2.1. Vector Algebra

2.1.1. Vectors. Basic Definitions and Concepts

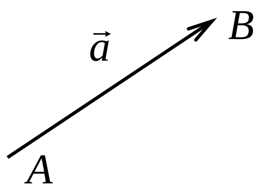

Figure 2

Definition. The vector is a directed segment (Fig.2).

Notations: $\vec{a}$ or $\bar{a}$ or $\overrightarrow{AB}$ or $\overline{AB}$.

Point *A* is called an origin of the vector.

Point *B* is called a terminus.

Definition. The distance between the origin and terminus is called a module or a length of this vector.

Notations: $|\vec{a}|$.

If the origin of the vector coincides with the terminus then $|\vec{a}| = 0$. Such a vector is called a zero-vector and denoted as $\vec{0}$ or just 0.

Definition. Two vectors $\vec{a}$ and $\vec{b}$ are called equal if they have the same module and the same direction.

From the last definition It follows that the vectors obtained one from another by parallel shift are equal.

Definition. Two vectors $\vec{a}$ and $\vec{b}$ are called collinear if they are parallel to the same straight line.

Definition. Three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are called coplanar (or complanar) if they are parallel to the same plane.

Definition. Vector of the unit length having the same direction with vector $\vec{a}$ is called the ort or the unit vector of the vector $\vec{a}$.

Notation: $\vec{a}^{\,o}$.

2.1.2. Linear Operations on Vectors

Linear operations on vectors are the multiplication of vector by scalar and the addition of the vectors.

Definition. The vector $\vec{b} = \lambda\vec{a}$ is called the multiplication of the vector $\vec{a}$ by scalar λ if:

1. $|\vec{b}| = |\lambda||\vec{a}|$;
2. $\vec{a}$ and $\vec{b}$ are collinear vectors;
3. $\vec{a}$ and $\vec{b}$ have the same direction for positive values of λ and the opposite directions for negative values of λ.

See examples on Fig.3.

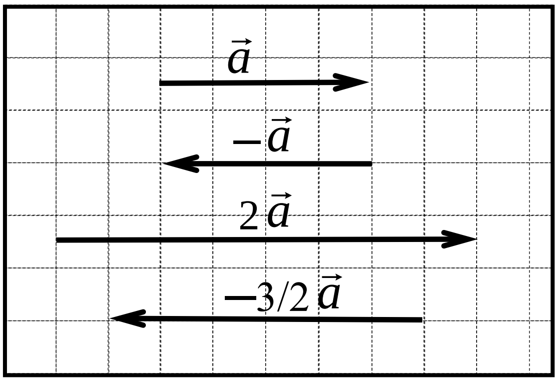

Figure 3

From definition It follows that two collinear vectors could be obtained one from another by multiplication by a scalar. So, we have the following *criterion of collinearity for two nonzero vectors*:

$$\vec{a} \parallel \vec{b} \Leftrightarrow \vec{a} = \lambda\vec{b} \Leftrightarrow \vec{b} = \mu\vec{a}, \quad \lambda, \mu \in R \setminus \{0\}$$

Definition. The sum of vectors $\vec{a}$ and $\vec{b}$ is called a vector $\vec{c} = \vec{a} + \vec{b}$ which origin coincides with the origin of $\vec{a}$ and terminus coincides with the terminus of $\vec{b}$ if the terminus of $\vec{a}$ and the origin of $\vec{b}$ are connected (Fig.4).

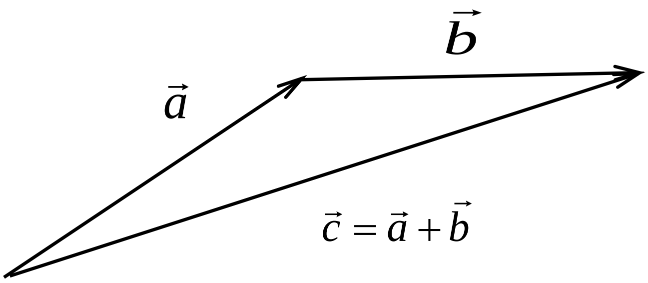

Figure 4. The rule of triangle

This rule to get sum is called the rule of triangle (Fig.4).

There is another rule to get sum called the rule of parallelogram (Fig.5). In this case you should construct a parallelogram on the vectors $\vec{a}$ and $\vec{b}$. The sum of the vectors coincides with the diagonal of this parallelogram directed from the origin of $\vec{a}$ to the terminus of $\vec{b}$.

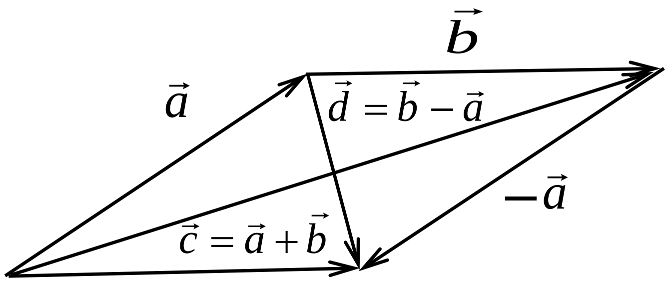

Figure 5. The rule of parallelogram

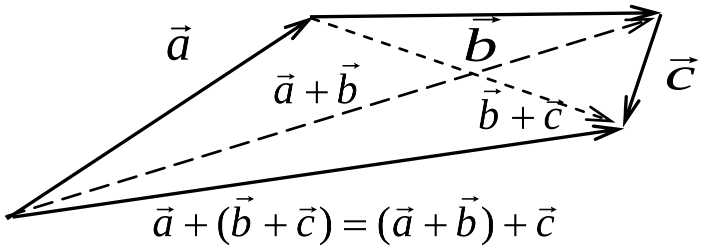

Figure 6

To get difference of vectors you should fulfill the following operations:

$$\vec{d} = \vec{b} - \vec{a} = \vec{b} + (-1)\vec{a} = \vec{b} + (-\vec{a}).$$

The difference of vectors coincides with the other diagonal of the parallelogram constructed on $\vec{a}$ and $\vec{b}$ (Fig.5). It is directed from minuend origin to subtrahend origin if their terminuses are connected.

Basic properties of linear operations:

1. $\vec{a} + \vec{b} = \vec{b} + \vec{a}$
2. $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$ (Fig.6)
3. $\lambda(\vec{a} + \vec{b}) = \lambda\vec{a} + \lambda\vec{b}$, $\lambda \in R$
4. $(\lambda + \mu)\vec{a} = \lambda\vec{a} + \mu\vec{a}$, $\lambda, \mu \in R$

Example. Let us find a vector $\vec{c}$ with direction coinciding with a bisector of an angle between the vectors $\vec{a}$ and $\vec{b}$ (Fig.7).

A diagonal bisects the angle of a parallelogram only if this parallelogram is a rhomb. That is why the vector $\vec{c}$ bisects an angle between two vectors only if their lengths are equal to each other.

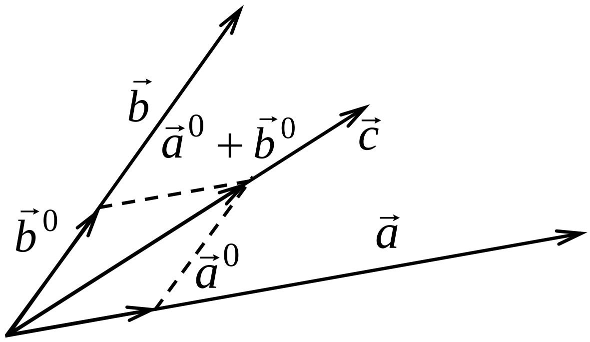

Figure 7

Let us consider the orts $\vec{a}^0$ and $\vec{b}^0$. Their lengths are equal to one and their directions coincide with directions of $\vec{a}$ and $\vec{b}$, relatively. Then the vector directed along the bisector of the angle between $\vec{a}$ and $\vec{b}$ has the same direction as $\vec{a}^0+\vec{b}^0$. Therefore,

$$\vec{c}=\lambda(\vec{a}^0+\vec{b}^0),\ \lambda>0.$$

Note, that there are several other ways to construct the vectors of the equal length. For example, we can find the bisector as

$$\vec{c}=\lambda\left(\vec{a}\cdot\left|\vec{b}\right|+\vec{b}\cdot\left|\vec{a}\right|\right),\ \lambda>0.$$

2.1.3. Concept of Linear Space

Definition. The set L of the elements $x, y, z, \ldots$ is called linear space (LS) if

I. There is an operation of multiplication of elements by scalar such that

$$\forall x\in L \Rightarrow \alpha x\in L\ \ \forall\alpha\in R;$$

II. There is an operation of addition such that

$$\forall x,y\in L \Rightarrow x+y\in L;$$

III. These operations satisfy 8 conditions:

1. $x+y=y+x \quad \forall x,y\in L;$
2. $(x+y)+z=x+(y+z) \quad \forall x,y,z\in L;$

3. $\exists 0 \in L: \; x + 0 = 0 + x = x \;\; \forall x \in L$;

4. $\forall x \in L \;\; \exists y = -x \in L: \; x + y = 0$;

5. $1 \cdot x = x \;\;\; \forall x \in L$;

6. $\lambda(x + y) = \lambda x + \lambda y \;\; \forall x, y \in L \; \forall \lambda \in R$;

7. $(\lambda + \mu)x = \lambda x + \mu x \;\;\; \forall x \in L \; \forall \lambda, \mu \in R$;

8. $(\lambda\mu)x = \lambda(\mu x) = \mu(\lambda x) \;\;\; \forall x \in L \; \forall \lambda, \mu \in R$.

Example 1. The set of continuous functions on segment $[a,b]$ is a linear space. Indeed, usual operations of addition and multiplication by number satisfy all conditions: the sum of continuous functions is continuous function, continuous function multiplied by a number is still continuous function, zero-element is zero function which is obviously continuous and so on.

Example 2. The set of real numbers is a linear space. But the set of integers is not *LS*, since, for example, multiplication of any integer by real number π makes this number not integer.

Example 3. The set of all matrices of the identical size is a linear space.

Example 4. The set of vectors is a linear space.

Below we are going to consider some properties of linear spaces only on the example of vector space since vectors are objects of our consideration. But these properties are the same for any linear space.

2.1.4. Concept of Basis. Decomposition of the Vector

Definition. The expression $\alpha_1\vec{a}_1 + \alpha_2\vec{a}_2 + \alpha_3\vec{a}_3 + \ldots + \alpha_n\vec{a}_n$, $\alpha_1, \alpha_2, \ldots \alpha_n \in R$ is called linear combination (LC) of the vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3, \ldots, \vec{a}_n$.

Definition. The vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3, \ldots, \vec{a}_n$ are called linearly independent (LI) if any their trivial (zero) linear combination has trivial coefficients, i.e.

$\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ are LI $\Leftrightarrow \left(\alpha_1\vec{a}_1 + \alpha_2\vec{a}_2 + \ldots + \alpha_n\vec{a}_n = 0 \;\;\; \Rightarrow \;\;\; \alpha_1 = \alpha_2 = \ldots = \alpha_n = 0\right)$.

In other case they are called linearly dependent (LD), i.e.

$\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ are LD $\Leftrightarrow \exists k : \alpha_k \neq 0$ and $\alpha_1\vec{a}_1 + \alpha_2\vec{a}_2 + \alpha_3\vec{a}_3 + \ldots + \alpha_n\vec{a}_n = 0$.

Theorem (Linear dependence of vectors) The vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3, \ldots, \vec{a}_n$ are linearly dependent if and only if one of these vectors is linear combination of other vectors.

Proof. Necessity. We know that vectors are linearly dependent. We should proof that one of them is linear combination of others. Suppose we have some zero linear combination of vectors. Then at least one coefficient of it is not equal to zero. Suppose it has number *k*, i.e. $\alpha_k \neq 0$. We divide zero expression by $-\alpha_k$ and express from obtained equation the vector $\vec{a}_k$:

$$\alpha_1\vec{a}_1 + \alpha_2\vec{a}_2 + \ldots + \alpha_k\vec{a}_k + \ldots + \alpha_n\vec{a}_n = 0 \Leftrightarrow$$

$$\Leftrightarrow -\frac{\alpha_1}{\alpha_k}\vec{a}_1 - \frac{\alpha_2}{\alpha_k}\vec{a}_2 - \ldots - \vec{a}_k - \ldots - \frac{\alpha_n}{\alpha_k}\vec{a}_n = 0 \Leftrightarrow$$

$$\Leftrightarrow \vec{a}_k = -\frac{\alpha_1}{\alpha_k}\vec{a}_1 - \frac{\alpha_2}{\alpha_k}\vec{a}_2 - \ldots - \frac{\alpha_{k-1}}{\alpha_k}\vec{a}_{k-1} - \frac{\alpha_{k+1}}{\alpha_k}\vec{a}_{k+1} - \ldots - \frac{\alpha_m}{\alpha_k}\vec{a}_m = 0 \Leftrightarrow$$

$$\Leftrightarrow \vec{a}_k = -\sum_{\substack{i=1 \\ i \neq k}}^{n} \frac{\alpha_i}{\alpha_k}\vec{a}_i .$$

Necessity is proven.

Sufficiency. Suppose $\vec{a}_k = \sum_{\substack{i=1 \\ i \neq k}}^{n} \gamma_i \vec{a}_i$. We should prove that vectors are linearly dependent. Let us put $\vec{a}_k$ to the right of the last equation. So we get

$$0 = \sum_{\substack{i=1 \\ i \neq k}}^{n} \gamma_i \vec{a}_i - \vec{a}_k ,$$

i.e. we have obtained zero linear combination of all vectors with the coefficient $\gamma_k = -1 \neq 0$. From definition of linear dependence it means that these vectors are LD. ***Theorem is proven.***

By means of this Theorem we will prove the following 3 statements:

Statement 1. Two vectors are linearly dependent if and only if they are collinear.

Proof. $\vec{a},\vec{b}$ are LD $\Leftrightarrow$ [by Theorem 1] $\Leftrightarrow \vec{a}=\alpha\vec{b} \Leftrightarrow \vec{a},\vec{b}$ are collinear.

Statement is proven.

Corollary. Two vectors are linearly independent if and only if they are not collinear.

Statement 2. Three vectors are linearly dependent if and only if they are coplanar.

Proof. Necessity. $\vec{a},\vec{b},\vec{c}$ are LD. By the Theorem 1 we get, for example, that $\vec{c}=\alpha\vec{a}+\beta\vec{b}$. Thus, $\vec{c}$ is a diagonal of the parallelogram constructed on $\alpha\vec{a}$ and $\beta\vec{b}$ and it belongs to the plane of this parallelogram as $\vec{a}$ and $\vec{b}$ do. So these vectors are coplanar.

To prove Sufficiency we need just to prove that for any three coplanar vectors one is linear combination of others.

Suppose $\vec{a},\vec{b}$ are collinear. Then

$$\vec{a}=\lambda\vec{b} \text{ or } \vec{a}=\lambda\vec{b}+0\cdot\vec{c},$$

so vectors are LD by Theorem 1. Suppose now that $\vec{a},\vec{b}$ are not collinear. Then, accordingly to the Fig.8,

$$\vec{c}=\overrightarrow{AD}=\overrightarrow{AB}+\overrightarrow{AC}=\lambda\vec{a}+\mu\vec{b},$$

since $ABDC$ is a parallelogram and $\overrightarrow{AB}\parallel\vec{a}$, $\overrightarrow{AC}\parallel\vec{b}$. ***Statement is proven.***

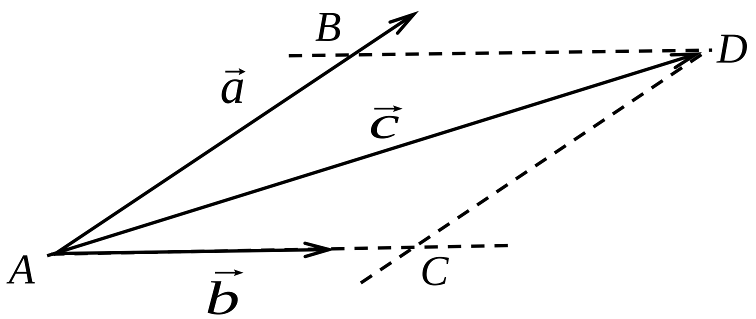

Figure 8

Statement 3. Any four vectors in space are linearly dependent.

Proof. Let us consider any four vectors in space. There are two cases.

Case 1. $\vec{a},\vec{b},\vec{c}$ are coplanar. Then they are LD, i.e. $\alpha\vec{a}+\beta\vec{b}+\gamma\vec{c}=\alpha\vec{a}+\beta\vec{b}+\gamma\vec{c}+0\cdot\vec{d}=0$ with not all zero coefficients. Thus $\vec{a},\vec{b},\vec{c},\vec{d}$ are LD as well.

Case 2. $\vec{a},\vec{b},\vec{c}$ are not coplanar. Then let us draw a straight line through the terminus of the vector $\vec{d}$ (point *F*) parallel to the vector $\vec{c}$ to find point *D* which is an intersection of constructed straight line and plane of the vectors $\vec{a},\vec{b}$ (Fig.9). Obtained vector $\overrightarrow{AD}$ is coplanar with not collinear vectors $\vec{a},\vec{b}$, i.e. it could be presented as their linear combination. In other case, vector $\overrightarrow{DF}=\overrightarrow{AE}$ is collinear to $\vec{c}$, i.e.

$$\overrightarrow{DF}=\lambda\vec{c},\ \lambda\neq 0.$$

So,

$$\vec{d}=\overrightarrow{AF}=\overrightarrow{AE}+\overrightarrow{AD}=\overrightarrow{AE}+\overrightarrow{AB}+\overrightarrow{AC}=\lambda\vec{c}+\alpha\vec{a}+\beta\vec{b},$$

where $\lambda\neq 0$. It means that these vectors are linearly dependent. ***Statement is proven.***

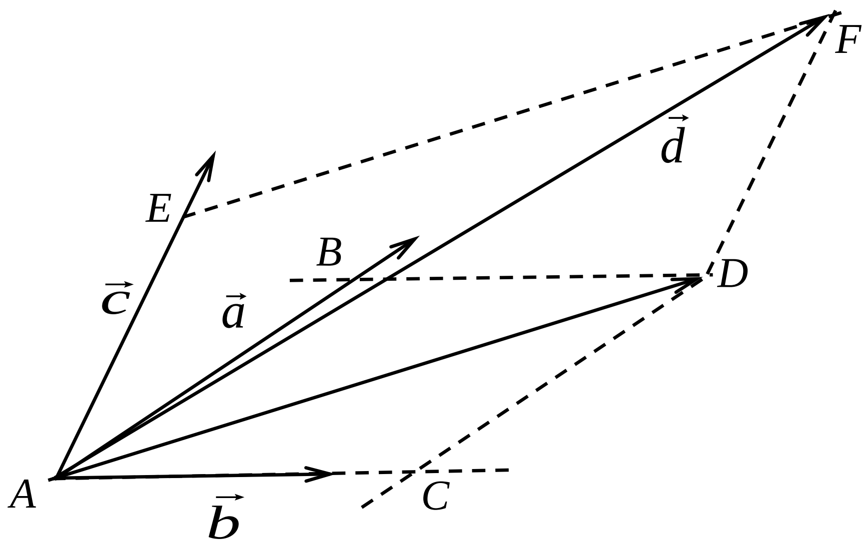

Figure 9

Definition. Linear space L is called n-dimensional if there are n linearly independent elements and any (n+1) are linearly dependent.

Definition. In n-dimensional linear space any n linearly independent elements are called basis of this space.

Note 1. There could be a lot of different bases in the same *LS*.

Note 2. From statements 1-3 follows that:

1. Plane is 2-dimensional *LS* and any two not collinear vectors form basis.
2. Space is 3-dimensional *LS* and any three not coplanar vectors form basis.

Definition. The basis is called orthogonal if every two vectors from basis are perpendicular to each other.

Definition. The basis is called orthonormal if it is orthogonal and the module of every vector is equal to 1.

Theorem (Decomposition of the vector in the basis) Any vector of n-dimensional linear space can be presented as linear combination of basic vectors and this presentation is unique.

Proof. Let us consider any basis of the n-dimensional linear space $\vec{e}_1, \vec{e}_2, \vec{e}_3, \ldots \vec{e}_n$ and an arbitrary vector $\vec{x}$. From definition of the n-dimensional linear space it follows that vectors $\vec{x}, \vec{e}_1, \vec{e}_2, \vec{e}_3, \ldots \vec{e}_n$ are linearly dependent, i.e. we have some zero linear combination of these vectors

$$\gamma_0 \vec{x} + \gamma_1 \vec{e}_1 + \gamma_2 \vec{e}_2 + \gamma_3 \vec{e}_3 + \ldots + \gamma_n \vec{e}_n = 0$$

with not all zero coefficients. There are two possible cases.

Case 1. $\gamma_0 = 0$. Then we found zero linear combination of the basis vectors with not all zero coefficients: $\gamma_1 \vec{e}_1 + \gamma_2 \vec{e}_2 + \gamma_3 \vec{e}_3 + \ldots + \gamma_n \vec{e}_n = 0$. It means that basis vectors are LD. We got a contradiction with definition of basis.

Case 2. $\gamma_0 \neq 0$. So we got the presentation of the vector through basis ones:

$$\vec{x} = -\frac{\gamma_1}{\gamma_0}\vec{e}_1 - \frac{\gamma_2}{\gamma_0}\vec{e}_2 - \frac{\gamma_3}{\gamma_0}\vec{e}_3 - \ldots - \frac{\gamma_n}{\gamma_0}\vec{e}_n.$$

Let us prove that this presentation is unique. We suppose the opposite statement, namely, that there are two different presentations:

$$\vec{x} = \alpha_1\vec{e}_1 + \alpha_2\vec{e}_2 + \alpha_3\vec{e}_3 + \ldots + \alpha_n\vec{e}_n \text{ and } \vec{x} = \alpha_1\vec{e}_1 + \alpha_2\vec{e}_2 + \alpha_3\vec{e}_3 + \ldots + \alpha_n\vec{e}_n.$$

After subtraction of two last equations one from another we have

$$0 = (\beta_1 - \alpha_1)\vec{e}_1 + (\beta_2 - \alpha_2)\vec{e}_2 + \ldots + (\beta_n - \alpha_n)\vec{e}_n.$$

Since the vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3, \ldots \vec{e}_n$ are linearly independent, we have

$$\begin{cases} \beta_1 - \alpha_1 = 0, \text{ i.e. } \beta_1 = \alpha_1; \\ \beta_2 - \alpha_2 = 0, \text{ i.e. } \beta_2 = \alpha_1; \\ \vdots \\ \beta_n - \alpha_n = 0, \text{ i.e. } \beta_n = \alpha_n. \end{cases}$$

So, any two presentations have the same coefficients, i.e. presentation is unique.***Theorem is proven.***

Definition. The presentation of the vector as linear combination of basic vectors is called the decomposition of the vector $\vec{x}$ in the basis $\vec{e}_1, \vec{e}_2, \vec{e}_3, \ldots \vec{e}_n$. At the same time the coefficients of this decomposition are called the coordinates of the vector $\vec{x}$ in this basis.

Note 1. If the basis is chosen, one can write only coordinates of vector instead of the whole decomposition, i.e. one can write that $\vec{x} = (\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n)$ instead of $\vec{x} = \alpha_1\vec{e}_1 + \alpha_2\vec{e}_2 + \alpha_3\vec{e}_3 + \ldots + \alpha_n\vec{e}_n$.

Note 2. Suppose we have 2 or 3-dimentional space. From Statements 2 and 3 and the last Theorem It follows the way to find decomposition of the vector in any chosen basis.

Note 3. Since each vector can be associated with row/column of its coordinates, linear dependence/independence of vectors coincides with linear dependence/independence of rows/columns. Thus, to check linear dependence of the vectors given by their coordinates It is enough and sufficient to check it for the rows/columns of their coordinates.

2.1.5. Linear Operations on Vectors Given by Their Coordinates in Some Basis

Suppose we consider some n-dimensional linear space with basis $\vec{e}_1,\vec{e}_2,\vec{e}_3,\ldots\vec{e}_n$ and vectors

$\vec{x}=\alpha_1\vec{e}_1+\alpha_2\vec{e}_2+\alpha_3\vec{e}_3+\ldots+\alpha_n\vec{e}_n=(\alpha_1,\alpha_2,\alpha_3,\ldots,\alpha_n)$,

$\vec{y}=\beta_1\vec{e}_1+\beta_2\vec{e}_2+\beta_3\vec{e}_3+\ldots+\beta_n\vec{e}_n=(\beta_1,\beta_2,\beta_3,\ldots,\beta_n)$.

Then

$$\lambda\vec{x}=\lambda(\alpha_1\vec{e}_1+\alpha_2\vec{e}_2+\ldots+\alpha_n\vec{e}_n)=\lambda\alpha_1\vec{e}_1+\lambda\alpha_2\vec{e}_2+\ldots+\lambda\alpha_n\vec{e}_n=(\lambda\alpha_1,\lambda\alpha_2,\lambda\alpha_3,\ldots,\lambda\alpha_n).$$

$$\vec{x}+\vec{y}=\alpha_1\vec{e}_1+\alpha_2\vec{e}_2+\alpha_3\vec{e}_3+\ldots+\alpha_n\vec{e}_n+\beta_1\vec{e}_1+\beta_2\vec{e}_2+\beta_3\vec{e}_3+\ldots+\beta_n\vec{e}_n=$$

$$=(\alpha_1+\beta_1)\vec{e}_1+(\alpha_2+\beta_2)\vec{e}_2+\ldots+(\alpha_n+\beta_n)\vec{e}_n=(\alpha_1+\beta_1,\alpha_2+\beta_2,\alpha_3+\beta_3,\ldots,\alpha_n+\beta_n).$$

Conclusions:

1. To multiply vector by scalar means to multiply all its coordinates by this scalar;
2. To add two vectors means to add their corresponding coordinates.

Example. Show that the vectors $\vec{a}_1,\vec{a}_2,\vec{a}_3$ form a basis and find decomposition $\vec{b}$ in this given basis if

$$\vec{a}_1=(1;-1;2),\vec{a}_2=(2;2;-1),\vec{a}_3=(2;1;0),\vec{b}=(3;7;-7).$$

To decompose the vector $\vec{b}$ in this basis $\vec{a}_1,\vec{a}_2,\vec{a}_3$ means to find the following its presentation

$$\vec{b}=\alpha_1\vec{a}_1+\alpha_2\vec{a}_2+\alpha_3\vec{a}_3.$$

The last equality is equivalent to the following:

$$(3;7;-7)=\alpha_1(1;-1;2)+\alpha_2(2;2;-1)+\alpha_3(2;1;0)=$$
$$=(\alpha_1+2\alpha_2+2\alpha_3;-\alpha_1+2\alpha_2+\alpha_3;2\alpha_1-\alpha_2)$$

or

$$\begin{cases}\alpha_1+2\alpha_2+2\alpha_3=3\\-\alpha_1+2\alpha_2+\alpha_3=7\\2\alpha_1-\alpha_2=-7\end{cases}.$$

Extended matrix of the obtained system is

$$\left(\begin{array}{ccc|c} 1 & 2 & 2 & 3 \\ -1 & 2 & 1 & 7 \\ 2 & -1 & 0 & -7 \end{array}\right).$$

Note, that the matrix of this system consists of the columns $\vec{a}_1, \vec{a}_2, \vec{a}_3$. If the vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3$ form the basis then they are linearly independent and a rank of the system matrix is equal to 3. Thus we can answer both questions of this example by solving this system. If the rank of system matrix is equal to 3 then the vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3$ form basis and solutions of the system are the coordinates of $\vec{b}$ in this basis.

Let us solve this inhomogeneous system of equations relatively to $\alpha_1, \alpha_2, \alpha_3$ by Jordan-Gauss method:

$\left(\begin{array}{ccc|c} 1 & 2 & 2 & 3 \\ -1 & 2 & 1 & 7 \\ 2 & -1 & 0 & -7 \end{array}\right)$~[We add multiplied by 1 to the second row and the first row multiplied by (−2) to the third one] $\sim \left(\begin{array}{ccc|c} 1 & 2 & 2 & 3 \\ 0 & 4 & 3 & 10 \\ 0 & -5 & -4 & -13 \end{array}\right)$~[We add the last row to the second one to get 1 in the second row]~

$\sim \left(\begin{array}{ccc|c} 1 & 2 & 2 & 3 \\ 0 & -1 & -1 & -3 \\ 0 & 4 & 3 & 10 \end{array}\right)$~[We add the multiplied by 2 to the first row and the second row multiplied by 4 to the third one]~$\left(\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & -1 & -2 \end{array}\right)$~[We add the third row to the second and multiply the third row by (−1)] $\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right)$.

The rank of the system matrix is equal to 3 and to the rank of the extended matrix. Thus, the vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3$ form the basis and the system is compatible. Here $\alpha_1 = -3; \alpha_2 = 1; \alpha_3 = 2.$

Therefore $\vec{b} = -3\vec{a}_1 + \vec{a}_2 + 2\vec{a}_3$ is a decomposition of the vector $\vec{b}$ in the basis $\vec{a}_1, \vec{a}_2, \vec{a}_3$.

2.1.7. Projection of the Vector on Axis

Let us consider an arbitrary vector $\overrightarrow{AB}$ and an axis with direction given by the vector $\vec{u}$ (Fig.10). To get points A^*, B^* we drop perpendiculars from the origin and terminus of the vector on the axis.

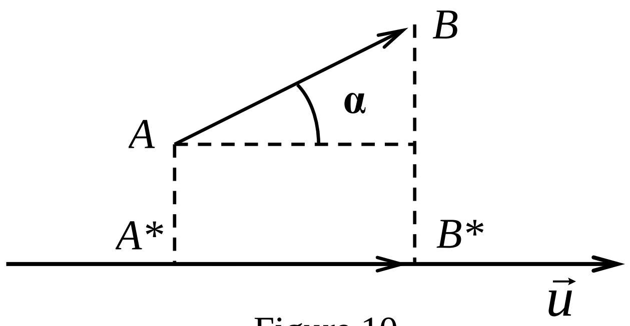

Figure 10

Definition. The length of the segment A^*B^* taken with sigh "+" if $\overrightarrow{A^*B^*}$ has the same direction with $\vec{u}$ or with the sigh "–" if $\overrightarrow{A^*B^*}$ has the opposite direction with $\vec{u}$ is called projection of the vector $\overrightarrow{AB}$ on $\vec{u}$ (or on the axis with direction $\vec{u}$). Notation: $pr_{\vec{u}}\overrightarrow{AB}$.

Note. From the definition and Fig.10 it follows that

$$pr_{\vec{u}}\overrightarrow{AB} = \left|\overrightarrow{A^*B^*}\right| = \left|\overrightarrow{AB}\right|\cos(\overrightarrow{AB},\hat{\ }\vec{u}\ ) = \left|\overrightarrow{AB}\right|\cos\alpha\,.$$

Properties of the projections:

1. $pr_{\vec u}\lambda\vec a = \lambda pr_{\vec u}\vec a$;
2. $pr_{\vec u}(\vec a+\vec b) = pr_{\vec u}\vec a + pr_{\vec u}\vec b$.
3. $pr_{\lambda\vec u}\vec a = pr_{\vec u}\vec a$ for positive λ;

   $pr_{\lambda\vec u}\vec a = -pr_{\vec u}\vec a$ for negative λ.

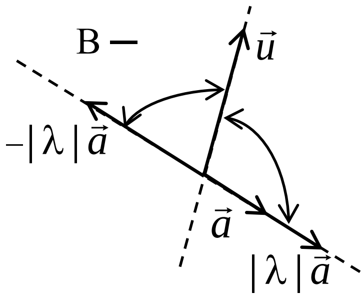

Figure 11

Proof. 1. Let $\alpha = (\hat{\vec a,\vec u})$ (Fig.11). Then

$$pr_{\vec u}\lambda\vec a = |\lambda\vec a|\cos(\hat{\lambda\vec a,\vec u}) = |\lambda||\vec a|\cos(\hat{\lambda\vec a,\vec u}) =$$

$$= \begin{cases} |\lambda||\vec a|\cos\alpha & if\ \ \lambda\ge 0 \\ |\lambda||\vec a|\cos(\pi-\alpha) & if\ \ \lambda<0 \end{cases} = \begin{cases} |\lambda||\vec a|\cos\alpha & if\ \ \lambda\ge 0 \\ -|\lambda||\vec a|\cos\alpha & if\ \ \lambda<0 \end{cases} = \lambda|\vec a|\cos\alpha = \lambda pr_{\vec u}\vec a\,.$$

2. Let us prove this property geometrically. There are six different cases (Fig.12).

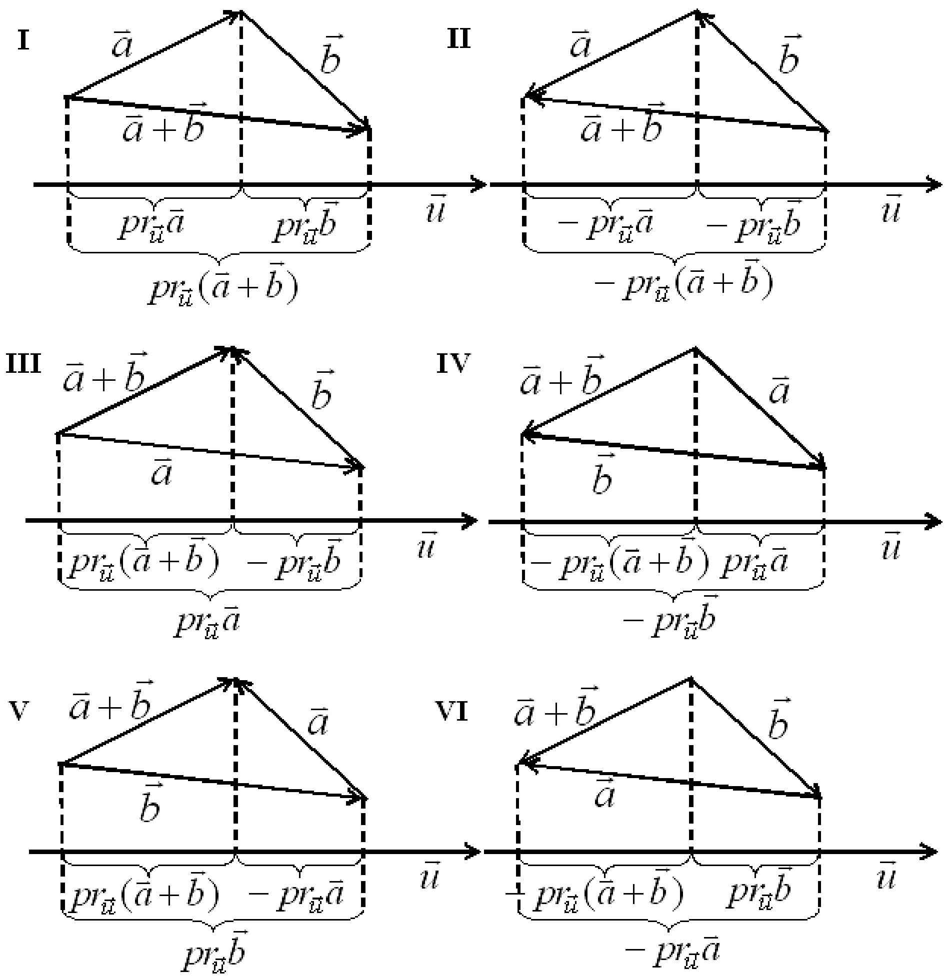

Figure 12

It follows from case I that

$$pr_{\vec{u}}(\vec{a}+\vec{b}) = pr_{\vec{u}}\vec{a} + pr_{\vec{u}}\vec{b}\ .$$

It follows from case II that

$$-pr_{\vec{u}}(\vec{a}+\vec{b}) = -pr_{\vec{u}}\vec{a} + -pr_{\vec{u}}\vec{b}\text{, i.e. } pr_{\vec{u}}(\vec{a}+\vec{b}) = pr_{\vec{u}}\vec{a} + pr_{\vec{u}}\vec{b}\ .$$

It follows from case III that

$$pr_{\vec{u}}\vec{a} = pr_{\vec{u}}(\vec{a}+\vec{b}) - pr_{\vec{u}}\vec{b}\text{, i.e. } pr_{\vec{u}}(\vec{a}+\vec{b}) = pr_{\vec{u}}\vec{a} + pr_{\vec{u}}\vec{b}\ .$$

It follows from case IV that

$$-pr_{\vec{u}}\vec{b} = -pr_{\vec{u}}(\vec{a}+\vec{b}) + pr_{\vec{u}}\vec{a}\text{, i.e. } pr_{\vec{u}}(\vec{a}+\vec{b}) = pr_{\vec{u}}\vec{a} + pr_{\vec{u}}\vec{b}\ .$$

It follows from case V that

$$pr_{\vec{u}}\vec{b} = pr_{\vec{u}}(\vec{a}+\vec{b}) - pr_{\vec{u}}\vec{a}\text{, i.e. } pr_{\vec{u}}(\vec{a}+\vec{b}) = pr_{\vec{u}}\vec{a} + pr_{\vec{u}}\vec{b}\ .$$

It follows from case VI that

$$-pr_{\vec{u}}\vec{a} = -pr_{\vec{u}}(\vec{a}+\vec{b}) + pr_{\vec{u}}\vec{b}\text{, i.e. } pr_{\vec{u}}(\vec{a}+\vec{b}) = pr_{\vec{u}}\vec{a} + pr_{\vec{u}}\vec{b}\ .$$

3. Since for the positive λ the direction of the axis stays the same, the projection of the vector saves its value. For the negative λ we obtain the opposite direction of the axis and therefore the opposite sign of the projection.

Properties are proven.

Note. One additional property of vector projection follows directly from the definition:

Example. It is known that $pr_{\vec{c}}\vec{a} = 10$, $pr_{\vec{c}}\vec{b} = 5$. Find $pr_{-\vec{c}}(3\vec{a} - 2\vec{b})$.

By the projection properties we have

$$pr_{-\vec{c}}(3\vec{a}-2\vec{b}) = -pr_{\vec{c}}(3\vec{a}-2\vec{b}) = -3pr_{\vec{c}}\vec{a} + 2pr_{\vec{c}}\vec{b} = -3\cdot 10 + 2\cdot 5 = -30 + 10 = -20\ .$$

Thus, the vector $-\vec{c}$ and the vector-projection of the vector $3\vec{a} - 2\vec{b}$ have the opposite directions.

2.1.8. Cartesian Coordinate System

Cartesian coordinate system consists on a point *O* called an origin and perpendicular directed coordinate axes passing through the origin.

Cartesian coordinate system with two (three) axes is called coordinate system in plane (space).

Traditionally, the axes in plane are called axis of abscissas (axis O*x*) and axis of ordinates (axis O*y*) and directed in the way that the shortest turn from positive semi-axis O*x* to positive semi-axis O*y* is made anticlockwise.

The axes in space are called axis of abscissas (axis O*x*), axis of ordinates (axis O*y*) and applicate axis (axis O*z*) and directed in the way that the shortest turn from positive semi-axis O*x* to positive semi-axis O*y* is made anticlockwise if you look from the positive semi-axis O*z*.

Natural bases in plane and in space are bases formed from the unit vectors directed along the positive semi-axes.

Namely, natural basis in plane is set of vectors

$$\vec{i}\,(1,0),\ \vec{j}\,(0,1);$$

natural basis in space is set of vectors

$$\vec{i}\,(1,0,0),\ \vec{j}\,(0,1,0), \vec{k}\,(0,0,1).$$

From Note 2 to the Theorem about vector decomposition (Section 2.1.4) it follows that to find coordinates of the vector in the mentioned above bases we should connect the origin of the vector with point *O* and drop perpendiculars on the exes to find the vector-projections of this vector on basis vectors. In this case the vector is equal to sum of obtained vector-projections (Fig.13).

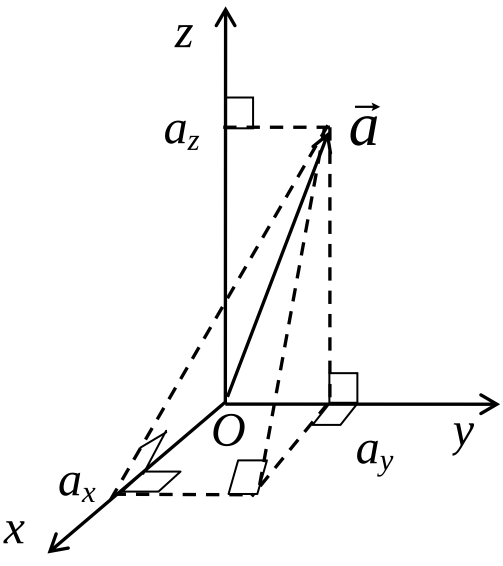

Figure13

Thus, we have:

in plane *Oxy*

$$\vec{a} = a_x\vec{i} + a_y\vec{j} = (a_x, a_y);$$

in space *Oxyz*

$$\vec{a} = a_x\vec{i} + a_y\vec{j} + a_z\vec{k} = (a_x, a_y, a_z),$$

where

$$a_x = pr_{\vec{i}}\vec{a} = |\vec{a}|\cos\alpha, \quad a_y = pr_{\vec{j}}\vec{a} = |\vec{a}|\cos\beta, \quad a_z = pr_{\vec{k}}\vec{a} = |\vec{a}|\cos\gamma,$$

$\alpha = (\vec{a}, \hat{\ } \vec{i})$, $\beta = (\vec{a}, \hat{\ } \vec{j})$, $\gamma = (\vec{a}, \hat{\ } \vec{k})$ are angles between the vector and positive semi-axes *Ox*,*Oy*,Oz.

$\cos\alpha, \cos\beta, \cos\gamma$ are called *the direction cosines* of the vector.

From Fig.13 it follows that

1) By Pythagorean Theorem

$$\boxed{|\vec{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}}$$

2) $1 = \dfrac{|\vec{a}|^2}{|\vec{a}|^2} = \dfrac{|\vec{a}|^2 \cos^2\alpha + |\vec{a}|^2 \cos^2\beta + |\vec{a}|^2 \cos^2\gamma}{|\vec{a}|^2} = \cos^2\alpha + \cos^2\beta + \cos^2\gamma$, i.e.

$$\boxed{\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1}$$

3) Vector $(\cos\alpha, \cos\beta, \cos\gamma)$ is a vector of unit length with the same with vector $\vec{a}$ direction. Thus, this vector is ort of the vector $\vec{a}$, i.e.

$$\boxed{\vec{a}^0 = (\cos\alpha, \cos\beta, \cos\gamma) = \frac{\vec{a}}{|\vec{a}|}}$$

Example. It is known that $|\vec{a}| = 2$, $\cos\alpha = 1/2$, $\cos\gamma = -1/2$ and an angle between the axis O*y* and $\vec{a}$ is acute. Find the coordinates of the vector $\vec{a}$.

Since the angle β is acute then $\cos\beta > 0$ and

$$\cos\beta = \sqrt{1 - \cos^2\alpha - \cos^2\gamma} = \sqrt{1 - \frac{1}{4} - \frac{1}{4}} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}.$$

Therefore

$a_x = |\vec{a}| \cos\alpha = 1, \quad a_y = |\vec{a}| \cos\beta = \sqrt{2}, \quad a_z = \vec{a} | \cos\gamma = -1;$

$\vec{a} = \left(1; \sqrt{2}; -1\right).$

2.1.9. Radius-vector of the Point

Definition. Suppose we have Cartesian coordinate system. Vector $\overrightarrow{OM}$ with origin in the point *O* and a terminus *M* is called a radius-vector of the point *M*.

Coordinates of the point in the Cartesian coordinate system by definition are coordinates of its radius-vector, i.e.

$$\text{If} \quad \overrightarrow{OM} = x\vec{i} + y\vec{j} + z\vec{k} = (x, y, z) \quad \text{then} \quad M(x, y, z).$$

Let us find coordinates of the vector $\overrightarrow{AB}$ through the coordinates of A and B (Fig.14).

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (x_B, y_B, z_B) - (x_A, y_A, z_A) = (x_B - x_A, y_B - y_A, z_B - z_A)$$

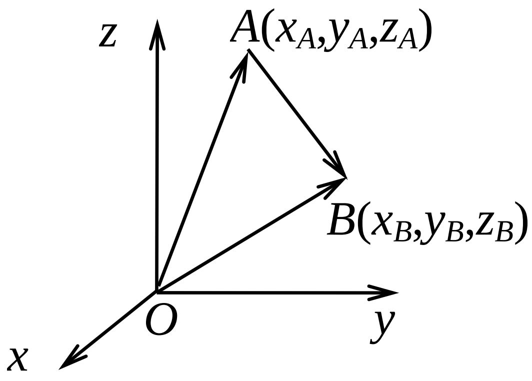

Figure 14

It means that to find coordinates of the vector we should subtract from the coordinates of the terminus the coordinates of the origin.

At the same time, since module of the vector $\overrightarrow{AB}$ is equal to the distance between two points, we state the following:

The distance between two points A and B is equal to

$$d = |\overrightarrow{AB}| = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}.$$

Example. It is known that $\vec{a} = \overrightarrow{AB} = (1;2;-1)$, $A(1;1;0)$. Find the coordinates of the point B and distance between the points A and B.

$a_x = x_B - x_A \Rightarrow x_B = a_x + x_A = 1 + 1 = 2$;

$a_y = y_B - y_A \Rightarrow y_B = a_y + y_A = 2 + 1 = 3$;

$a_z = z_B - z_A \Rightarrow z_B = a_z + z_A = -1 + 0 = -1$.

Therefore, $B(2;3;-1)$.

The distance between the points A and B is equal to the length of the vector $\overrightarrow{AB}$:

$$d = \sqrt{1^2 + 2^2 + (-1)^2} = \sqrt{6}.$$

2.1.10. Division of the Segment in the Given Ratio

Let us find coordinates of the point C which divides the segment AB in the ratio λ:μ, i.e. $|\overrightarrow{AC}|:|\overrightarrow{CB}|=\lambda:\mu$.

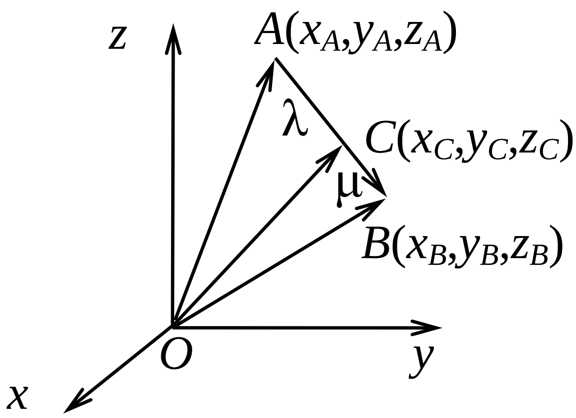

Figure 15

From Fig.15 it follows that

$$\overrightarrow{OC}=\overrightarrow{OA}+\overrightarrow{AC}=\overrightarrow{OA}+\overrightarrow{AB}\frac{\lambda}{\lambda+\mu}=$$

$$=\overrightarrow{OA}+\left(\overrightarrow{OB}-\overrightarrow{OA}\right)\frac{\lambda}{\lambda+\mu}=\frac{\mu}{\lambda+\mu}\overrightarrow{OA}+\frac{\lambda}{\lambda+\mu}\overrightarrow{OB}=$$

$$=\left(\frac{\mu}{\lambda+\mu}x_A+\frac{\lambda}{\lambda+\mu}x_B,\ \frac{\mu}{\lambda+\mu}y_A+\frac{\lambda}{\lambda+\mu}y_B,\ \frac{\mu}{\lambda+\mu}z_A+\frac{\lambda}{\lambda+\mu}z_B\right)$$

Thus

$$C:\quad x_C=\frac{\mu x_A+\lambda x_B}{\lambda+\mu},\ y_C=\frac{\mu y_A+\lambda y_B}{\lambda+\mu},\ z_C=\frac{\mu z_A+\lambda z_B}{\lambda+\mu}.$$

Example 1. Let point M be a middle of the segment. In this case $\lambda=\mu=1$. Therefore

$$x_M=\frac{x_A+x_B}{2},\ y_M=\frac{y_A+y_B}{2},\ z_M=\frac{z_A+z_B}{2}.$$

Example 2. Find the point M of median intersection in the triangle with vertices $A(x_A,y_A,z_A)$, $B(x_B,y_B,z_B)$, $C(x_C,y_C,z_C)$ (Fig.16).

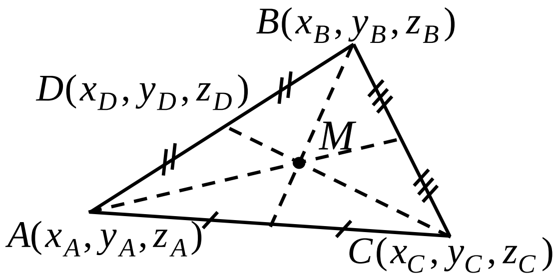

Figure 16

Since M is a point of median intersection, it divides each median in the ration 2:1. Therefore, the coordinates of this point could be found through the coordinates of the points C and D in the following way:

$$M: \quad x_M = \frac{x_C + 2x_D}{2+1}, \; y_M = \frac{y_C + 2y_D}{2+1}, \; z_M = \frac{z_C + 2z_D}{2+1},$$

where D is a middle of the side AB and therefore

$$x_D = \frac{x_A + x_B}{2}, \; y_D = \frac{y_A + y_B}{2}, \; z_D = \frac{z_A + z_B}{2}.$$

Thus,

$$M: \quad x_M = \frac{x_C + 2\frac{x_A + x_B}{2}}{2+1} = \frac{x_A + x_B + x_C}{3},$$

$$y_M = \frac{y_C + 2\frac{y_A + y_B}{2}}{2+1} = \frac{y_A + y_B + y_C}{3},$$

$$z_M = \frac{z_C + 2\frac{z_A + z_B}{2}}{2+1} = \frac{z_A + z_B + z_C}{3}.$$

Example 3. Find the center of the gravity of the triangle with vertices $A(1;2;3)$, $B(-1;3;4)$, $C(3,0,-2)$. Since the center of the gravity in triangle coincides with the point of median intersection, the coordinates of the center are:

$$x_o = \frac{x_A + x_B + x_C}{3} = \frac{1-1+3}{3} = 1, \; y_o = \frac{y_A + y_B + y_C}{3} = \frac{2+3+0}{3} = \frac{5}{3},$$

$$z_o = \frac{z_A + z_B + z_C}{3} = \frac{3+4-2}{3} = \frac{5}{3}.$$

Note. All obtained above formulas are valid for the points in plane, as well.

Except linear operations on vectors, such as addition and multiplication by scalar, there is an operation of vector multiplication. Moreover, It is possible to multiply vectors in three ways, namely in scalar, vector and mixed ways.

2.1.11. Scalar Product

Definition. Scalar product (or *dot product*) of two vectors $\vec{a}$ and $\vec{b}$ is a number (scalar) equal to $|\vec{a}||\vec{b}|\cos\alpha$, where α is an angle between vectors $\vec{a}$ and $\vec{b}$.

We denote the scalar product in two ways: $(\vec{a},\vec{b})$ or just $\vec{a}\vec{b}$. So,

$$(\vec{a},\vec{b})=|\vec{a}||\vec{b}|\cos\alpha.$$

Since

$$|\vec{b}|\cos\alpha = pr_{\vec{a}}\vec{b}, \qquad |\vec{a}|\cos\alpha = pr_{\vec{b}}\vec{a},$$

we have

$$(\vec{a},\vec{b})=|\vec{a}|pr_{\vec{a}}\vec{b}=|\vec{b}|pr_{\vec{b}}\vec{a},$$

$$pr_{\vec{b}}\vec{a}=\frac{(\vec{a},\vec{b})}{|\vec{b}|}.$$

Statement (Criterion of the perpendicularity) Two non-zero vectors are perpendicular if and only if their scalar product is equal to zero, i.e.

$$\bar{a}\perp\bar{b} \Leftrightarrow (\bar{a},\bar{b})=0.$$

Indeed,

$$\bar{a}\perp\bar{b} \Leftrightarrow \alpha=\frac{\pi}{2} \Leftrightarrow \cos\alpha=0 \Leftrightarrow (\bar{a},\bar{b})=|\vec{a}||\vec{b}|\cos\alpha=0.$$

Algebraic properties of the scalar product:

1) $(\vec{a},\vec{b})=(\vec{b},\vec{a})$**;**

2) $\left(\lambda\vec{a},\vec{b}\right)=\left(\vec{a},\lambda\vec{b}\right)=\lambda\left(\vec{a},\vec{b}\right)$;

3) $\left(\vec{a}+\vec{b},\vec{c}\right)=\left(\vec{a},\vec{c}\right)+\left(\vec{b},\vec{c}\right)$.

Proof. 1) $\left(\vec{a},\vec{b}\right)=|\vec{a}||\vec{b}|\cos\alpha=|\vec{b}||\vec{a}|\cos\alpha=\left(\vec{b},\vec{a}\right)$;

2) $\left(\lambda\vec{a},\vec{b}\right)=|\vec{b}|\,pr_{\vec{b}}\lambda\vec{a}=\lambda|\vec{b}|\,pr_{\vec{b}}\vec{a}=\lambda\left(\vec{a},\vec{b}\right)$;

3) $\left(\vec{a}+\vec{b},\vec{c}\right)=|\vec{c}|\;np_{\vec{c}}\,(\vec{a}+\vec{b})=|\vec{c}|\left(pr_{\vec{c}}\vec{a}+pr_{\vec{c}}\vec{b}\right)=\left(\vec{a},\vec{c}\right)+(\vec{b},\vec{c})$.

Properties are proven.

From the definition It follows that

$$\left(\vec{a},\vec{a}\right)=|\vec{a}|^2 \quad \text{or} \quad |\vec{a}|=\sqrt{(\vec{a},\vec{a})}\,.$$

Thus, we obtain an additional fourth property of scalar product:

4) $\left(\vec{a},\vec{a}\right)\geq 0$ and $\left(\vec{a},\vec{a}\right)=0 \Leftrightarrow \vec{a}=\vec{0}$.

Example 1. It is known that $\vec{a}=5\vec{p}+2\vec{q}$, $\vec{b}=\vec{p}-3\vec{q}$, $|\vec{p}|=1$, $|\vec{q}|=2$, $\varphi=\left(\widehat{\vec{p},\vec{q}}\right)=\dfrac{\pi}{3}$. Find $|\overline{a}+\overline{b}|$.

By the last formula

$$|\vec{a}+\vec{b}|^2=(\vec{a}+\vec{b},\vec{a}+\vec{b})=(5\vec{p}+2\vec{q}+\vec{p}-3\vec{q},5\vec{p}+2\vec{q}+\vec{p}-3\vec{q})=(6\vec{p}-\vec{q},6\vec{p}-\vec{q})=$$

$=36(\vec{p},\vec{p})-6(\vec{p},\vec{q})-6(\vec{q},\vec{p})+(\vec{q},\vec{q})=$[By properties of scalar product]=

$$=36\,|\,\vec{p}\,|^2-12(\vec{p},\vec{q})+|\,\vec{q}\,|^2=36\cdot 1-12\cdot 1\cdot 2\cdot\cos\frac{\pi}{3}+2^2=36-12+4=28\,.$$

Thus,

$$|\overline{a}+\overline{b}|=\sqrt{28}=2\sqrt{7}\,.$$

Example 2. Find the ort of the vector.

From the definition of ort it follows that $\vec{a}^\circ=\lambda\vec{a}$, where $\lambda>0$. Therefore

$$1=\left(\vec{a}^\circ,\vec{a}^\circ\right)=\left(\lambda\vec{a},\lambda\vec{a}\right)=\lambda^2|\overline{a}|^2\Rightarrow\lambda=\frac{1}{|\overline{a}|}\Rightarrow$$

$$\boxed{\vec{a}^\circ=\frac{\vec{a}}{|\overline{a}|}}$$

Note, that we have obtained the same formula as obtained above through the direction cosines.

Let us find the formula to calculate the scalar product of vectors given by their coordinates in the orthonormal basis $\vec{i}, \vec{j}, \vec{k}$.

Since
$$|\vec{i}| = |\vec{j}| = |\vec{k}| = 1$$
and
$$\vec{i} \perp \vec{j}, \vec{i} \perp \vec{k}, \vec{j} \perp \vec{k}, \text{ i.e. } (\vec{i}, \vec{j}) = (\vec{i}, \vec{k}) = (\vec{j}, \vec{k}) = 0,$$
we have
$$(\vec{a}, \vec{b}) = (a_x\vec{i} + a_y\vec{j} + a_z\vec{k}, b_x\vec{i} + b_y\vec{j} + b_z\vec{k}) = a_x b_x + a_y b_y + a_z b_z.$$

It means that to find the scalar product we should multiply the corresponding coordinates of vectors and then summarize these products.

Note. This formula is valid for the vectors in plane (case, when $a_z = b_z = 0$).

Example. It is known that $\vec{a}(1,2,3)$, $\vec{b}(-1,1,2)$, $\vec{c}(0,1,4)$. Find a value k such that $\vec{a} \perp \left(\vec{b} - k\vec{c}\right)$. By the criterion of the perpendicularity we have
$$0 = \left(\vec{a}, \vec{b} - k\vec{c}\right) = \left(\vec{a}, \vec{b}\right) - k\left(\vec{a}, \vec{c}\right) =$$
$$= 1(-1) + 2 \cdot 1 + 3 \cdot 2 - k(1 \cdot 0 + 2 \cdot 1 + 3 \cdot 4) =$$
$$= 7 - 14k = 0.$$
Thus
$$k = \frac{7}{14} = \frac{1}{2}.$$

Note. The two and three-dimensional vector spaces with scalar product, satisfying four properties written above, are called *Euclidean vector spaces*.

2.1.13. Vector Product

Definition. The ordered triple of uncomplanar vectors $\vec{a}, \vec{b}, \vec{c}$ form *a right–hand triple* if the shortest turn from the vector $\vec{a}$ to the vector $\vec{b}$ is made anticlockwise when their origins are connected and you observe this turn from the terminus of $\vec{c}$. In other case they form *a left – hand triple*.

Definition. Vector product (or *cross product*) of vectors $\vec{a}$ and $\vec{b}$ is a vector $\vec{c}$ satisfying the following three conditions:

1) $\vec{c} \perp \vec{a}$, $\vec{c} \perp \vec{b}$;

2) $|\vec{c}| = |\vec{a}||\vec{b}|\sin\alpha$, where α is an angle between $\vec{a}$ and $\vec{b}$;

3) $\vec{a}, \vec{b}, \vec{c}$ form the right–hand triple.

We denote vector product in two ways, namely $\vec{c} = \vec{a} \times \vec{b}$ or $\vec{c} = [\vec{a}, \vec{b}]$.

Algebraic properties of the vector product:

1) $[\vec{a}, \vec{b}] = -[\vec{b}, \vec{a}]$ (Property of anti-symmetry);

2) $[\lambda\vec{a}, \vec{b}] = \lambda[\vec{a}, \vec{b}] = [\vec{a}, \lambda\vec{b}]$;

3) $[\vec{a}, \vec{b} + \vec{c}] = [\vec{a}, \vec{b}] + [\vec{a}, \vec{c}]$.

Proof. Properties 1)-2) follow directly from conditions 2 and 3 of definition.

To prove the property 3) let us show first that there is another way to plot the result of vector product (Fig. 17). We connect the origins of two vectors, project

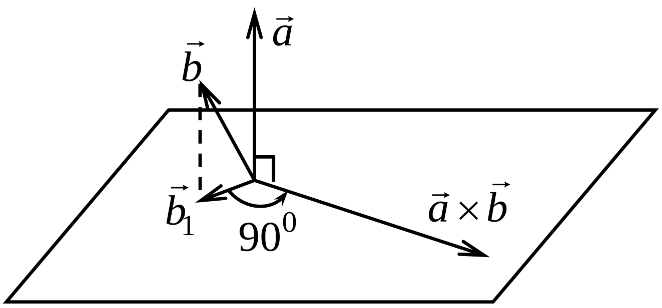

Figure 17

the vector $\vec{b}$ on the plane perpendicular to the vector $\vec{a}$. Then we turn the obtained vector $\vec{b}_1$ anticlockwise on 90 degrees and multiply by $|\vec{a}|$. The result is $\vec{a} \times \vec{b}$ since it satisfies all conditions from the definition.

We are going to use this procedure to prove the third property. Consider the parallelogram I from the Fig.18 and project it on the plane perpendicular to $\vec{a}$.

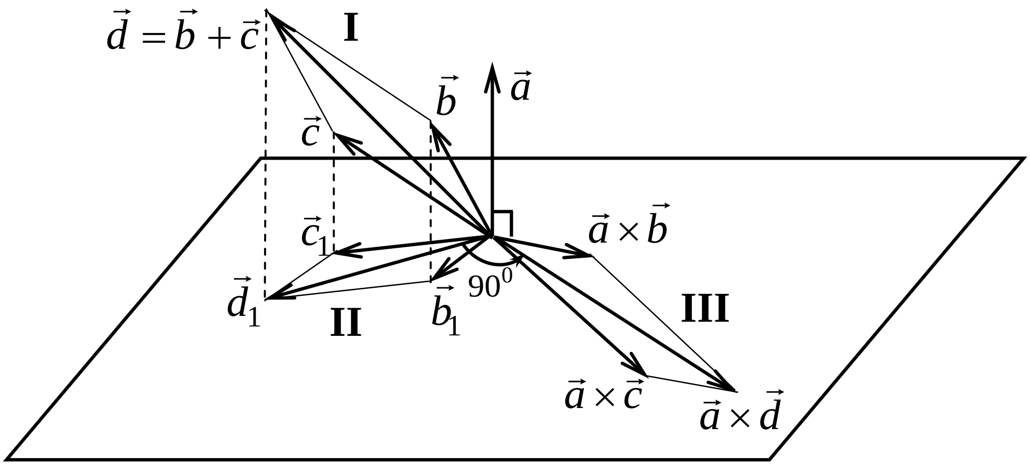

Figure 18

Obtained figure II is also parallelogram and, moreover, the diagonal $\vec{d} = \vec{b} + \vec{c}$ of the figure I is projected into the diagonal $\vec{d}_1 = \vec{b}_1 + \vec{c}_1$ of the figure II. To obtain the figure III we turn the figure II anticlockwise on 90 degrees and stretch it in $|\vec{a}|$ times. At that we again obtain the parallelogram where the diagonal of III is obtained by turn and stretching of the diagonal of II. It means that the obtained diagonal is the vector $\vec{a} \times \vec{d} = \vec{a} \times (\vec{b} + \vec{c})$ equal to the sum of the parallelogram sides, i.e.

$$\vec{a} \times \vec{d} = \vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}\,.$$

Properties are proven.

Geometrical properties of the vector product:

1) $\vec{a} \parallel \vec{b} \Leftrightarrow \vec{a} \times \vec{b} = 0$ (Criterion of collinearity of two non-zero vectors)

Indeed, $\vec{a} \parallel \vec{b} \Leftrightarrow \alpha = (\widehat{\vec{a}, \vec{b}}) = \left[\begin{array}{l} 0 \\ \pi \end{array}\right. \Leftrightarrow \sin\alpha = 0 \Rightarrow |\vec{a} \times \vec{b}| = 0 \Leftrightarrow \vec{a} \times \vec{b} = \vec{0}$.

Note. Another criterion of collinearity follows from definition, namely,

$$\vec{a} \parallel \vec{b} \Leftrightarrow \vec{a} = \lambda\vec{b} \Leftrightarrow a_x = \lambda b_x, a_y = \lambda b_y, a_z = \lambda b_z \Leftrightarrow \frac{a_x}{b_x} = \frac{a_y}{b_y} = \frac{a_z}{b_z},$$

i.e. *the coordinates of collinear vectors are proportional*.

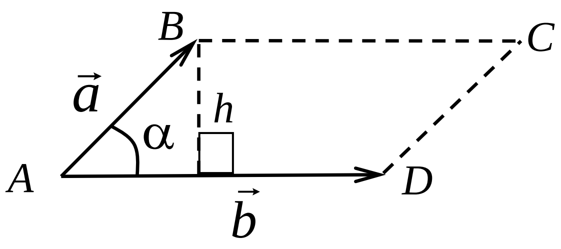

Figure 19

2) $S_{par} = |\bar{a} \times \bar{b}|$, i.e. the area of the parallelogram constructed on the vectors $\vec{a}$ and $\vec{b}$ is equal to the module of their vector product.

Indeed, from Fig. 19 we have

$$S_{par} = AB \cdot AD \cdot \sin\alpha = |\vec{a}||\vec{b}|\sin\left(\widehat{\vec{a},\vec{b}}\right) = |\vec{a} \times \vec{b}|.$$

3) The altitude of the parallelogram is equal to

$$h = \frac{|\vec{a} \times \vec{b}|}{|\vec{b}|} \quad .$$

Indeed, from Fig.19 It follows that:

$$S_{par} = h \cdot AD \Rightarrow h = \frac{S_{par}}{AD} = \frac{|\vec{a} \times \vec{b}|}{|\vec{b}|} \quad .$$

4) The area of the triangle, constructed on the vectors $\vec{a}$ and $\vec{b}$, is equal to a half of the module of their vector product. At the same time, the formula for the altitude dropped on the vector $\vec{b}$ is the same as for the parallelogram. So

$$S_{tr} = \frac{1}{2}|\bar{a} \times \bar{b}|, \qquad h = \frac{|\vec{a} \times \vec{b}|}{|\vec{b}|} \quad .$$

5) **Finding the vector perpendicular to the plane of any two uncollinear vectors.** Suppose, $\vec{a}$ and $\vec{b}$ are not collinear. Then some parallelogram which is planar figure can be constructed on them. Vector $\vec{a} \times \vec{b}$ is a vector perpendicular to both $\vec{a}$ and $\vec{b}$ and thus to the plane of the parallelogram. Therefore, any vector, perpendicular to the plane of two uncollinear vectors $\vec{a}$ and $\vec{b}$ is collinear to $\vec{a} \times \vec{b}$. So, we state for uncollinear non-zero vectors $\vec{a}$ and $\vec{b}$

$$\begin{cases} \vec{c} \perp \vec{a} \\ \vec{c} \perp \vec{b} \end{cases} \Rightarrow \vec{c} \parallel \vec{a} \times \vec{b} \Leftrightarrow \vec{c} = \lambda \vec{a} \times \vec{b},\ \lambda \in R \setminus \{0\}.$$

Let us find the formula to calculate the vector product of vectors given by their coordinates in the orthonormal basis $\vec{i}, \vec{j}, \vec{k}$. Since

$$\vec{i} \times \vec{i} = 0 \qquad \vec{i} \times \vec{j} = \vec{k} \qquad \vec{i} \times \vec{k} = -\vec{j}$$

$$\vec{j} \times \vec{i} = -\vec{k} \qquad \vec{j} \times \vec{j} = 0 \qquad \vec{j} \times \vec{k} = \vec{i}$$

$$\vec{k} \times \vec{i} = \vec{j} \qquad \vec{k} \times \vec{j} = -\vec{i} \qquad \vec{k} \times \vec{k} = 0$$

the vector product of vectors $\vec{a} = a_x\vec{i} + a_y\vec{j} + a_z\vec{k}$ and $\vec{b} = b_x\vec{i} + b_y\vec{j} + b_z\vec{k}$ is equal to

$$\left[\vec{a}, \vec{b}\right] = \left[a_x\vec{i} + a_y\vec{j} + a_z\vec{k}, b_x\vec{i} + b_y\vec{j} + b_z\vec{k}\right] = a_xb_x\vec{i} \times \vec{i} + a_xb_y\vec{i} \times \vec{j} + a_xb_z\vec{i} \times \vec{k} + a_yb_x\vec{j} \times \vec{i} +$$

$$+ a_yb_y\vec{j} \times \vec{j} + a_yb_z\vec{j} \times \vec{k} + +a_zb_x\vec{k} \times \vec{i} + a_zb_y\vec{k} \times \vec{j} + a_zb_z\vec{k} \times \vec{k} =$$

$$= \left(a_xb_y - a_yb_x\right)\vec{i} \times \vec{j} + \left(-a_xb_z + a_zb_x\right)\vec{k} \times \vec{i} + \left(a_yb_z - a_zb_y\right)\vec{j} \times \vec{k} =$$

$$= \vec{i}\left(a_yb_z - a_zb_y\right) + (-1)\vec{j}\left(a_xb_z - a_zb_x\right) + \vec{k}\left(a_xb_y - a_yb_x\right) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}.$$

So,

$$\bar{a} \times \bar{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}.$$

Example. Find area of the triangle with vertices in the points $A(1,1), B(2,-1), C(0,3)$ and vector $\vec{h}$ collinear to the altitude dropped on side *AB*. Since the problem is formulated in plane we can not calculate vector product to find area. That is why before solving this problem we reformulate the task by expanding the coordinates of points to spatial case, i.e. we suppose that vertices have the following coordinates:

$$A(1,1,0), B(2,-1,0), C(0,2,0).$$

Then $\overrightarrow{AB} = (1,-2,0), \quad \overrightarrow{AC} = (-1,1,0)$,

$$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & -2 & 0\\ -1 & 1 & 0\end{vmatrix}=0\vec{i}-0\vec{j}+(-1)\vec{k}=(0,0,-1),$$

$$S_{tr}=\frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|=\frac{1}{2}\sqrt{0^2+0^2+(-1)^2}=\frac{1}{2}\cdot 1=\frac{1}{2}.$$

Vector $\vec{h}$ is perpendicular to the vector $\overrightarrow{AB}$ and to the vector $\overrightarrow{AB}\times\overrightarrow{AC}$ (since this vector is perpendicular to any vector in the plane of triangle). It means that

$$\vec{h}=\left[\overrightarrow{AB},\overrightarrow{AB}\times\overrightarrow{AC}\right]=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & -2 & 0\\ 0 & 0 & -1\end{vmatrix}=2\vec{i}+\vec{j}+0\vec{k}=(2,1,0).$$

These coordinates are coordinates in space. To get final answer we should save only the first two coordinates, i.e. $\vec{h}(2,1)$.

2.1.14. Mixed product

Definition. Mixed product of vectors $\vec{a},\vec{b},\vec{c}$ is equal to the value obtained after scalar multiplication of the vector $\vec{c}$ by the vector product of vectors $\vec{a}$ and $\vec{b}$, i.e.

$$\left(\vec{a},\vec{b},\vec{c}\right)=\left(\vec{a}\times\vec{b},\vec{c}\right).$$

Theorem (Criterion of complanarity of three non-zero vectors) $\left(\vec{a},\vec{b},\vec{c}\right)=0\Leftrightarrow\vec{a},\vec{b},\vec{c}$ are complanar.

Proof. $\left(\vec{a},\vec{b},\vec{c}\right)=\left(\vec{a}\times\vec{b},\vec{c}\right)=0\Leftrightarrow\left[\begin{array}{l}\vec{a}\times\vec{b}\perp\vec{c}\\ \vec{a}\times\vec{b}=0\end{array}\right.$. It means that either $\vec{c}$ is parallel to the plane of $\vec{a}$ and $\vec{b}$ or $\vec{a}$ and $\vec{b}$ are collinear. In all these cases the vectors $\vec{a},\vec{b},\vec{c}$ are complanar. ***Theorem is proven.***

Proof. If at least two factors coincide in the mixed product, this product is equal to zero. That is $(\vec{a},\vec{a},\vec{b})=0$.

Theorem (Mixed product of the right-hand triple) $\vec{a},\vec{b},\vec{c}$ form the right-hand triple if and only if $(\overline{a},\overline{b},\overline{c})>0$.

Proof. From Fig.20 it follows that if $\vec{a},\vec{b},\vec{c}$ form the right-hand triple then an angle α is acute. Thus

$$(\overline{a}\times\overline{b},\vec{c})=(\overline{a},\overline{b},\vec{c})>0.$$

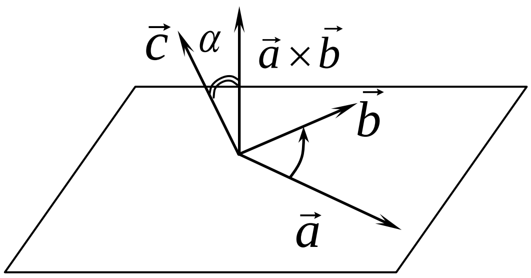

Figure 20

From the other hand, if $(\overline{a}\times\overline{b},\vec{c})>0\Rightarrow\cos\alpha>0\Rightarrow$

$\Rightarrow\alpha$ is acute $\Rightarrow\overline{a},\overline{b},\overline{c}$ form the right-hand triple. ***Theorem is proven.***

Corollary.

$(\overline{a},\overline{b},\overline{c})<0\Leftrightarrow\vec{a},\overline{b},\vec{c}$ form the left-hand triple.

Theorem (Geometrical meaning of the mixed product)

$V_{parallelepiped}=|(\overline{a},\overline{b},\vec{c})|$, i.e. the volume of the parallelepiped, constructed on the vectors $\vec{a},\vec{b},\vec{c}$, is equal to the module of their mixed product.

Proof. Suppose $\vec{a},\vec{b},\vec{c}$ is a right-hand triple (Fig.21). Then

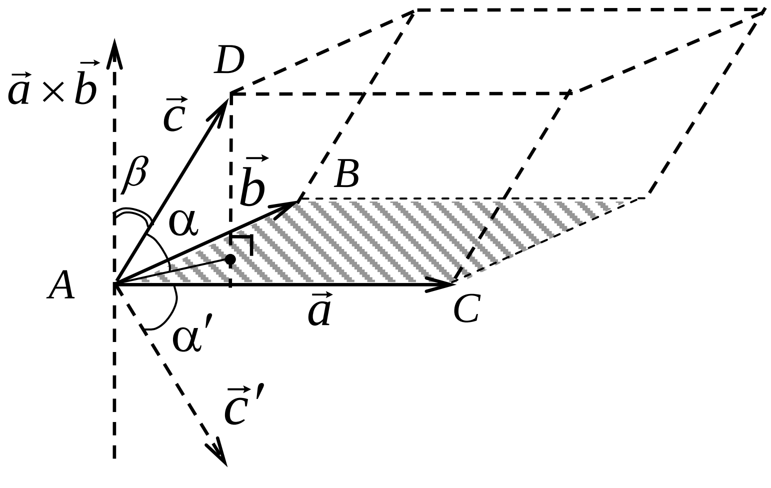

Figure 21

$V=S\cdot AD\sin\alpha=|\vec{a}\times\vec{b}||\vec{c}|\sin\alpha=$

$$=|\vec{a}\times\vec{b}||\vec{c}|\sin\left(\frac{\pi}{2}-\beta\right)=|\vec{a}\times\vec{b}||\vec{c}|\cos\beta=(\vec{a}\times\vec{b},\vec{c})=(\vec{a},\vec{b},\vec{c})=|(\overline{a},\overline{b},\vec{c})|.$$

If $\vec{a},\vec{b},\vec{c}$ form the left-hand triple (for this case $\vec{c}$ and α are shown as $\vec{c}\,',\alpha'$ on Fig.21) then

$$\sin\alpha = \sin\left(\beta - \frac{\pi}{2}\right) = -\sin\left(\frac{\pi}{2} - \beta\right) = -\cos\beta .$$

Therefore $V = -\left(\vec{a},\vec{b},\vec{c}\right) = \left|\left(\vec{a},\vec{b},\vec{c}\right)\right|$. ***Theorem is proven.***

Note. It is simple to check that if $\vec{a},\vec{b},\vec{c}$ is a right-hand triple then $\vec{c},\vec{a},\vec{b}$ and $\vec{b},\vec{c},\vec{a}$ form the right-hand triples, as well. Hence,

$$V = (\vec{a},\vec{b},\vec{c}) = (\vec{c},\vec{a},\vec{b}) = (\vec{b},\vec{c},\vec{a}) .$$

In the same way it can be shown that

$$V = -(\vec{b},\vec{a},\vec{c}) = -(\vec{c},\vec{b},\vec{a}) = -(\vec{a},\vec{c},\vec{b}) .$$

Moreover, from the obtained above it follows that

$$(\vec{a},\vec{b},\vec{c}) = (\vec{a}\times\vec{b},\vec{c}) = (\vec{b},\vec{c},\vec{a}) = (\vec{b}\times\vec{c},\vec{a}) = (\vec{a},\vec{b}\times\vec{c}),$$

i.e. to find mixed product we can multiply any two neighbour vectors in the vector way and then multiply the result vector by the third one in the scalar way.

Algebraic properties of the mixed product:

1) *a*) $\left(\vec{a},\vec{b},\vec{c}\right) = \left(\vec{c},\vec{a},\vec{b}\right) = \left(b,\vec{c},\vec{a}\right)$,

b) $\left(\vec{a},\vec{b},\vec{c}\right) = -\left(\vec{b},\vec{a},\vec{c}\right) = -\left(\vec{a},\vec{c},\vec{b}\right) = -\left(\vec{c},\vec{b},\vec{a}\right)$;

i.e. cyclic transposition of vectors does not change the value of the mixed product, but the transposition of any two neighbour vectors changes the sign of the mixed product. It follows from the last Note or from the properties of scalar and vector products.

2) $\left(\lambda\vec{a},\vec{b},\vec{c}\right) = \lambda\left(\vec{a},\vec{b},\vec{c}\right) = \left(\vec{a},\lambda\vec{b},\vec{c}\right) = \left(\vec{a},\vec{b},\lambda\vec{c}\right)$;

3) $\left(\vec{a}+\vec{b},\vec{c},\vec{d}\right) = \left(\vec{a},\vec{c},\vec{d}\right) + \left(\vec{b},\vec{c},\vec{d}\right)$.

Last two properties follow directly from the properties of scalar and vector products.

Geometrical properties of the mixed product:

1) $V_{parallelepiped} = \left|\left(\vec{a},\vec{b},\vec{c}\right)\right|$ (Fig.22)

2) The altitude of the parallelepiped dropped on the base of vectors $\vec{a}$ and $\vec{b}$ is

$$h=\frac{V}{S}=\frac{\left|\left(\vec{a},\vec{b},\vec{c}\right)\right|}{\left|\vec{a}\times\vec{b}\right|}.$$

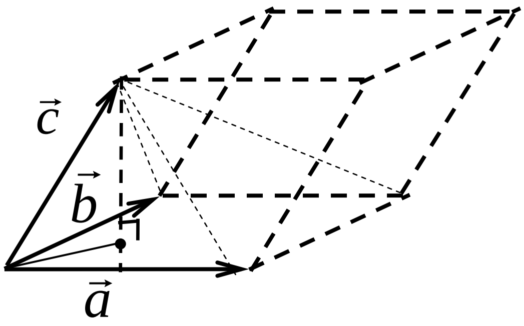

Figure 22

3) The volume of the tetrahedron constructed on vectors $\vec{a},\vec{b},\vec{c}$ (Fig.22) is equal to

$$V_{tetrahedron}=\frac{1}{6}V_{par.}=\frac{1}{6}\left|\left(\vec{a},\vec{b},\vec{c}\right)\right|.$$

The altitude of the tetrahedron coincides with the altitude of the parallelepiped, so it could be found by the same formula.

Let us find the formula to calculate the mixed product of vectors given by their coordinates in the orthonormal basis $\vec{i},\vec{j},\vec{k}$.

Suppose,

$$\vec{a}=a_x\vec{i}+a_y\vec{j}+a_z\vec{k}=\left(a_x,a_y,a_z\right)$$

$$\vec{b}=b_x\vec{i}+b_y\vec{j}+b_z\vec{k}=\left(b_x,b_y,b_z\right)$$

$$\vec{c}=c_x\vec{i}+c_y\vec{j}+c_z\vec{k}=\left(c_x,c_y,c_z\right)$$

Let us evaluate $\left(\overline{a},\overline{b},\vec{c}\right)=\left(\overline{a},\overline{b}\times\overline{c}\right)$:

$$\overline{b}\times\vec{c}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ b_x & b_y & b_z\\ c_x & c_y & c_z\end{vmatrix}=\vec{i}\begin{vmatrix}b_y & b_z\\ c_y & c_z\end{vmatrix}-\vec{j}\begin{vmatrix}b_x & b_z\\ c_x & c_z\end{vmatrix}+\vec{k}\begin{vmatrix}b_x & b_y\\ c_x & c_y\end{vmatrix}=$$

$$\left(\begin{vmatrix}b_y & b_z\\ c_y & c_z\end{vmatrix},-\begin{vmatrix}b_x & b_z\\ c_x & c_z\end{vmatrix},\begin{vmatrix}b_x & b_y\\ c_x & c_y\end{vmatrix}\right).$$

$$\left(\vec{a},\vec{b}\times\vec{c}\right)=a_x\begin{vmatrix}b_y & b_z\\ c_y & c_z\end{vmatrix}-a_y\begin{vmatrix}b_x & b_z\\ c_x & c_z\end{vmatrix}+a_z\begin{vmatrix}b_x & b_y\\ c_x & c_y\end{vmatrix}=\begin{vmatrix}a_x & a_y & a_z\\ b_x & b_y & b_z\\ c_x & c_y & c_z\end{vmatrix}.$$

Therefore,

$$\left(\vec{a},\vec{b},\vec{c}\right)=\begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}.$$

Example 1. Find the coordinates of the vertex *D* of the tetrahedron *ABCD* if the volume of this tetrahedron is equal to 10, *D* is situated on the positive semi-axis O*z* and $A(1;2;3),\ B(-1;0;2),\ C(0,4,1)$.

From condition it follows that *D* has coordinates $D(0;0;z_D)$ and

$$V=10=\frac{1}{6}\left|\left(\overrightarrow{AB},\overrightarrow{AC},\overrightarrow{AD}\right)\right|,\text{ i.e. }\left|\left(\overrightarrow{AB},\overrightarrow{AC},\overrightarrow{AD}\right)\right|=60.$$

But

$$\left(\overrightarrow{AB},\overrightarrow{AC},\overrightarrow{AD}\right)=\begin{vmatrix} -2 & -2 & 1 \\ -1 & 2 & -2 \\ -1 & -2 & z_D-3 \end{vmatrix}=-4(z_D-3)-4+2+2+8-2(z_D-3)=$$

$$=-6z_D+26.$$

Therefore

$$-6z_D+26=\pm 60 \Leftrightarrow \left[\begin{array}{l} -6z_D=34 \\ -6z_D=-86 \end{array}\right. \Leftrightarrow \left[\begin{array}{l} z_D=-34/6=-17/3 \\ z_D=43/3 \end{array}\right.$$

Since *D* is situated on the positive semi-axis O*z* the answer is $D(0;0;43/3)$.

Example 2. Prove that four points are situated on the same plane if their coordinates are $A(1;1;1),\ B(1;2;3),\ C(2;3;4),\ D(0;2;4)$.

These points are from the same plane if and only if the vectors $\overrightarrow{AB},\overrightarrow{AC},\overrightarrow{AD}$ are complanar. Let us check this statement.

$$\left(\overrightarrow{AB},\overrightarrow{AC},\overrightarrow{AD}\right)=\begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ -1 & 1 & 3 \end{vmatrix}=0-3+2+4-3-0=0.$$

Therefore the vectors are complanar and points are situated on the same plane.

Example 3. Find $\left(\vec{a}+\vec{b},\vec{b}+\vec{c},\vec{c}\right)$ if $\left(\vec{a},\vec{b},\vec{c}\right)=1$. By mixed product properties:

$$(\vec{a}+\vec{b},\vec{b}+\vec{c},\vec{c})=(\vec{a},\vec{b},\vec{c})+(\vec{a},\vec{c},\vec{c})+(\vec{b},\vec{b},\vec{c})+(\vec{b},\vec{c},\vec{c})=1+0+0+0=1.$$