2.8. The Linear Differential Equation of the Second Order with Constant Coefficients and Special Right Part of the first type.

Let us consider the inhomogeneous differential equation of the second order

$$y'' + p(x)y' + q(x)y = f(x). \qquad (1)$$

Theorem 3. The general solution of INHLDE (1) is equal to the sum of general solution of corresponding homogeneous differential equation $\bar{y}$ and some particular solution y^* of this equation .That's

$$y = \bar{y} + y^* \qquad (1a)$$

Proof of this theorem is presented in Lecture or textbook "L.V.Kurpa, O.S.Mazur, T.V.Shmatko. Differential equations and Series"

Conclusion. *In order to find the general solution of the inhomogeneous linear differential equation (INHLDE) we should find the general solution of corresponding homogeneous differential equation and add some particular solution of the given INHLDE*

Let' s begin with solution of t*he Linear Differential Equation of the Second Order with Constant Coefficients and Special Right Part*

Let us consider the differential equation

$$y'' + py' + qy = f(x), \qquad (2)$$

where $p = \text{const}$ and $q = \text{const}$. Since the general solution $\bar{y}$ of the corresponding homogeneous equation we can find at once, then the integration of this equation is reduced to finding any particular solution $y^*(x)$ of this equation. It may be done enough simple for some types of the right part.

I. The Right Side of the Differential Equation of the First Kind. Let the right part have the following form:

$$f(x) = e^{\alpha x} P_n(x), \tag{3}$$

where $P_n(x)$ is polynomial of *n*-th power. For example polynomial of the **zero power is** just constant $P_0(x) = A$,

Polynomial of the **first power** is defined as

$P_1(x) = Ax + B$.

Polynomial of the **second power** is

$P_2(x) = Ax^2 + Bx + C$, and so on.

There are possible three cases for function (3).

1) *Number* α ***is not root*** *of the characteristic equation oh the homogeneous linear differential equation*

$$\lambda^2 + p\lambda + q = 0.$$

In this case the particular solution should search for in the form

$$y^* = e^{\alpha x} Q_n(x), \tag{3a}$$

where $Q_n(x)$ is unknown polynomial of the *n*-th power.

Example. Consider the equation

$$y'' + y = 4xe^x. \tag{4}$$

Solution. The general solution of the (4) is

$$y = \bar{y} + y^*$$

In order to find the general solution of the homogeneous equation $\bar{y}$, let us construct the characteristic equation

$$\lambda^2 + 1 = 0.$$

Its roots are $\lambda_{1,2} = \pm i$, therefore the general solution of the HLDE has the following form:

$$\bar{y} = C_1 \cos x + C_2 \sin x.$$

Let’s find the partial solution of the initial differential equation y^*.
The right side of the given equation has the form

$$f(x) = e^{1\cdot x} P_1(x).$$

In the given case $\alpha = 1$. Since $\lambda_1 \neq 1$ and $\lambda_2 \neq 1$, then we will search for the solution $y^*(x)$ in the form

$$y^* = e^{1\cdot x} Q_1(x),$$

where $Q_1(x)$ is polynomial of the first power, that’s $Q_1(x) = Ax + B$.
and

$$y^* = (Ax + B)e^x. \quad (5)$$

In order to find the unknown coefficients *A* and *B*, we should calculate derivatives of the first and second order of function (5), substitute into initial equation (4) and equate the coefficients at the same degree of the polynomials of the right and left sides.
Whence

$$(y^*)' = Ae^x + (Ax + B)e^x, \quad (6)$$

$$(y^*)'' = 2Ae^x + (Ax + B)e^x. \quad (7)$$

After substitution the obtained expressions (5),(6),(7) into equation (4) and reducing by e^x we obtain

$$2A + Ax + B + Ax + B = 4x.$$

$$2Ax + 2A + B = 4x$$

You can see that coefficient before *x* in the left hand side is 2A, and in the right hand side is just 2. Coefficient before zero power of the polynomial in the left hand side is *2A+B*, and in right hand side the free term is absent, this means that coefficient is equal to zero.
Thus, we go to the following system

$$\begin{cases} 2A = 4 \\ 2A + 2B = 0 \end{cases},$$

whence $A=2,\ B=-2$, it means that

$$y^{*}=(2x-2)e^{x}.$$

Therefore the general solution has the following form:

$$y=C_1\cos x+C_2\sin x+2(x-1)e^{x}.$$

2) ***Let number*** α ***be simple root of the characteristic equation.*** **This means that** $\alpha=\lambda_1, \alpha\neq\lambda_2$ or $\alpha=\lambda_2, \alpha\neq\lambda_1$

In the considering case we have to take as particular solution the following function:

$$y^{*}=e^{\alpha x}x\,Q_n(x). \qquad (8)$$

Example. Find the general solution of the equation:

$$y''+y'-2y=e^{x}. \qquad (9)$$

Solution. The general solution of the (9) is

$$y=\bar{y}+y^{*} \qquad (10)$$

In order to find the general solution of the homogeneous equation $\bar{y}$, let us construct the characteristic equation of the corresponding homogeneous differential equation

$$\lambda^2+\lambda-2=0$$

This equations has the following roots $\lambda_1=1,\ \lambda_2=-2$ and hence the general solution of an appropriate homogeneous equation can be written in the following form:

$$\bar{y}=C_1e^{x}+C_2e^{-2x}. \qquad (11)$$

Let's find the partial solution of the initial differential equation y^{*}. So, since here function

$$f(x)=e^{1\cdot x}P_0(x), \quad (P_0=1)$$

has the special form of the first type and here $\alpha=1$ $\lambda_1\neq\lambda_2$, but $\alpha=\lambda_1$ then we can put

$$y^* = e^{1\cdot x} x\, Q_0(x),$$

Where $Q_0(x)$ is polynomial of the zero power and its general form is defined just as $Q_0(x) = A$. So partial solution we can search for as

$$y^* = Ae^x x. \qquad (12)$$

Whence we can find derivative of the function (12)

$$\left(y^*\right)' = Ae^x x + Ae^x, \ \left(y^*\right)'' = Ae^x x + 2Ae^x. \qquad (13)$$

Substituting these expressions (12,13) into equation (9) and reducing by e^x we obtain

$$Ax + 2A + Ax + A - 2Ax = 1,$$

that is $3A = 1$, whence $A = \frac{1}{3}$ and hence

$$y^* = \frac{1}{3} e^x x.$$

The general solution (10)is presented in form:

$$y = C_1 e^x + C_2 e^{-2x} + \frac{1}{3} e^x x.$$

3) ***The number*** α ***is multiple (double) root of the characteristic equation*** ($\lambda_1 = \lambda_2 = \alpha$), then partial solution y^* is defined as

$$y^* = e^{\alpha x} Q_n(x) x^2 \quad . \qquad (14)$$

Remark. If the right side of the equation (1) is sum of some terms $f(x) = f_1(x) + f_2(x)$ of the kind $e^{\alpha x} P_n(x)$, then $y^* = y_1^* + y_2^*$,

Where y_1^*, y_2^* are relatively partial solutions of the equations

$$y'' + py' + qy = f_1(x), \qquad (15)$$

$$y'' + py' + qy = f_2(x) \ . \qquad (16)$$

Example. Find the general solution of the equation

$$y'' - 4y' + 4y = e^{2x} - 4. \qquad (17)$$

Solution. The general solution is

$$y = \bar{y} + y_1^* + y_2^*, \quad (18)$$

Because the right part of the equation (17) is sum, of two terms, each of them is function of the special type of the first kind.

First let's find the general solution of the homogeneous differential equation. Construct the characteristic equation

$$\lambda^2 - 4\lambda + 4 = 0,$$

whence $\lambda_1 = \lambda_2 = 2$ and following general solution of HLDE:

$$\bar{y} = C_1 e^{2x} + C_2 e^{2x} x. \quad (19)$$

Now form the partial solution for differential equations

$$y'' - 4y' + 4y = e^{2x} \quad (20)$$

You can see that here $\alpha = 2 = \lambda_1 = \lambda_2$. The degree of the polynomial $P(x)$ is equal to zero. So the particular solution for equation (20) y_1^* is defined as

$$y_1^* = Ae^{2x}x^2 \quad (21)$$

Particular solution for equation

$$y'' - 4y' + 4y = -4, \quad (22)$$

is defined as $y_2^* = B$, because in this case $\alpha = 0, \alpha \neq \lambda_1 \neq \lambda_2$. And we have polynomial $P(x)$of the zero degree in formula (22). That's we have the first case for construction of the partial solution. Therefore

$$y^* = Ae^{2x}x^2 + B.$$

Whence

$$\left(y^*\right)' = 2Ae^{2x}x^2 + 2Ae^{2x}x,$$

$$\left(y^*\right)'' = 4Ae^{2x}x^2 + 8Ae^{2x}x + 2Ae^{2x}.$$

Let us substitute these expressions into initial equation (here we cannot reduce by e^{2x}). As result we get

$$4Ae^{2x}x^2+8Ae^{2x}x+2Ae^{2x}-8Ae^{2x}x^2-8Ae^{2x}x+$$
$$+4Ae^{2x}x^2+4B=e^{2x}-4,$$

i.e.

$$2Ae^{2x}+4B=e^{2x}-4.$$

Whence

$$\begin{cases} 2A=1 \\ 4B=-4 \end{cases},$$

that is $A=\frac{1}{2}$, $B=-1$. It means that

$$y^*=\frac{1}{2}e^{2x}x^2-1.$$

Then the general solution we can write down in the following form:

$$y=C_1e^{2x}+C_2e^{2x}+\frac{1}{2}e^{2x}x^2-1.$$

Example . Find the general solution of the given differential equation:

$$y''-y=x^3e^{2x}.$$

Solution. $P_n(x)=x^3 \Rightarrow n=3; \alpha=2$.

Let us consider the corresponding homogeneous equation

$$y''-y=0.$$

The characteristic equation is $\lambda^2-1=0$, the roots of this equation are $\lambda_1=1,\ \lambda_2=-1$.

Thus the value $\alpha=2$ is not a root of the characteristic equation.

The particular solution of LIDE is searched for as follows:

$$v(x)=(A_0+A_1x+A_2x^2+A_3x^3)e^{2x}.$$

Putting the function $v(x)$ and its derivative $v''(x)$ into the given differential equation we obtain the following identity:

$$2A_2+6A_3x+4(A_1+2A_2x+3A_3x^2)+4(A_0+A_1x+A_2x^2+A_3x^3)-$$
$$-(A_0+A_1x+A_2x^2+A_3x^3)=x^3.$$

Equating coefficients at the same powers of x, we obtain the system of 4 equations:

$$\begin{array}{l|l}
x^3 & 3A_3=1, \\
x^2 & 12A_3+3A_2=0, \\
x^1 & 6A_3+8A_2+3A_1=0, \\
x^0 & 2A_2+4A_1+3A_0=0.
\end{array}$$

Example 2. Find the general solution of LIDE

$$y''+2y'-3y=8xe^x.$$

Solution. $P_n(x)=8x \Rightarrow n=1;\ \alpha=1$.

Let us consider the corresponding LHDE

$$y''+2y'-3y=0.$$

The characteristic equation is $\lambda^2+2\lambda-3=0$, the roots of the equation are $\lambda_1=1,\ \lambda_2=-3$.

Thus the value $\alpha=1$ is a simple real root of the characteristic equation.

The particular solution is searched for as follows:

$$y^*(x)=x\left(A_1+A_0x\right)e^x$$

Let us put the function $v(x)$ and its derivatives $y^{*/}(x)$ and $y^{*//}(x)$ into the given differential equation and obtain the following identity:

$$2A_0+8A_0x+4A_1=8x.$$

From here

$$\begin{array}{l|l}
x^1 & 8A_0=8, \\
x^0 & 2A_0+4A_1=0.
\end{array}$$

$$A_0=1, A_1=-\frac{1}{2}.$$

Then $y^*(x)=\left(x^2-\frac{1}{2}x\right)e^x$. The general solution of the given differential equation is:

$$y(x)=c_1e^x+c_2xe^{-3x}+\left(x^2-\frac{1}{2}x\right)e^x.$$

Examples. Now you can train to construct the particular solution for equation

$y''-3y'+2y=f(x)$, if $f(x)$ is presented by the different expressions. Let's find $\bar{y}$. Characteristic equation is

$\lambda^2-3\lambda+2=0, \quad \Rightarrow \lambda_1=1; \quad \lambda_2=2,$

$\bar{y}=C_1e^x+C_2e^{2x}$

If

1) $f(x)=10e^{-x}, \quad y^{*}=Ae^{-x};$

2) $f(x)=3e^{2x}, \quad y^{*}=A\,x\,e^{2x}$

3) $f(x)=2x^{3}-30, \quad y^{*}=A\,x^{3}+Bx^{2}+Cx+D$

4) $f(x)=x+e^{-2x}+1, \quad f_1(x)=x+1; \quad f_2(x)=e^{-2x},$
$y_1^{*}=Ax+B, \quad y_2^{*}=Ce^{-2x}$

5) $f(x)=e^{x}(3-4x), \quad y^{*}=x\,e^{2x}(Ax+B)$

Self-Service Examples

Task	**Answers**
Find the general solution of the equations:	
10.1. $2y''+y'-y=2e^{x}$.	$y=C_1e^{-x}+C_2e^{\frac{x}{2}}+e^{x}$.
10.3. $y''-6y'+9y=2x^{2}-x+3$.	$y=(C_1+C_2x)e^{3x}+\frac{2}{9}x^{2}+\frac{5}{27}x+\frac{11}{27}$.
10.4. $y''+4y'-5y=1$.	$y=C_1e^{x}+C_2e^{-5x}-0{,}2$.
Find the particular solution of the equations satisfying given initial conditions:	
10.6. $4y''+16y'+15y=4e^{-\frac{3}{2}x}$, $y(0)=3,\ y'(0)=-5{,}5$.	$y=(1+x)e^{-\frac{3}{2}x}+2e^{-\frac{5}{2}x}$.
10.7. $y''-y'=2(1-x)$, $y(0)=1,\ y'(0)=1$.	$y=e^{x}+x^{2}$.
10.11. $y''-2y'+2y=2x$	$y=e^{x}(C_1\cos x+C_2\sin x)+x+1$
10.15. $y''-2y'=e^{x}(x^{2}+x-3)$, $y(0)=2,\ y'(0)=2$.	$y=e^{x}(e^{x}-x^{2}-x+1)$.

The Linear Differential Equation of the Second Order with Constant Coefficients and Special Right Part of the second type.

II. The Right Side of the Differential Equation of the Second Kind. Let the right side of the differential equation

$$y'' + p(x)y' + q(x)y = f(x) \qquad (1)$$

have the form

$$f(x) = e^{\alpha x}[P_n(x)\cos\beta x + Q_m(x)\sin\beta x], \qquad (2)$$

where $P_n(x)$ and $Q_m(x)$ are polynomials *n-th* and *m-th* degree relatively. It may be shown that in this case the form of the particular solution is determined as follows:

1) *If the number* $\alpha+\beta i$ ***is not a root of the characteristic equation***, then the particular solution of the equation (1) should be sought for in the form

$$y^* = e^{\alpha x}(U_s(x)\cos\beta x + V_s(x)\sin\beta x), \qquad (3)$$

where $U_s(x)$ and $V_s(x)$ are polynomials of the same degree equal to the highest degree of the polynomials $P_n(x)$ and $Q_m(x)$, that is $s = \max(n,m)$.

2) *If the number* $\alpha+\beta i$ *is a root of the characteristic equation*, then we write the particular solution in the form

$$y^* = x\,e^{\alpha x}(U_s(x)\cos\beta x + V_s(x)\sin\beta x). \qquad (4)$$

Note. Here in order to avoid mistakes we must remember that expressions (3) and (4) are retained when one of the polynomials $P_n(x)$ or $Q_m(x)$ on the right side of equation (2.23) is identically zero; that is when the right side has the following form

$$f(x) = P_n(x)e^{\alpha x}\cos\beta x$$

or

$$f(x) = Q_m(x)e^{\alpha x}\sin\beta x.$$

Consider now an important *special case*. Let the right side of the second order LDE has the form:

$$f(x) = M\cos\beta x + N\sin\beta x, \tag{5}$$

where *M* and *N* are constants:

1) *if βi is not a root of the characteristic equation,* then the particular solution should be sought for in the form

$$y^* = A\cos\beta x + B\sin\beta x. \tag{6}$$

2) *if βi is a root of the characteristic equation,* then the particular solution should be sought for in the form

$$y^* = x(A\cos\beta x + B\sin\beta x). \tag{7}$$

We remark that the function (5) is a special case of the right side of the second kind $(P(x) = M,\ Q(x) = N,\ \alpha = 0)$, the functions (6) and (7) are special cases of the function (3) and (4).

Example. Find the general solution of the INHLDE.

$$y'' + 2y' + 5y = 2\cos x. \tag{8}$$

Solution. The general solution of the (8) is

$$y = \bar{y} + y^*$$

In order to find the general solution of the *homogeneous equation* $\bar{y}$, let us construct the characteristic equation

$$\lambda^2 + 2\lambda + 5 = 0$$

and find roots of the square equation

$$\lambda_1 = -1 \pm 2i.$$

The general solution of *homogenous equation* has the following form

$$\bar{y} = e^{-x}(c_1\cos 2x + c_2\sin 2x).$$

Let's analyze the right part of the given equation (8) and compare with function (2). There is only polynomial *P(x)* of the zero power $P(x)=2$, polynomial $Q(x)=0$. So *max(m,n)= 0* . Parameter $\alpha=0$, and parameter $\beta=1$. Construct number $\alpha+\beta i=0+i=i$. Number *i* is not a root of the characteristic equation, therefore we will seek for the particular solution in the form:

$$y^{*}=A\cos x+B\sin x,$$

where A and B are unknown constant coefficients, which should be determined. Putting y^{*} into the given equation we get

$$-A\cos x-B\sin x+2(-A\sin x+B\cos x)+$$
$$+5(A\cos x+B\sin x)=2\cos x.$$

Equating the coefficients of $\cos x$ and $\sin x$ we obtain two equations for determining A and B:

$$-A+2B+5A=2,$$
$$-B-2A+5B=0.$$

Whence $A=\frac{2}{5}$; $B=\frac{1}{5}$. The general solution of the given equation is $y=\bar{y}+y^{*}$, that is,

$$y=e^{-x}(c_1\cos 2x+c_2\sin 2x)+\frac{2}{5}\cos x+\frac{1}{5}\sin x.$$

Example 2. Find the general solution of the given INHLIDE

$$y''+y=x\cos x+\sin x. \qquad (9)$$

Solution. Let us consider the corresponding homogeneous differential equation

$$y''+y=0.$$

Characteristic equation is

$$\lambda^2+1=0,\ \lambda_1=i,\ \lambda_2=-i. \qquad (9a)$$

The general solution of the homogeneous differential equation $\bar{y}$ is defined as:

$$\bar{y}=C_1\cos x+C_2\sin x, \qquad (10)$$

The value $\alpha + i\beta = i$ is a root of the characteristic equation. Let 's analyse the right part of the differential equation (9)

$$f(x) = x\cos x + \sin x,$$

$$\Rightarrow P_1(x) = x, \quad Q_0(x) = 1; \Rightarrow \max(n,m) = 1; \quad \alpha = 0, \beta = 1.$$

Form the expression $\alpha \pm \beta i = i$ and compare with roots of the characteristic equation. You can see that this expression coincides with roots of the characteristic equation (9a). So the particular solution we will search, applying the formula (4), that's as follows

$$y^*(x) = x((Ax+B)\cos x + (Cx+D)\sin x).$$

Putting the function $y^*(x)$ and its derivatives $y^{*/}(x)$ and $y^{*//}(x)$ into the given differential equation, we obtain the following identity

$$2C\sin x + 2A\cos x + (4Cx + 2D)\cos x - (4Ax + 2B)\sin x = x\cos x + \sin x.$$

From here

$$\begin{array}{r|c} \cos x & 2A + 2D = 0, \\ x\cos x & 4C = 1, \\ \sin x & 2C - 2B = 1, \\ x\sin x & -4A = 0. \end{array}$$

$$A = 0,\ D = 0, C = \frac{1}{4},\ B = -\frac{1}{4}.$$

Then, $y^*(x) = \frac{1}{4}x^2\sin x - \frac{1}{4}x\cos x$. The general solution of the differential equation is:

$$y(x) = c_1\cos x + c_2\sin x + \frac{1}{4}x^2\sin x - \frac{1}{4}x\cos x.$$

Self-Service Examples

10.2. $y'' - 7y' + 6y = \sin x$.	$y = C_1e^{6x} + C_2e^x + \dfrac{5\sin x + 7\cos x}{74}$.
10.10. $y'' + 2y' + 5y = -\dfrac{7}{12}\cos 2x$	$y = e^{-x}(C_1\cos 2x + C_2\sin 2x) - -\dfrac{1}{2}\cos 2x - 2\sin 2x$.
$y'' + 9y = \cos 3x$	
$y'' + y = \cos x\cos 2x$	