Vector Algebra. Practice 1.

Task 1. Find coordinates of vector $\overline{M_1M_2}$ if $M_1(-1;1;3)$, $M_2(0;-7;4)$. Find lenght of this vector.

Solution. $\overline{M_1M_2} = (0-(-1), -7-1, 4-3) =$

$$= \overset{x}{(2}, \overset{y}{-8}, \overset{z}{1})$$

$$|\overline{M_1M_2}| = \sqrt{x_t^2 + y_t^2 + z_t^2} =$$

$$= \sqrt{2^2 + (-8)^2 + 1^2} = \sqrt{4+64+1} = \sqrt{69}$$

Task 2. Find decomposition of vector $\vec{x}$ in the basis of vectors $\vec{p}, \vec{q}, \vec{r}$.

$\vec{x}(2;7;5)$ $\vec{p}(1;0;1)$ $\vec{q}(1;-2;0)$, $\vec{r}(0;3;1)$

Solution.

First of all let's check that vectors $\vec{p}, \vec{q}, \vec{r}$ make a basis. In this case their linear combination must be $= 0$ when all coefficients are equal to 0.

Let's present their linear combination

$$\alpha_1 \overline{p} + \alpha_2 \overline{q} + \alpha_3 \overline{r} = 0.$$

$$\alpha_1\begin{pmatrix}1\\0\\1\end{pmatrix}+\alpha_2\begin{pmatrix}1\\-2\\0\end{pmatrix}+\alpha_3\begin{pmatrix}0\\3\\1\end{pmatrix}=0.$$

$$\begin{cases}\alpha_1+\alpha_2=0\\ -2\alpha_2+3\alpha_3=0\\ \alpha_1+\alpha_3=0\end{cases}$$

this equation has the only solution if ~~det A~~ main Δ is not equal to 0

$$\begin{vmatrix}1&1&0\\-2&0&3\\1&0&1\end{vmatrix}=1\cdot(-1)^3\begin{vmatrix}-2&3\\1&1\end{vmatrix}=-(-5)\neq 0.$$

Let's present decomposition of ~~[illegible]~~ vector $\vec{x}$ in the basis.

$$\beta_1\vec{p}+\beta_2\vec{q}+\beta_3\vec{r}=\vec{x}$$

$$\beta_1\begin{pmatrix}1\\0\\1\end{pmatrix}+\beta_2\begin{pmatrix}1\\-2\\0\end{pmatrix}+\beta_3\begin{pmatrix}0\\3\\1\end{pmatrix}=\begin{pmatrix}2\\7\\5\end{pmatrix}$$

$$\begin{cases}\beta_1+\beta_2=2\\ -2\beta_2+3\beta_3=7\\ \beta_1+\beta_3=5\end{cases}$$

$$\left(\begin{array}{ccc|c}1&1&0&2\\0&-2&3&7\\1&0&1&5\end{array}\right)\sim[e_1-e_2\to e_2]\sim$$

$$\sim\left(\begin{array}{ccc|c}1&1&0&2\\0&-2&3&7\\0&1&-1&-3\end{array}\right)\sim[e_2\leftrightarrow e_3]\sim\left(\begin{array}{ccc|c}1&1&0&2\\0&1&-1&-3\\0&-2&3&7\end{array}\right)\sim$$

$$\sim[e_2(2)+e_3\to e_3]\sim\left(\begin{array}{ccc|c}1&1&0&2\\0&1&-1&-3\\0&0&1&1\end{array}\right)$$

$$\begin{cases} \beta_3 = 1 \\ \beta_2 - \beta_3 = -3 \\ \beta_1 + \beta_2 = 2 \end{cases} \qquad \begin{cases} \beta_3 = 1 \\ \beta_2 - 1 = -3 \Rightarrow \beta_2 = -2 \\ \beta_1 - 2 = 2 \Rightarrow \beta_1 = 4 \end{cases}$$

Decomposition is

$$\vec{x} = 4\vec{p} - 2\vec{q} + \vec{r}.$$

Scalar product of vectors.

Let vector $\vec{a}$ and $\vec{b}$ are given then scalar product is scalar

$$\bar{a} \cdot \bar{b} = |\bar{a}| \cdot |\bar{b}| \cdot \cos\alpha.$$

If vectors are given by coordinates then

$$\bar{a} \cdot \bar{b} = a_x b_x + a_y b_y + a_z b_z$$

<u>Problem 2</u>. Find cosine between vectors $\overline{AB}$ and $\overline{AC}$.

$A(-4;0,4)$ and $B(-1,6,7)$, $C(1,10,9)$

Solution.

Let's define coordinates of vectors.

$\overline{AB}\ (3, 6, 3)$

$\overline{AC} = (5; 10; 5)$.

To define cosine of angle between them

$$\cos\alpha = \frac{\overline{AB} \cdot \overline{AC}}{|\overline{AB}| \cdot |\overline{AC}|}$$

$$|\overline{AB}| = \sqrt{x^2 + y^2 + z^2} = \sqrt{3^2 + 6^2 + 3^2} = \sqrt{9 + 36 + 9} =$$

$$= \sqrt{54}$$

$|\overline{AC}| = \sqrt{5^2+10^2+5^2} = \sqrt{25+100+25} = \sqrt{150}.$

Scalar product is equal to

$\overline{AB} \cdot \overline{AC} = x_{AB} \cdot x_{AC} + y_{AB}\, y_{AC} + z_{AB} \cdot z_{AC} =$

$= 3 \cdot 5 + 6 \cdot 10 + 3 \cdot 5 = 30 + 60 = 90$

$$\cos\alpha = \frac{90}{\sqrt{150} \cdot \sqrt{54}} = \frac{\cancel{90}^{\cancel{30}\,6}}{\cancel{5}\sqrt{6} \cdot \cancel{3}\sqrt{6}} = 1.$$

$\underline{\cos\alpha = 1}$

Problem 3. Find projection of vector $\vec{a}$ on the direction of the vector $\vec{b}$.

$\vec{a} = 2\vec{p} - 3\vec{q}$, $\vec{b} = 5\vec{p} + \vec{q}$ $\quad |\vec{p}| = 2$

$|\vec{q}| = 3 \quad (\vec{p}\,\widehat{\ }\,\vec{q}) = \frac{\pi}{2}.$

Solution.

$$pr_{b}\,\vec{a} = \frac{\bar{a} \cdot \bar{b}}{|\bar{b}|}.$$

Let's find scalar product.

$\bar{a} \cdot \bar{b} = (2\bar{p} - 3\bar{q}) \cdot (5\bar{p} + \bar{q}) =$

$= 10\, \bar{p} \cdot \bar{p} + 2\, \bar{p} \cdot \bar{q} - 15\, \bar{q} \cdot \bar{p} - 3\, \bar{q} \cdot \bar{q} \;=$

$\bar{p} \cdot \bar{q} = \bar{q} \cdot \bar{p}$

$\bar{p} \cdot \bar{p} = \bar{p}^2 = |\bar{p}|^2 \cdot \cos 0^\circ = |\bar{p}|^2$

$\bar{q} \cdot \bar{q} = \bar{q}^2 = |\bar{q}|^2 \cdot \cos 0^\circ = |\bar{q}|^2$

$= 10|\bar{p}|^2 + 2\bar{p}\cdot\bar{q} - 15\bar{p}\cdot\bar{q} - 3|\bar{q}|^2 =$

$= 10|\bar{p}|^2 - 13\bar{p}\cdot\bar{q} - 3|\bar{q}|^2 =$

$= 10\cdot 2^2 - 13\cdot|\bar{p}|\cdot|\bar{q}|\cdot \underbrace{\cos\frac{\pi}{2}}_{=0} - 3\cdot 3^2 =$

$= 40 - 13\cdot 2\cdot 3\cdot 0 - 3\cdot 3^2 = 40 - 27 = 13.$

$|\bar{b}|^2 = \bar{b}^2 = (5\bar{p} + \bar{q})^2 = 5\bar{p}^2 + 10\bar{p}\cdot\bar{q} + \bar{q}^2 =$

$= 5\cdot|\bar{p}|^2 + 10\cdot\underbrace{|\bar{p}|\cdot|\bar{q}|\cos\frac{\pi}{2}}_{=0} + |\bar{q}|^2 =$

$= 5\cdot 4 + 9 = 29$

$pr_{\bar{b}}\,\bar{a} = \frac{13}{29}$