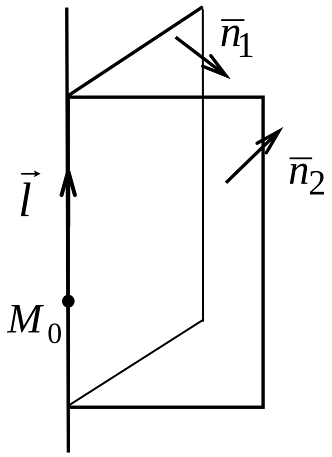

Figure 33

General Equation of Straight Line in Space

If two planes are not parallel then they intersect and a line of their intersection is called the straight line (Fig.33).

From definition it follows that all points of straight line satisfy the following system of equations

$$\begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0 \end{cases}$$

This system is called *the general equation of the straight line*.

Definition. Vector $\vec{l}$ parallel to the straight line is called the direction (or directing) vector of this straight line.

Vector $\vec{l}\,(m, n, p)$ determines *an orientation* of this straight line.

Since there are several straight lines with the same orientation, to describe straight line in the unique way we should determine any point $M_0(x_0, y_0, z_0)$ of this straight line (Fig.33). This point determines *a location* of this straight line.

There is a question: how to find the coordinates of point $M_0(x_0, y_0, z_0)$ and the direction vector $\vec{l}\,(m, n, p)$ from the general equation of the straight line?

Since two planes are not parallel their normal vectors are not collinear. Therefore

$$\begin{cases} \vec{l} \perp \vec{n}_1 \\ \vec{l} \perp \vec{n}_2 \end{cases} \Rightarrow \vec{l} = \lambda \vec{n}_1 \times \vec{n}_2 .$$

Moreover, at least one of equalities $\dfrac{A_1}{A_2} = \dfrac{B_1}{B_2}$, $\dfrac{A_1}{A_2} = \dfrac{C_1}{C_2}$, $\dfrac{B_1}{B_2} = \dfrac{C_1}{C_2}$ is false, that is at least one of numbers $\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}$ $\begin{vmatrix} A_1 & C_1 \\ A_2 & C_2 \end{vmatrix}$ $\begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix}$ is not 0 and therefore rank of the matrix $\begin{pmatrix} A_1 & \mathrm{B}_1 & C_1 \\ A_2 & B_2 & C_2 \end{pmatrix}$ is equal to 2 and the system $\begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0 \end{cases}$ has infinite number of solutions.

So we can evaluate coordinates of point M_0 by assigning to one of coordinates some constant value and evaluating other coordinates.

Example. Find the direction vector and one point of the straight line

$$\begin{cases} 2x - y + 2z = 1 \\ -x + y + 3z = 4 \end{cases}.$$

Here

$$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 2 \\ -1 & 1 & 3 \end{vmatrix} = -5\vec{i} - 8\vec{j} + \vec{k} = (-5,-8,1).$$

Then as direction vector we can take, for example, the vector $\vec{l} = -\vec{n}_1 \times \vec{n}_2 = (5,8,-1)$.

Let us write down the matrix of the system:

$$\begin{pmatrix} 2 & -1 & 2 \\ -1 & 1 & 3 \end{pmatrix}.$$

Since $\begin{vmatrix} 2 & 2 \\ -1 & 3 \end{vmatrix} \neq 0$ then variable y is free.

$$y := 4: \quad \begin{cases} 2x + 2z = 5 \\ -x + 3z = 0 \end{cases} \Leftrightarrow \begin{matrix} x = 3z \\ 6z + 2z = 5 \end{matrix} \Leftrightarrow z = \frac{5}{8},\ x = \frac{15}{8}.$$

Thus we obtained the point $M_0\left(\frac{15}{8}, 4, \frac{5}{8}\right)$.

Canonical Equations of Straight Line

Suppose we know the coordinates of the point $M_0(x_0, y_0, z_0)$ and the direction vector $\vec{l}\,(m,n,p)$ of the straight line. Then for any point $M(x,y,z)$ of this straight line we have (Fig.34):

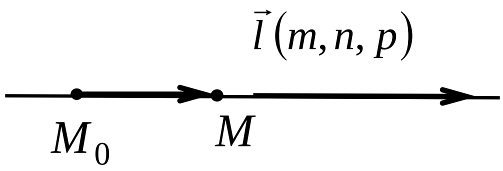

Figure 34

$$\overrightarrow{M_0M} \parallel \vec{l} \Leftrightarrow \boxed{\frac{x - x_0}{m} = \frac{y - y_0}{n} = \frac{z - z_0}{p}}$$

These equations are called *the canonical equations of the straight line*.

Note. One or two coordinates of the direction vector could be equal to zero. Canonical equations just show the proportionality of the coordinates of collinear vectors.

Example. Find the canonical equations of the straight line passing through the point $M_0(1;-3;0)$ and perpendicular to the vectors $\vec{a}(1,0,-3)$ and $\vec{b}(2,1,5)$.

The direction vector of this straight line is perpendicular to these vectors as well and therefore

$$\vec{l} = \lambda \vec{a} \times \vec{b} .$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -3 \\ 2 & 1 & 5 \end{vmatrix} = 3\vec{i} - 11\vec{j} + \vec{k} = (3,-11,1).$$

Let $\vec{l} = 1 \cdot \vec{a} \times \vec{b} = (3,-11,1)$. Then the canonical equations of this straight line are

$$\frac{x-1}{3} = \frac{y-(-3)}{-11} = \frac{z-0}{1} \text{ or } \frac{x-1}{3} = \frac{y+3}{-11} = \frac{z}{1}.$$

Equation of Straight Line Passing through Two Points

Suppose we know the coordinates of two points $M_1(x_1, y_1, z_1)$ and $M_2(x_2, y_2, z_2)$ of the given straight line. Then

$$\vec{l} = (m,n,p) = \overrightarrow{M_1M_2} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$$

and the canonical equations of this straight line have the following form

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}.$$

Example. Find the canonical equations of the straight line passing through the points $M_1(1;-3;0)$ and $M_2(2;-1;4)$. By the given formula we have

$$\frac{x-1}{2-1} = \frac{y-(-3)}{-1-(-3)} = \frac{z-0}{4-0} \text{ or } \frac{x-1}{1} = \frac{y+3}{2} = \frac{z}{4}.$$

Parametric Equations of Straight Line

From the canonical equations of straight line It follows that for any point of the straight line three values are equal to the same value (value of some parameter).

Let us denote this value as *t*. Then

$$\frac{x-x_0}{m}=\frac{y-y_0}{n}=\frac{z-z_0}{p}=t\in R \quad \Leftrightarrow \quad \begin{cases} x=mt+x_0 \\ y=nt+y_0 \quad t\in R \\ z=pt+z_0 \end{cases}$$

The last equations are called *the parametric equations of the straight line*.

Note. Parameter *t* plays a role of continuous index by means of which all points of straight line are numbered.

Example. Find the canonical equations of the straight line passing through the point $M_0(1;2;1)$ and parallel to the vector $\vec{a}(1,0,-3)$. Vector $\vec{a}$ can be taken as the direction vector of this straight line and therefore we have

$$\begin{cases} x=1t+1=t+1 \\ y=0t+2=2 \qquad t\in R. \\ z=-3t+1 \end{cases}$$

Distance from Point to Straight Line

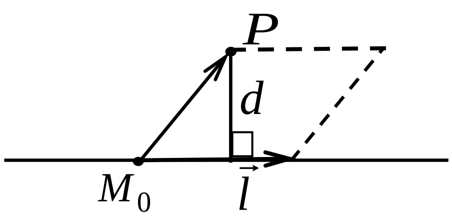

Figure 35

To find distance from the point $P(x_P, y_P, z_P)$ to the straight line It is enough to find the altitude of the parallelogram constructed on vectors $\vec{l}$ and $\overrightarrow{M_0P}$, where $\vec{l}=(m,n,p)$ is the direction vector of the straight line, $M_0(x_0, y_0, z_0)$ is any point of this straight line (Fig.35). Therefore

$$d=\frac{\left|\vec{l}\times\overrightarrow{M_0P}\right|}{\left|\vec{l}\right|}.$$

Example. Find the distance from the origin to the straight line $\frac{x-1}{1}=\frac{y+3}{2}=\frac{z}{4}$. Here

$$P(0,0,0),\ \vec{l}=(1,2,4),\ M_0(1,-3,0),\ \overrightarrow{M_0P}=(-1,3,0),$$

$$\left|\vec{l}\right| = \sqrt{1^2+2^2+4^2} = \sqrt{1+4+16} = \sqrt{21},$$

$$\vec{l}\times\overrightarrow{M_0P} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 4 \\ -1 & 3 & 0 \end{vmatrix} = -12\vec{i} - 4\vec{j} + 5\vec{k} = (-12,-4,5),$$

$$\left|\vec{l}\times\overrightarrow{M_0P}\right| = \sqrt{(-12)^2+(-4)^2+5^2} = \sqrt{144+16+25} = \sqrt{185}.$$

Thus

$$d = \frac{\left|\vec{l}\times\overrightarrow{M_0P}\right|}{\left|\vec{l}\right|} = \sqrt{\frac{185}{21}}.$$

Positional Relationship of Straight Lines in Space

Definition. An angle between two straight lines is the angle between their direction vectors.

As It is shown on Fig.36

$$\cos\alpha = \frac{\left(\vec{l}_1,\vec{l}_2\right)}{\left|\vec{l}_1\right|\left|\vec{l}_2\right|}$$

then

$$\alpha = \arccos\frac{\left(\vec{l}_1,\vec{l}_2\right)}{\left|\vec{l}_1\right|\left|\vec{l}_2\right|}.$$

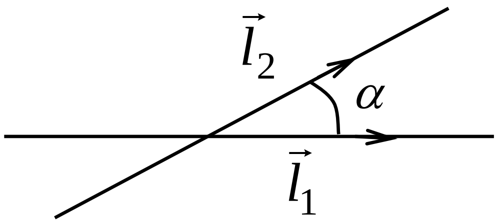

Fig.36

Moreover, for any two straight lines L_1, L_2 we have:

$$L_1 \parallel L_2 \Leftrightarrow \vec{l}_1 \parallel \vec{l}_2 \Leftrightarrow \frac{m_1}{m_2} = \frac{n_1}{n_2} = \frac{p_1}{p_2};$$

$$L_1 \perp L_2 \Leftrightarrow \vec{l}_1 \perp \vec{l}_2 \Leftrightarrow m_1m_2 + n_1n_2 + p_1p_2 = 0.$$

Note. The angle between two straight lines does not give to us full information about positional relationship between these straight lines.

We have three different situations, namely:

a) crossing straight lines (Fig.37*a*);

b) parallel or coinciding straight lines (Fig.37*b*);

c) skew straight lines (Fig.37*c*).

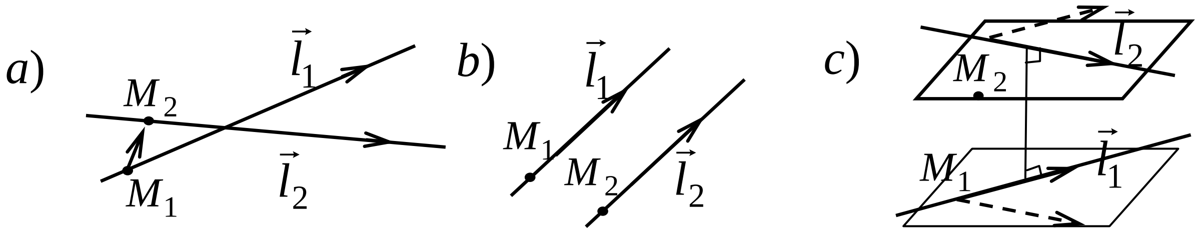

Figure 37

In cases *a*)-*b*) these straight lines lie in one plane, in case c) they lie in different planes.

It is simple to check that:

1) Straight lines lie in the same plane if and only if their direction vectors $\vec{l}_1, \vec{l}_2$ and vector $\overrightarrow{M_1M_2}$ connecting two different points of these straight lines are complanar, i.e.

$$\left(\vec{l}_1, \vec{l}_2, \overline{M_1M_2}\right) = 0 .$$

2) Two parallel lines coincide if and only if at least one point of one straight line belongs to other one.

It follows from discussed above that:

$$\left(\vec{l}_1, \vec{l}_2, \overline{M_1M_2}\right) \neq 0 \;\Leftrightarrow\; \text{straight lines are skew straight lines;}$$

$$\left(\vec{l}_1, \vec{l}_2, \overline{M_1M_2}\right) = 0 \Leftrightarrow \begin{cases} \text{straight lines are crossing, if } \ \vec{l}_1 \times \vec{l}_2 \neq 0 \\ \text{straight lines are parallel, if } \ \vec{l}_1 \times \vec{l}_2 = 0 \end{cases}$$

Example. Determine the positional relationship of two straight lines $\frac{x}{1} = \frac{y}{2} = \frac{z-1}{2}$ and $\frac{x-1}{-1} = \frac{y-3}{2} = \frac{z}{2}$. Since the direction vectors of these straight lines $\vec{l}_1(1,2,2)$, $\vec{l}_2(-1,2,2)$ are not collinear, these straight lines are not parallel.

Let us find the angle between them and determine either they are intersecting straight lines or skew ones.

$$\alpha = \arccos\frac{\left(\vec{l}_1,\vec{l}_2\right)}{\left|\vec{l}_1\right|\left|\vec{l}_2\right|} = \arccos\frac{-1+4+4}{\sqrt{1+4+4}\sqrt{1+4+4}} = \arccos\frac{7}{9}.$$

Here

$$M_1(0;0;1),\ M_2(1;3;0),$$

$$\overrightarrow{M_1M_2} = (1,3,-1),$$

$$\left(\vec{l}_1,\vec{l}_2,\overline{M_1M_2}\right) = \begin{vmatrix} 1 & 2 & 2 \\ -1 & 2 & 2 \\ 1 & 3 & -1 \end{vmatrix} = -2+4-6-4-2-6 = -16 \neq 0.$$

Therefore, these straight lines are skew straight lines.

Distance between Two Straight Lines

Case 1. Suppose we have two crossing straight lines (Fig.37*a*). Then the distance between them is equal to zero, i.e.

$$d = 0.$$

Case 2. Suppose we have two parallel or coinciding straight lines (Fig.37*b*). Then the distance between them is equal to the distance from any point of one straight line to other straight line. Thus it can be calculated by formula:

$$d = \frac{\left|\overrightarrow{M_1M_2} \times \vec{l}_1\right|}{\left|\vec{l}_1\right|}.$$

Case 3. Suppose we have two skew straight lines (Fig.37*c*). Let us plot from the points M_1 and M_2 direction vectors $\vec{l}_1,\vec{l}_2$ of both straight lines as it shown on Fig. 37*c*. We have formed two parallel planes. Common perpendicular of straight lines is perpendicular to these planes. Therefore to find distance between two skew straight lines It is enough to find distance between two parallel planes or just the altitude of the parallelepiped constructed on vectors $\vec{l}_1,\vec{l}_2,\overrightarrow{M_1M_2}$. Thus

$$d=\frac{\left|\left(\vec{l}_1,\vec{l}_2,\overrightarrow{M_1M_2}\right)\right|}{\left|\vec{l}_1\times\vec{l}_2\right|}.$$

Note. If two straight lines intersect then the numerator of the fraction in the last formula is equal to zero, while the denominator is not equal to zero. Therefore, we can use the last formula to find distance between any two unparallel straight lines (either intersecting or skew ones).

Example. Find the distance between two straight lines

$$\frac{x}{1}=\frac{y}{2}=\frac{z-1}{2}\text{ and }\frac{x-1}{-1}=\frac{y-3}{2}=\frac{z}{2}.$$

Since the direction vectors of these straight lines $\vec{l}_1(1,2,2)$, $\vec{l}_2(-1,2,2)$ are not collinear, these straight lines are not parallel. So we should use the last formula to find distance. Here

$$M_1(0;0;1),\ M_2(1;3;0),\ \overrightarrow{M_1M_2}=(1,3,-1),$$

$$\vec{l}_1\times\vec{l}_2=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\1&2&2\\-1&2&2\end{vmatrix}=0\vec{i}-4\vec{j}+4\vec{k}=(0,-4,4),$$

$$\left|\vec{l}_1\times\vec{l}_2\right|=\sqrt{0^2+(-4)^2+4^2}=\sqrt{2\cdot16}=4\sqrt{2},$$

$$\left(\vec{l}_1,\vec{l}_2,\overrightarrow{M_1M_2}\right)=\left(\vec{l}_1\times\vec{l}_2,\overrightarrow{M_1M_2}\right)=0-12-4=-16.$$

Therefore we obtain

$$d=\frac{\left|\left(\vec{l}_1,\vec{l}_2,\overrightarrow{M_1M_2}\right)\right|}{\left|\vec{l}_1\times\vec{l}_2\right|}=\frac{16}{4\sqrt{2}}=2\sqrt{2}.$$