Individual Task #3

Example of solution.

1) Find indefinite integral

- $\displaystyle \int \frac{5x+1}{\sqrt{9x^2+1}}\,dx = \int \frac{5x}{\sqrt{9x^2+1}}\,dx + \int \frac{1}{\sqrt{(3x)^2+1}}\,dx = \left[\begin{matrix} d(9x^2+1)= \\ =18x\,dx \end{matrix}\right] =$

$$= \frac{5}{18}\int \frac{\overset{d(9x^2+1)}{18x\,dx}}{\underset{\int\frac{du}{\sqrt{u}}}{\sqrt{9x^2+1}}} + \int \frac{dx}{\sqrt{(\underset{ax+b}{3x})^2+1}} = \frac{5}{18}\cdot 2\sqrt{9x^2+1} + \frac{\ln\left|3x+\sqrt{(3x)^2+1}\right|}{3} + c$$

or

- $\displaystyle \int \frac{2x\cos x - x^2\sin x}{(x^2\cos x)^3}\,dx = \left[\begin{matrix} d(x^2\cos x) = \\ =(2x\cos x - x^2\sin x)\,dx \\ \Rightarrow u = x^2\cos x \end{matrix}\right] = \int \frac{du}{u^3} =$

$$= \int u^{-3}\,du = \frac{u^{-2}}{-2} + c = -\frac{1}{2u^2} + c = -\frac{1}{2(x^2\cos x)^2} + c$$

2) Find ind. integral

- $\displaystyle \int \arctan\sqrt{5x+1}\,dx = \left[\begin{matrix} 5x+1 = t^2 \\ d(5x+1) = dt^2 \\ 5\,dx = 2t\,dt \end{matrix}\right] = \int \underbrace{\arctan t}_{u} \cdot \underbrace{\frac{2t}{5}\,dt}_{dv} =$

$$= \left[\begin{matrix} du = \frac{1}{1+t^2}\,dt \\ v = \frac{2}{5}\frac{t^2}{2} = \frac{t^2}{5} \end{matrix}\right] = \frac{t^2}{5}\arctan t - \frac{1}{5}\int \frac{t^2+1-1}{1+t^2}\,dt =$$

$$= \frac{t^2}{5}\arctan t - \frac{1}{5}\int\left(1 - \frac{1}{1+t^2}\right)dt = \frac{t^2}{5}\arctan t -$$

$$- \frac{1}{5}(t - \arctan t) + c = \frac{(5x+1)}{5}\arctan\sqrt{5x+1} -$$

$$- \frac{1}{5}\sqrt{5x+1} + \frac{1}{5}\arctan\sqrt{5x+1} + c.$$

or

- $\displaystyle \int \underbrace{x}_{u}\,\underbrace{\cos x\,dx}_{dV} = \left[\begin{matrix} du = 1\,dx \\ v = \sin x \end{matrix}\right] = x\sin x - \int \sin x\,dx = x\sin x + \cos x + c$

3) • $\int \frac{2x+1}{\sqrt{25x^2+10x+2}}dx = \left[\begin{array}{l} \underbrace{25x^2}_{A^2} + \underbrace{10x}_{2AB} + 1^2 - 1^2 + 2 \\ A = 5x \quad B = \frac{10x}{2A} = 1 \end{array}\right] =$

$= \int \frac{2x+1}{\sqrt{(5x+1)^2+1}}dx = \left[\begin{array}{l} 5x+1 = t \\ 5\,dx = dt \\ x = (t-1)/5 \end{array}\right] = \int \frac{2\frac{t-1}{5}+1}{\sqrt{t^2+1}} \frac{dt}{5} =$

$= \frac{1}{25}\int \frac{2t-2+5}{\sqrt{t^2+1}}dt = \frac{1}{25}\int \frac{\overbrace{2t\,dt}^{d(t^2+1)}}{\sqrt{t^2+1}} + \frac{3}{25}\int \frac{dt}{\sqrt{t^2+1}} =$

$= \frac{1}{25}\cdot 2\sqrt{t^2+1} + \frac{3}{25}\ln\left|t+\sqrt{t^2+1}\right| + c =$

$= \frac{2}{25}\sqrt{(5x+1)^2+1} + \frac{3}{25}\ln\left|5x+1+\sqrt{(5x+1)^2+1}\right| + c$

4) • $\int \frac{(x^3+2)\,dx}{x^3-x} = \int\left(1 + \frac{x+2}{x^3-x}\right)dx \ominus$

$\begin{array}{r|l} x^3+2 & x^3-x \\ -\;x^3-x & 1 \\ \hline x+2 & \end{array}$

$\frac{x+2}{x(x^2-1)} = \frac{x+2}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$

(x-1)(x+1)   x(x+1)   x(x-1)

$x+2 = A(x-1)(x+1) + Bx(x+1) + Cx(x-1)$

$x=0: \; 2 = -A \quad A=-2$

$x=1: \; 3 = B\cdot 1\cdot 2 \quad B = \frac{3}{2}$

$x=-1: \; 1 = C\cdot(-1)(-2) \quad C = \frac{1}{2}$

$\ominus \int\left(1 + \frac{-2}{x} + \frac{\frac{3}{2}}{x-1} + \frac{\frac{1}{2}}{x+1}\right)dx = x - 2\ln|x| + \frac{3}{2}\ln|x-1| +$
$+ \frac{1}{2}\ln|x+1| + c.$

• $\int \frac{(3x+2)\,dx}{x^3+x} \ominus \quad \frac{3x+2}{x^3+x} = \frac{3x+2}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$

x^2+1 x

$3x+2 = A(x^2+1) + Bx^2 + Cx$

$x^2: \; 0 = A+B \Rightarrow B = -A = -2$

$x^1: \; 3 = C$

$x^0: \; 2 = A$

$\ominus \int\left(\frac{2}{x} + \frac{-2x+3}{x^2+1}\right)dx = \int \frac{2}{x}dx - \int\frac{\overbrace{2x\,dx}^{d(x^2+1)}}{x^2+1} + 3\int\frac{dx}{x^2+1} =$

$= 2\ln|x| - \ln|x^2+1| + 3\arctan x + c$

5) • $\int \frac{dx}{4\cos x + \sin x + 1} = \left[\begin{array}{ll} t = \tan\frac{x}{2} & dx = \frac{2dt}{1+t^2} \\ \cos x = \frac{1-t^2}{1+t^2} & \sin x = \frac{2t}{1+t^2} \end{array}\right] =$

$= \int \frac{\frac{2dt}{1+t^2}}{\frac{4-4t^2}{1+t^2} + \frac{2t}{1+t^2} + \frac{1+t^2}{1+t^2}} = \int \frac{2\,dt}{4-4t^2+2t+1+t^2} = \int \frac{2\,dt}{-3t^2+2t+5} =$

$= -2\int \frac{dt}{\underbrace{3t^2}_{A^2} - \underbrace{2t}_{2AB} + \left(\frac{1}{\sqrt{3}}\right)^2 - \left(\frac{1}{\sqrt{3}}\right)^2 + 5} = -2\int \frac{dt}{\left(\sqrt{3}t - \frac{1}{\sqrt{3}}\right)^2 + \underbrace{\frac{14}{3}}_{a^2}} =$ $\quad a = \sqrt{\frac{14}{3}}$

$A = \sqrt{3}t \quad B = \frac{2t}{2A} = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}}$

$= -2 \cdot \frac{1}{\sqrt{\frac{14}{3}}} \arctan \frac{\sqrt{3}t - \frac{1}{\sqrt{3}}}{\frac{\sqrt{14}}{\sqrt{3}}} + C = -\frac{2\sqrt{3}}{\sqrt{14}} \arctan\left(\frac{3\tan\frac{x}{2} - 1}{\sqrt{14}}\right) + C$

6) • $\int \cos^5(3x)\,dx = \left[\begin{array}{l} t = \sin 3x \\ dt = 3\cos 3x\,dx \\ \cos^4 3x = (1-\sin^2 3x)^2 \end{array}\right] = \int \underbrace{\cos^4 3x}_{(1-t^2)^2} \cdot \underbrace{\cos 3x\,dx}_{\frac{dt}{3}}$

$= \frac{1}{3}\int (1 - 2t^2 + t^4)\,dt = \frac{1}{3}\left(t - 2\frac{t^3}{3} + \frac{t^5}{5}\right) + C =$

$= \frac{1}{3}\left(\sin 3x - \frac{2}{3}\sin^3 3x + \frac{1}{5}\sin^5 3x\right) + C$

• $\int \cos^4 5x\,dx = \int (\cos^2 5x)^2\,dx = \int \left(\frac{1+\cos 10x}{2}\right)^2 dx =$

$= \frac{1}{4}\int (1 + 2\cos 10x + \cos^2 10x)\,dx = \frac{1}{4}\int \left(1 + 2\cos 10x + \frac{1+\cos 20x}{2}\right) dx =$

$= \frac{1}{4}\left(x + 2\frac{\sin 10x}{10} + \frac{1}{2}x + \frac{1}{2}\frac{\sin 20x}{20}\right) + C$

7) $\int_0^{1/\sqrt{2}} \frac{dx}{\sqrt{(1-x^2)^3}} = \left[\begin{array}{l} x = \sin t \\ dx = \cos t\,dt \\ 1 - x^2 = \cos^2 t \\ x = 0 \Leftrightarrow t = 0 \\ x = 1/\sqrt{2} \Leftrightarrow t = \frac{\pi}{4} \end{array}\right] = \int_0^{\pi/4} \frac{\cos t\,dt}{\cos^3 t} = \int_0^{\pi/4} \frac{dt}{\cos^2 t} =$

$= \tan t \Big|_0^{\pi/4} = \tan\frac{\pi}{4} - \tan 0 = 1 - 0 = 1$

8) Find area of figure bounded by lines:

$$y = x\sqrt{25-x^2}, \quad y=0 \quad (0 \le x \le 5).$$

Since $y(0)=0\cdot\sqrt{25-0^2}=0$ and $y(5)=5\cdot\sqrt{25-5^2}=0$

and $x\sqrt{25-x^2}>0$ for $x\in(0,5)$, then

$$S=\int_0^5\Big(\underbrace{x\sqrt{25-x^2}}_{y_2(x)}-\underbrace{0}_{y_1(x)}\Big)dx=\int_0^5 x\sqrt{25-x^2}\,dx=\begin{bmatrix} d(25-x^2)= \\ =-2x\,dx\end{bmatrix}=$$

$$=-\frac12\int_0^5\sqrt{25-x^2}\cdot\underbrace{(-2x)dx}_{d(25-x^2)}=-\frac12\,\frac{(25-x^2)^{\frac12+1}}{\frac12+1}\Big|_0^5=$$

$$=-\frac{1}{\cancel{2}}\cdot\frac{\cancel{2}(25-25)^{3/2}}{3}-\left(-\frac{1}{\cancel{2}}\right)\frac{\cancel{2}(25-0)^{3/2}}{3}=\frac{125}{3}.$$

9) Calculate the arc length if $y=\frac{x^2}{4}-\frac{\ln x}{2}$. $(1\le x\le 9)$

$$L=\int_1^9\sqrt{1+(y')^2}\,dx$$

$y'=\frac{2x}{4}-\frac{1}{2x}$; $1+(y')^2=1+\left(\frac x2-\frac{1}{2x}\right)^2=1+\frac{x^2}{4}-\frac24+\frac{1}{4x^2}=$

$$=\frac{x^2}{4}+\frac12+\frac{1}{4x^2}=\left(\frac x2+\frac{1}{2x}\right)^2$$

$$L=\int_1^9\sqrt{\left(\frac x2+\frac{1}{2x}\right)^2}\,dx=\int_1^9\Big|\underbrace{\frac x2+\frac{1}{2x}}_{>0}\Big|\,dx=\int_1^9\left(\frac x2+\frac12\,\frac1x\right)dx=$$

$$=\left(\frac{x^2}{4}+\frac12\ln|x|\right)\Big|_1^9=\frac{81}{4}-\frac14+\frac12\ln 9-\frac12\ln 1=20+\ln 3$$

10) Calculate the volume of the body obtained by revolution of a ~~curve~~ figure around the axis Ox (or OY).

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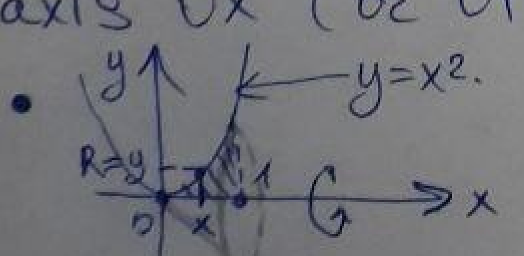
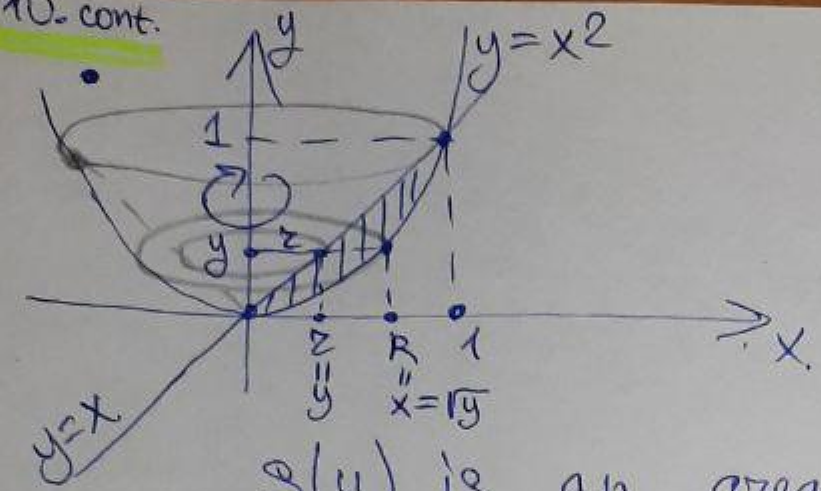


$0\le x\le 1$ $\quad D_x$ is a circle, obtained by intersection of ~~the~~ body and plane $x=\text{const}$.

$$S(x)=S_{D_x}=\pi R^2=\pi(y(x))^2=\pi(x^2)^2=\pi x^4$$

$$V=\int_0^1\pi x^4\,dx=\pi\frac{x^5}{5}\Big|_0^1=\frac{\pi}{5}.$$

10. cont.



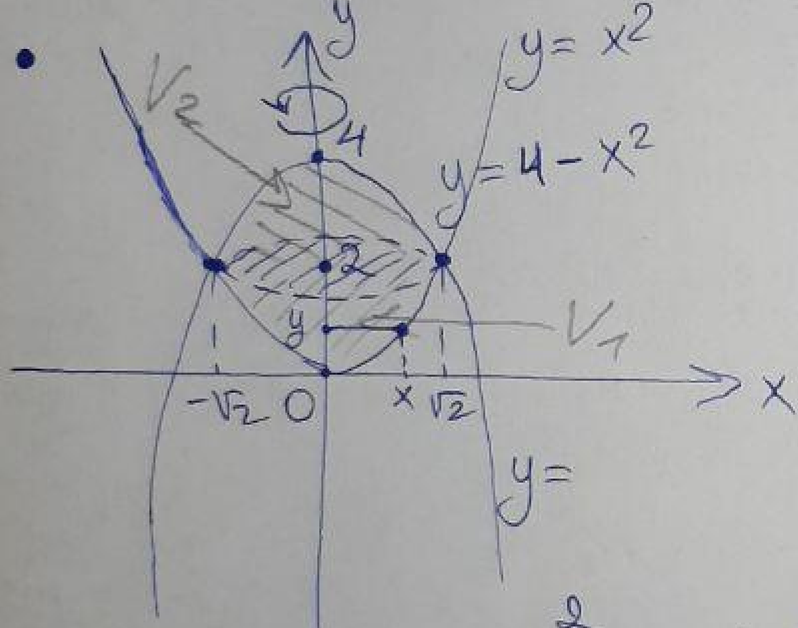
Let us consider sections of this body by plane $y = \text{const}$. Then

$$V = \int_0^1 S(y)\,dy.$$

$S(y)$ is an area of the ring: 

$$S(y) = \pi R^2 - \pi r^2 = \pi(\sqrt{y})^2 - \pi(y)^2 = \pi y - \pi y^2.$$

$$V = \pi\int_0^1 (y - y^2)\,dy = \pi\left(\frac{y^2}{2} - \frac{y^3}{3}\right)\Big|_0^1 = \pi\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{\pi}{6}.$$



$$V = V_1 + V_2$$

Let us find points of intersection of these lines:

$$\begin{cases} y = x^2 \\ y = 4 - x^2 \end{cases} \Rightarrow 2y = 4, \quad y = 2, \quad x = \pm\sqrt{2}.$$

So, $V = \int_0^2 S_1(y)\,dy + \int_2^4 S_2(y)\,dy$.

$$S_1(y) = \pi R_1^2 = \left[\begin{array}{l} R_1 = x \text{ on the curve } y = x^2, \\ \text{i.e. } R_1 = \sqrt{y} \end{array}\right] = \pi(\sqrt{y})^2 = \pi y.$$

$$S_2(y) = \pi R_2^2 = \left[\begin{array}{l} R_2 = x \text{ on the curve } y = 4 - x^2, \\ \text{thus } R_2 = \sqrt{4-y} \end{array}\right] = \pi(\sqrt{4-y})^2 = \pi(4-y)$$

$$V = \int_0^2 \pi y\,dy + \int_2^4 \pi(4-y)\,dy = \pi\frac{y^2}{2}\Big|_0^2 + \pi\left(4y - \frac{y^2}{2}\right)\Big|_2^4 =$$

$$= 2\pi - 0 + \pi\left(16 - \frac{16}{2}\right) - \pi\left(8 - \frac{4}{2}\right) = 2\pi + 8\pi - 6\pi = 4\pi \text{ (cub. units)}$$