Vector Algebra. Practice 3.

Mixed product.

$(\bar{a}, \bar{b}, \bar{c})$ is scalar.

Task 1. Are the vectors $\bar{a}, \bar{b}, \bar{c}$ complanar or not? $\vec{a}(-7, 10, -5)$, $\bar{b}(0, -2, -1)$ $\bar{c}(-2; 4; -1)$

Solution. If vectors are complanar then their mixed product is equal to 0.

So our task is check value of mixed product of vectors $\bar{a}, \bar{b}, \bar{c}$.

If vectors are given by their coordinates then

$$(\bar{a}, \bar{b}, \bar{c}) = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$$

$$(\bar{a}, \bar{b}, \bar{c}) = \begin{vmatrix} -7 & 10 & -5 \\ 0 & -2 & -1 \\ -2 & 4 & -1 \end{vmatrix} = \begin{vmatrix} e_2(-5) + e_1 \to e_1 \\ e_2(-1) + e_3 \to e_3 \end{vmatrix} =$$

$$= \begin{vmatrix} -7 & 20 & 0 \\ 0 & -2 & -1 \\ -2 & 6 & 0 \end{vmatrix} = (-1) \cdot (-1)^5 \begin{vmatrix} -7 & 20 \\ -2 & 6 \end{vmatrix} =$$

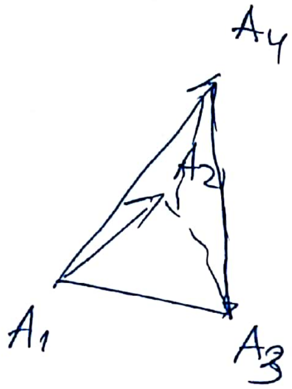
$$= -42 + 40 + 0$$

Vectors $\bar{a}, \bar{b}, \bar{c}$ are not complanar.

Task 2. Find the volume of the tetrahedron with vertices A_1, A_2, A_3, A_4 and its altitude dropped from A_4 on the base $A_1A_2A_3$.

$A_1(2;-4,-3)$, $A_2(5,-6,0)$, $A_3(-1,3,-3)$, $A_4(-10,-8,7)$.

Solution.



$$V_{tetr} = \frac{1}{6} V_{par} = \frac{1}{6} \left| (\bar{a}, \bar{b}, \bar{c}) \right|.$$

We take modulus because vectors can make left hand triple.

Find coordinates of vectors $\overline{A_1A_4}$, $\overline{A_1A_2}$, $\overline{A_1A_3}$

$\overline{A_1A_4} = (-12, -4, 10)$

$\overline{A_1A_2} = (3;-2,3)$

$\overline{A_1A_3} = (-3, 7, 0)$.

Mixed product

$$(\overline{A_1A_2}, \overline{A_1A_3}, \overline{A_1A_4}) = \begin{vmatrix} 3 & -2 & 3 \\ -3 & 7 & 0 \\ -12 & -4 & 10 \end{vmatrix} =$$

$$= \left| \begin{matrix} l_1 + l_2 \to l_2 \\ l_1 \cdot 4 + l_3 \to l_3 \end{matrix} \right| = \begin{vmatrix} 3 & -2 & 3 \\ 0 & 5 & 3 \\ 0 & -12 & 22 \end{vmatrix} = 3 \cdot (-1)^2 \begin{vmatrix} 5 & 3 \\ -12 & 22 \end{vmatrix} =$$

$$= 3 \cdot 146$$

Then volume of tetrahedron is equal to ③

$$V = \frac{1}{\cancel{6}}\left(\overline{A_1A_2},\ \overline{A_1A_3},\ \overline{A_1A_4}\right) =$$
$$= \frac{1}{2\cancel{6}} \cdot \cancel{3} \cdot 146 = \frac{146}{2} = 73.$$

To find altitude let's present volume as

$$V = \frac{1}{3} S_\Delta \cdot h$$

S_Δ is area of base. As this triangle constructed on vectors $\overline{A_1A_2}$ and $\overline{A_1A_3}$ then we can calculate it as $\frac{1}{2}$ of modulus of corresponding vector product

$$|\overline{A_1A_2} \times \overline{A_1A_3}| \div$$

Vector product is

$$\overline{A_1A_2} \times \overline{A_1A_3} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -2 & 3 \\ -3 & 7 & 0 \end{vmatrix} = \vec{i}\begin{vmatrix} -2 & 3 \\ 7 & 0 \end{vmatrix} - \vec{j}\begin{vmatrix} 3 & 3 \\ -3 & 0 \end{vmatrix} +$$

$$+ \vec{k}\begin{vmatrix} 3 & -2 \\ -3 & 7 \end{vmatrix} = -21\vec{i} - 9\vec{j} + 15\vec{k}$$

$$|\overline{A_1A_2} \times \overline{A_1A_3}| = \sqrt{(-21)^2 + (-9)^2 + 15^2} = \sqrt{441 + 81 + 225}$$
$$= \sqrt{747}$$

$$S_\Delta = \frac{1}{2}\sqrt{746}$$

$$h = \frac{3V}{S_\Delta} = \frac{3 \cdot 73}{\frac{1}{2}\sqrt{746}} = \frac{6 \cdot 73}{\sqrt{746}}$$