Homogeneous Linear Equations of the Second Order with Constant Coefficients

Let us consider an equation

$$y'' + py' + qy = 0, \qquad (1)$$

where $p = const$ and $q = const$.

In order to solve this equation we should construct characteristic equations. Characteristic equation may be obtained by change derivatives into (1) by corresponding power of the parameter λ. Note that function y we can consider as derivative of the zero power. So we get characteristic equation:

$$\lambda^2 + p\lambda + q = 0 \qquad (2)$$

While solving a characteristic equation the following cases are

possible:

1. Roots of a characteristic equation λ_1, λ_2 **are *real and different,*** then the differential equation (1) has two linearly independent, particular solutions $y_1 = e^{\lambda_1 x}, y_2 = e^{\lambda_2 x}$.

The general solution of the differential equation (1) is as follows

$$y(x) = c_1 y_1 + c_2 y_2$$

$$y(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}. \qquad (3)$$

2. Roots of a characteristic equation *are real and equal each to other* . Let $\lambda_1 = \lambda_2 = \lambda$, i.e. λ is a real root of the multiplicity 2.

Then 2 particular linearly independent solutions of the differential equation (1) correspond to the real root λ of the multiplicity 2:

$$y_1 = e^{\lambda x}, y_2 = xe^{\lambda x}.$$

In this case the general solution of the differential equation (1) is as follows:

$$y(x) = c_1 y_1 + c_2 y_2$$

or

$$y(x) = c_1 e^{\lambda x} + c_2 x e^{\lambda x}. \qquad (4)$$

3. Roots of a characteristic equation *are complex conjugate ones*

Let $\lambda_1 = \alpha + i\beta, \lambda_2 = \alpha - i\beta$.

Two particular, linearly independent solutions of the differential equation correspond to these complex conjugate roots:

$$y_1 = e^{\alpha x}\cos\beta x, y_2 = e^{\alpha x}\sin\beta x.$$

In this case the general solution of the differential equation (1) looks like:

$$y(x) = c_1 e^{\alpha x}\cos\beta x + c_2 e^{\alpha x}\sin\beta x. \qquad (5)$$

Or

$$y(x) = e^{\alpha x}\left(c_1\cos\beta x + c_2\sin\beta x\right). \qquad (6)$$

Examples. ***Find the general solution of the following equations:***

1. $y'' + y' - 2y = 0$

Solution. Let's form the characteristic equations and find its roots

$\lambda^2 + \lambda - 2 = 0$

$\lambda_1 = -2, \quad \lambda_1 = 1.$

Now we can construct fundamental system solution (two linear independent partial solution):

$$y_1 = e^{-2x}; \quad y_2 = e^{x}. \qquad (7)$$

If we know fundamental system of the solutions, then general solution is their linear combination. That's

$$y = C_1 y_1 + C_2 y_2 \quad (8)$$

Substitute (7) into (8) and get the general solution

$$y = C_1 e^{-2x} + C_2 e^{x}$$

2. $3y'' - 2y' - 8y = 0$

Solution. Let's form the characteristic equations and find its roots

$3\lambda^2 - 2\lambda - 8 = 0$

$\lambda_{1,2} = \frac{1 \pm \sqrt{1+24}}{3}, \quad \Rightarrow \quad \lambda_1 = \frac{1+5}{3} = 2, \ \lambda_2 = \frac{1-5}{3} = -\frac{4}{3}$

$\lambda_1 = 2, \quad \lambda_2 = -\frac{4}{3}$.

Now we can construct fundamental system solution (two linear independent partial solution) taking into account that **roots are real and different :**

$$y_1 = e^{2x}; \quad y_2 = e^{-\frac{4}{3}x}. \quad (9)$$

General solution is linear combination of the obtained solutions. That's

$$y = C_1 y_1 + C_2 y_2 \quad (10)$$

Substitute (9) into (10) and get the general solution

$$y = C_1 e^{2x} + C_2 e^{-\frac{4}{3}x}$$

3. $y'' + 6y' + 9y = 0$

Solution. Let’s form the characteristic equations and find its roots

$\lambda^2 + 6\lambda + 9 = 0$

$(\lambda + 3)^2 = 0, \quad \Rightarrow \quad \lambda_1 = -3, \; \lambda_2 = -3$

The roots of the characteristic equation are the same. The fundamental system in the case is defined as:

$$y_1 = e^{-3x}; \quad y_2 = x\, e^{-3x}. \qquad (11)$$

General solution is linear combination of the obtained solutions. That’s

$$y = C_1 y_1 + C_2 y_2 \qquad (12)$$

Substitute (11) into (12) and get the general solution

$$y = C_1 e^{-3x} + C_2 x\, e^{-3x}$$

or

$$y = e^{-3x}(C_1 + C_2 x).$$

4. $y'' + 6y' + 13y = 0$

Solution. Let’s form the characteristic equations and find its roots

$\lambda^2 + 6\lambda + 13 = 0$

$\lambda_{1,2} = -3 \pm \sqrt{9 - 13}, \quad \Rightarrow \quad \lambda_1 = -3 + \sqrt{-4} = -3 + 2\sqrt{-1} = -3 + 2i = -3 + 2i,$
$\lambda_2 = -3 - 2i$

You see that given characteristic equation has a complex conjugate root. **Real part of the root is** $\alpha = -3,$ **and Image part is** $\beta = 2,$

In this case fundamental system may be written as :

$$y_1 = e^{\alpha x} \cos \beta x; \quad y_2 = e^{\alpha x} \sin \beta x. \qquad (13)$$

Or $y_1 = e^{-3x}\cos 2x; \quad y_2 = e^{-3x}\sin 2x$

General solution is linear combination of the obtained solutions. That's

$$y = C_1 y_1 + C_2 y_2 \tag{14}$$

Substitute (13) into (14) and get the general solution

$$y = C_1 e^{-3x}\cos 2x + C_2\, e^{-3x}\sin 2x$$

Or

or

$$y = e^{-3x}(C_1\cos 2x + C_2\sin 2x).$$

5. Find the particular solution of the equations

$$y'' + 4y' + 29y = 0,$$

satisfying given initial conditions.

$$y(0) = 0, \; y'(0) = 15$$

Solution. Let's form the characteristic equations and find its roots

$\lambda^2 + 4\lambda + 29 = 0$

$\lambda_{1,2} = -2 \pm \sqrt{4-29}, \quad \Rightarrow \quad \lambda_1 = -2 + \sqrt{-25} = -2 + 5\sqrt{-1} = -2 + 5i,$
$\lambda_2 = -2 - 5i$

You see that given characteristic equation has a complex conjugate root. **Real part of the root is** $\alpha = -2,$ **and Image part is** $\beta = 5,$

In this case fundamental system may be written as :

$y_1 = e^{-2x}\cos 5x; \quad y_2 = e^{-2x}\sin 5x$

General solution is linear combination of the obtained solutions. That's

$$y = C_1 y_1 + C_2 y_2 \qquad (15)$$

Substitute (13) into (14) and get the general solution

$$y = e^{-2x}(C_1 \cos 5x + C_2 \sin 5x) \qquad (16)$$

In order to find partial solution, satisfying initial condition, we should substitute values $y(0) = 0$, $y'(0) = 15$ into (16)

We have

$$e^{-2\times 0}(C_1 \cos 5\times 0 + C_2 \sin 5\times 0) = 0, \quad \Rightarrow C_1 = 0;$$

So

$$y = C_2 e^{-2x} \sin 5x.$$

Calculate derivative

$$y' = (C_2 e^{-2x} \sin 5x)' = C_2\left(-2e^{-2x} \sin 5x + 5e^{-2x} \cos 5x\right) =$$
$$C_2 e^{-2x}(5\cos 5x - 2\sin 5x)$$

$$y' = C_2 e^{-2x}(5\cos 5x - 2\sin 5x),$$

$$y'(0) = 15, \quad \Rightarrow C_2 e^{-2\times 0}(5\cos 5\times 0 - 2\sin 5\times 0) = 15, \quad 5C_2 = 15; \quad C_2 = 3$$

The partial solution is

$$y = 3e^{-2x} \sin 5x$$

Below we present tasks for train

Self-Service Examples

Task	Answers
Find the general solution of the equations:	
9.2. $y'' - 4y' = 0$.	$y = C_1 e^{4x} + C_2$.
9.5. $y'' - 2y' + y = 0$.	$y = e^x (C_1 + C_2 x)$.
9.6. $2y'' + y' +$ $+ 2\sin^2 15^\circ \cos^2 15^\circ\, y = 0$.	$y = e^{-\frac{1}{4}x} (C_1 + C_2 x)$.
9.8. $y'' - 9y = 0$.	$y = C_1 e^{3x} + C_2 e^{-3x}$.
9.9. $y'' - 2y' - y = 0$.	$y = C_1 e^{(1+\sqrt{2})x} + C_2 e^{(1-\sqrt{2})x}$.
9.11 $4y'' - 8y' + 5y = 0$	$y = e^x \left(C_1 \cos\frac{x}{2} + C_2 \sin\frac{x}{2} \right)$
9.12. $4\frac{d^2x}{dt^2} - 20\frac{dx}{dt} + 25x = 0$.	$x = e^{2.5t} (C_1 + C_2 t)$.
Find the particular solution of the equations satisfying given initial conditions:	
9.7. $y'' - 4y' + 3y = 0$, $y(0) = 6$, $y'(0) = 10$.	$y = 4e^x + 2e^{3x}$.
9.14. $4y'' + 4y' + y = 0$, $y(0) = 2$, $y'(0) = 0$.	$y = e^{-\frac{x}{2}} (2 + x)$.