Lecture 4 (complete)
CONTINUITY OF A FUNCTION

3.1. Definition of Continuous Function at a Point

The function is called *continuous at the point* $x = x_0$ if it is defined in some neighborhood of the point x_0 and at the point x_0 as well and if

$$\lim_{x \to x_0} f(x) = f(x_0). \tag{3.1}$$

Continuity condition (3.1) may be also formulated in more detail:

The function $f(x)$ is called continuous at the point x_0 if:

a) it is defined at this point and some its neighborhood (that is, there exists such $\delta > 0$, that $C_\delta(x_0) \subset D_f$);

b) there are exist the right and left limits of the function $f(x)$ at the point x_0;

c) these one-sided limits are equal to each other;

d) these limits are equal to the value of the function $f(x)$ at the point x_0.

Example. Let show that the function $y = \sin x$ is continuous function $\forall\, x \in R$. We can see that $\lim\limits_{x \to x_0} \sin x = \sin x_0$ for all x_0. Earlier it was shown that $\lim\limits_{x \to 0} \sin x = 0$; $\lim\limits_{x \to 0} \cos x = 1$.

$$\lim_{x \to x_0} \sin x = \left\| \begin{matrix} y = x_0 - x \\ x = x_0 - y \end{matrix} \right\| = \lim_{y \to 0} \sin(x_0 - y) =$$

$$= \lim_{y \to 0} (\sin x_0 \cos y - \cos x_0 \sin y) = \sin x_0 .$$

Consequently due to definition of continuity it means that the function $\sin x$ is continuous one on set of real numbers.

A function $f(x)$ is called *continuous on closed interval* if it is continuous at each inside point of this interval and on the left end it is continuous on the right and at the right end it is continuous on the left.

3.2. Another Definitions of the Function Continuity

The definition of a function continuity at a point expressed by equality (3.1) may be formulated by language "ε, δ".

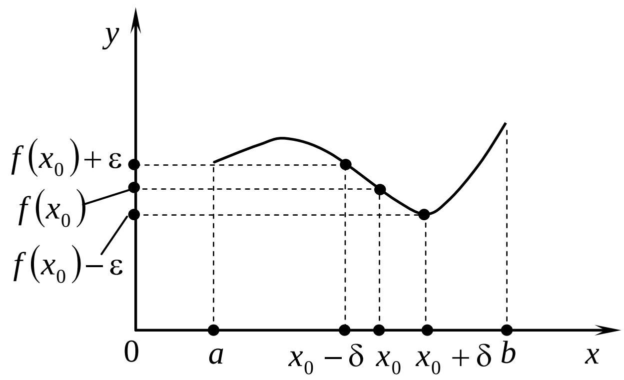

Fig. 3.1

Definition. *A function $f(x)$ is called continuous at a point x_0 (Fig. 3.1) if for any however small positive $\varepsilon > 0$ it is possible to indicate such number $\delta(\varepsilon) > 0$, that the inequality*

$$|f(x) - f(x_0)| < \varepsilon$$

is fulfilled for all x, satisfying condition $|x - x_0| < \delta$. This definition in symbol form may be written as

$$\forall \varepsilon > 0\ \exists \delta(\varepsilon) > 0 \,|\, (|x - x_0| < \delta) \Rightarrow (|f(x) - f(x_0)| < \varepsilon)$$

To formulate the following definition let us introduce some concepts.

Let us consider the function $y = f(x)$ and investigate it at some point x_0. Let us give to number x_0 such increment Δx that

$[x_0, x_0 + \Delta x] \subset D_f$. Let us denote the difference $f(x_0 + \Delta x) - f(x_0)$ by Δy, i.e. $\Delta y = f(x_0 + \Delta x) - f(x_0)$.

This difference is called *increment of the function* $y = f(x)$ at the point x_0 and is designated as $\Delta f(x_0)$ or Δy (Fig. 3.2). It should be noted that the value Δx might be both negative and positive.

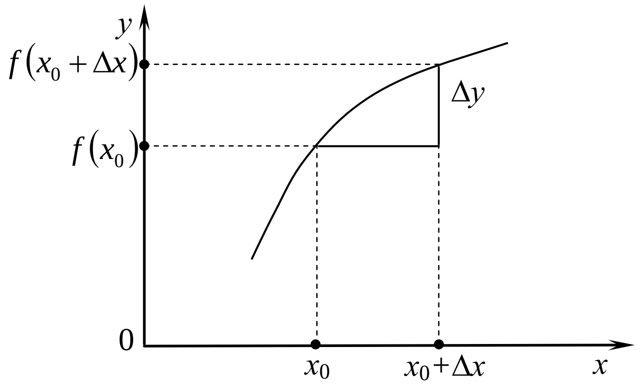

Fig. 3.2

Let a function $y = f(x)$ be continuous at a point x_0. Then it is possible to rewrite equality (3.1) in the following form

$$\lim_{x - x_0 \to 0} [f(x) - f(x_0)] = 0.$$

Designating $x - x_0 = \Delta x,\ f(x) - f(x_0) = \Delta y$, we obtain

$$\lim_{\Delta x \to 0} \Delta y = 0. \tag{3.2}$$

So if a function is continuous at the point then an infinitesimal increment of the function corresponds to infinitesimal increment of the argument.

Reasoning in contrary way we can obtain assertion: *if the function* $y = f(x)$ *satisfies equality (3.2) at the point then it is continuous at this point. So equality (3.2) is necessary and sufficient condition for the function to be continuous at the given point.*

Example. Show that function $f(x)=\cos x^2$ is continuous $\forall\ x \in R$.

Solution. Let us use definition of a continuous function on the language of «increments».

$$\Delta f = \cos(x+\Delta x)^2 - \cos x^2 = 2\sin\frac{x^2-(x+\Delta x)^2}{2}\sin\frac{x^2+(x+\Delta x)^2}{2}.$$

The value $\left|\sin\frac{x^2+(x+\Delta x)^2}{2}\right| \le 1$, i.e. is bounded, and $\left|\sin\frac{\Delta x(2x+\Delta x)}{2}\right| < \varepsilon$, i.e. it is the infinitesimal as $\Delta x \to 0$, $\lim\limits_{\Delta x\to 0}\Delta f = 0$, i.e. the increment of the function is also infinitesimal and therefore $f(x)=\cos x^2$ is continuous everywhere.

Note. Let the function $f(x)$ be continuous at the point x_0. Then equality (3.1) is fulfilled, which may be written as

$$\lim_{x\to x_0} f(x) = f\left(\lim_{x\to x_0} x\right). \qquad (3.3)$$

It means that to calculate the limit of continuous function it is enough to replace its argument by its limiting value. That is why equality (3.3) is called rule of limit passing under sign of the continuous function.

Formally equality (3.3) means the following: in case of continuous function the symbols *lim* and f may be replaced.

3.3. Arithmetic Operations on Continuous Functions

Theorem 1. The algebraic sum of the continuous functions at the given point is continuous function at this point.

■ Indeed, let the functions $f(x)$ and $\varphi(x)$ be continuous at the point x_0. Then

$$\lim_{x \to x_0} f(x) = f(x_0), \ \lim_{x \to x_0} \varphi(x) = \varphi(x_0).$$

Designate $F(x) = f(x) + \varphi(x)$. Then

$$\lim_{x \to x_0} F(x) = \lim_{x \to x_0} [f(x) + \varphi(x)] =$$

$$= \lim_{x \to x_0} f(x) + \lim_{x \to x_0} \varphi(x) = f(x_0) + \varphi(x_0) = F(x_0),$$

which is what had to be proved. □

Theorem 2. The product of two continuous functions at the given point is continuous function at this point as well.

Theorem 3. The quotient of two continuous functions is a continuous function if the denominator does not vanish at the point under consideration.

These theorems may be proved similarly.

3.4. Continuity of the Composite Function

Theorem. (Continuity of a composite function). Let two functions $y = f(u)$ and $u = \varphi(x)$ be given. If the function $\varphi(x)$ is continuous at the point x_0 and $\varphi(x_0) = u_0$, and the function $f(u)$ is continuous at the point u_0, then the composite function $F(x) = f[\varphi(x)]$ is continuous function as well at the point x_0 under consideration.

■ The first let us show that a function $F(x)$ is defined in same neighbourhood of the point x_0. Indeed by continuity of the function $f(u)$ at the point u_0 it follows an existence of the neighbourhood $C_\varepsilon(u_0)$, where the function $f(u)$ is defined. By virtue of continuity function $\varphi(x)$ at the point x_0 there exists such neighborhood $C_\delta(x_0)$, that from relation $x \in C_\delta(x_0)$ it follows that $|\varphi(x) - \varphi(x_0)| < \varepsilon$, i.e.

$u \in C_\varepsilon(u_0)$. But if $u \in C_\varepsilon(u_0)$, then $f(u)$ has sense. So there exists such neighbourhood $C_\delta(x_0)$, in which the function $f[\varphi(x)]$ has sense.

Let us take an arbitrary $\varepsilon > 0$. By continuity of the function $f(u)$ at the point u_0 we can indicate such number $\eta(\varepsilon) > 0$ that

$$(|u - u_0| < \eta) \Rightarrow (|f(u) - f(u_0)| < \varepsilon).$$

Further since the function $\varphi(x)$ is continuous function at the point x_0, then for obtained value of η we can indicate such $\delta(\eta) > 0$, that

$$(|x - x_0| < \delta) \Rightarrow (|\varphi(x) - \varphi(x_0)| < \eta), \text{ that is}$$

$$\forall(|x - x_0| < \delta) \Rightarrow (|u - u_0| < \eta).$$

But

$$(|u - u_0| < \eta) \Rightarrow (|f(u) - f(u_0)| < \varepsilon),$$

that is

$$(|u - u_0| < \eta) \Rightarrow (|f[\varphi(x)] - f[\varphi(x_0)]| < \varepsilon),$$

from this it follows that the function $F(x)$ is continuous at the point x_0. □

3.5. Theorem about Retaining Sign of a Continuous Function

Theorem. If the function $f(x)$ is continuous at the point $x = a$ and if $f(a) \neq 0$, then there exists such δ-neighbourhood of the point *a* that for all values of the argument of the mentioned δ-neighbourhood the function $f(x)$ does not vanish and preserves its sign at some neighborhood of the point. The proof of this theorem follows from limits properties of the function.

3.6. Continuity of the Inverse Function

Let the function $y = f(x)$ be given on the segment $[a,b]$ and range of values of this function is the segment $[\alpha,\beta]$. Let each $y \in [\alpha,\beta]$ has the only corresponding value $x \in [a,b]$ for which $f(x) = y$. Then on the segment $[\alpha,\beta]$ we can define the function $x = f^{-1}(y)$ putting in correspondence to each value $y \in [\alpha,\beta]$ such value $x \in [a,b]$ that $f(x) = y$. The function $x = f^{-1}(y)$ is called *inverse* to the function $y = f(x)$.

Theorem. Let strictly defined monotonic continuous function $y = f(x)$ be given on the segment $[a,b]$ and $\alpha = f(a)$, $\beta = f(b)$ $(\alpha < \beta)$. Then this function has strictly monotonic and continuous *inverse* function $x = f^{-1}(y)$ or $x = \varphi(y)$ on the segment $[\alpha,\beta]$. The proof of this theorem you can find in books [4].

3.7. Continuity of the Basic Elementary Functions

The first let us consider the *exponent function* $y = a^x$ and show that

$$\lim_{x \to 0} a^x = 1.$$

For the sake of definiteness let us suppose that $a > 1$. Take an arbitrary $\varepsilon > 0$ and require that inequality

$$\left| a^x - 1 \right| < \varepsilon$$

holds true or

$$-\varepsilon < a^x - 1 < \varepsilon.$$

This inequality is fulfilled if

$$1 - \varepsilon < a^x < 1 + \varepsilon,$$

whence we get

$$\log_a(1-\varepsilon) < x < \log_a(1+\varepsilon).$$

Since $\log_a(1+\varepsilon) < |\log_a(1-\varepsilon)|$ then we can put $\delta = \log_a(1+\varepsilon)$; so we obtain that if $|x| < \delta$, then $|a^x - 1| < \varepsilon$. From this it follows that $\lim\limits_{x\to 0} a^x = 1$. Which is what had to be proved.

Let us take an arbitrary number x_0 and calculate limit of the function increment at this point. We have that

$$\lim_{x\to x_0}\left(a^x - a^{x_0}\right) = \lim_{x\to x_0} a^{x_0}\left(a^{x-x_0} - 1\right) = a^{x_0} \lim_{x-x_0\to 0}\left(a^{x-x_0} - 1\right) =$$

$$= a^{x_0} \lim_{u\to 0}\left(a^u - 1\right),$$

but $\lim\limits_{u\to 0} a^u = 1$, so

$$\lim_{x\to x_0}\left(a^x - a^{x_0}\right) = 0,$$

that is.

$$\lim_{x\to x_0} a^x = a^{x_0}.$$

So the exponent function is continuous one for all $x \in R$.

Let us consider the function $y = \log_a x$, which is inverse one to exponent function $y = a^x$. Obviously logarithmic function is monotone increasing one (if $a > 1$) or monotone decreasing one (if $a < 1$) with range $(0,+\infty)$.

By virtue of theorem about inverse function the *logarithmic function* $y = \log_a x$ is monotonic and continuous on the interval $(0,+\infty)$.

Further since the *power function* may be represented in the following form

$$x^a = e^{a\ln x},$$

then continuity of the power function follows from continuity of the exponent and logarithmic functions $\forall x > 0$ and theorem about continuity of the composite function. If value of a is such that x^a has the sense for $x < 0$ (or for $x = 0$), then the function x^a is continuous for such values x. It is easy to check. For example, consider function $y = x^{\frac{2}{3}}$:

$$\lim_{x\to x_0}\left(x^{2/3} - x_0^{2/3}\right) = \lim_{x\to x_0}\left(\sqrt[3]{x^2} - \sqrt[3]{x_0^2}\right) = \lim_{x\to x_0}\frac{x^2 - x_0^2}{\sqrt[3]{x^4} + \sqrt[3]{x_0^2 x^2} + \sqrt[3]{x_0^4}} =$$

$$= \lim_{x\to x_0}(x - x_0)\frac{x + x_0}{\sqrt[3]{x^4} + \sqrt[3]{x_0^2 x^2} + \sqrt[3]{x_0^4}} = 0.$$

So for all x

$$\lim_{x\to x_0} x^{2/3} = x_0^{2/3},$$

whence the continuity of the function $y = x^{2/3}$ follows for all $x \in R$.

Let us consider the *trigonometric functions*. The first we will analyze the function $y = \sin x$. Let us take an arbitrary point $x = x_0$ and calculate

$$\left|\sin x - \sin x_0\right| = \left|2\sin\frac{x - x_0}{2}\cos\frac{x + x_0}{2}\right| \le 2\left|\sin\frac{x - x_0}{2}\right| < \left|x - x_0\right| < \delta = \varepsilon.$$

So $\forall \varepsilon > 0, \exists \delta = \varepsilon,$ that if $\left|x - x_0\right| < \delta \Rightarrow \left|\sin x - \sin x_0\right| < \varepsilon$.

It means that the function $y = \sin x$ is continuous. Since $\cos x = \sin\left(\frac{\pi}{2} - x\right)$, then using the theorem about the composite function we can conclude that the function $y = \cos x$ is continuous as well. The function $y = \tan x$ is continuous one at all points except of points at which $\cos x = 0$, that is, except $x = \pm\frac{\pi}{2}$, $x = \pm\frac{3\pi}{2}$, $x = \pm\frac{5\pi}{2}$,..., because in this case we have quotient of two continuous functions. In similar way the function $y = \cot x$ is continuous one at all points, where it is defined.

From theorem about continuity of the inverse function it follows that functions $\arcsin x$ and $\arccos x$ are continuous ones on the interval $[-1,1]$. The functions $\arctan x$ and $\operatorname{arccot} x$ are continuous ones $\forall x \in R$.

Analogously it may be shown the continuity of hyperbolic and inverse hyperbolic functions.

So all elementary functions are continuous ones in the domain of their definition.

3.8. Classification of Discontinuity Points

As we noted equality (3.1) is true if the following five conditions hold true:

1. The function $f(x)$ is defined at the point $x = x_0$ and some its neighborhood.

2. There exists the right limit of the function

$$\lim_{x\to x_0+0} f(x) = f(x_0+0) = b_1.$$

3. There exists the left limit of the function

$$\lim_{x\to x_0-0} f(x) = f(x_0-0) = b_2.$$

4. One-sided limits are equal and their general value coincides with the limit of the function at the point $x = x_0$.

5. Limit of the function is equal to value of the function at the point $x = x_0$, i.e. $\lim\limits_{x \to x_0} f(x) = f(x_0)$.

If at least one of these conditions is false then it is said that the function has *discontinuity* at the point $x = x_0$.

There we will distinguish points of discontinuity of the *I-st and II-nd* kinds.

If a function has finite one-sided limits at a point of discontinuity then it is the point of discontinuity of the I-st kind.

If at least one of one-sided limits tends to infinity or does not exist in essence then the point $x = x_0$ is called the discontinuity point of the II-nd kind.

In particular, if the property 1 and naturally 5 are false then the point $x = x_0$ is called the point of *removable discontinuity* (the I-st kind). In this case it is possible to determine the value function at the point of discontinuity additionally and obtain the continuous function.

If there exist *finite one-sided limits* but they are *not equal to* each other, i.e. $b_1 \neq b_2$, then the point $x = x_0$ is called the *point of discontinuity of the I-st kind «jump»* ($|b_2 - b_1|$ is value of the function jump).

So to investigate the nature of discontinuity point of the function $f(x)$ it is necessary:

1. Find points at which the function might have breaks.
2. Calculate the following one-sided limits:

$\lim\limits_{x \to x_0 - 0} f(x) = b_1$ and $\lim\limits_{x \to x_0 + 0} f(x) = b_2$.

3. To define kind of the discontinuity taking into account the obtained values of these limits.

Example 1. Let us consider the function $y = \frac{\sin x}{x}$. Obviously this function is not defined at the point $x = 0$, but you know that the function has limit at the point $x = 0$ and this limit is equal to 1, i.e. $\lim\limits_{x \to 0} \frac{\sin x}{x} = 1$. Hence the given function has break at the point $x = 0$. But this break is removable (I-st kind). Indeed if we define the function in the form

$$y(x) = \begin{cases} \frac{\sin x}{x}, & \text{if} \quad x \neq 0, \\ 1, & \text{if} \quad x = 0, \end{cases}$$

then such function will continuous at the point $x = 0$.

Example 2. Investigate the function $f(x) = \frac{x^2 - 4}{x - 2}$ for continuity.

Solution.

1. The function is defined for every x except $x_0 = 2$.

2. $\lim\limits_{x \to 2+0} \frac{x^2 - 4}{x - 2} = \lim\limits_{x \to 2+0} (x + 2) = 4$, $\lim\limits_{x \to 2-0} \frac{x^2 - 4}{x - 2} = \lim\limits_{x \to 2-0} (x + 2) = 4$.

3. One-sided limits coincide, i.e. there exists $\lim\limits_{x \to 2} \frac{x^2 - 4}{x - 2} = 4$, but the equality $\lim\limits_{x \to 2} f(x) = f(2)$ is false as the value $f(2)$ has no sense.

There is a removable discontinuity at the point $x_0 = 2$. Let us see how to make the point $x_0 = 2$ be the continuous point.

Let us consider $F(x)$ given as follows:

$$F(x) = \begin{cases} \frac{x^2 - 4}{x - 2}, & \text{if} \quad x \neq 2 \\ 4, & \text{if} \quad x = 2 \end{cases} .$$

The value $F(x) \equiv f(x)$ everywhere except $x_0 = 2$. At the point $x_0 = 2$ the function $f(x)$ is undetermined, and $F(x)$ is equal to 4, that coincides with the value $\lim\limits_{x \to 2} f(x) = 4$, i.e. $F(x)$ is continuous at the point $x_0 = 2$.

We should mention that the main reason allowing us to obtain the continuous function $F(x)$ from the discontinuous function $f(x)$ is equality of its one-sided limits: $\lim\limits_{x \to 2-0} f(x) = \lim\limits_{x \to 2+0} f(x)$.

Example 3. Investigate the function $f(x) = \begin{cases} x^2 + 2, & x \le 0, \\ -2x, & x > 0. \end{cases}$ for continuity.

Solution. The given function changes type of an analytical expression at the point $x = 0$. So at this point it might have a break. The value of the function is equal to 2 at this point. Let us calculate

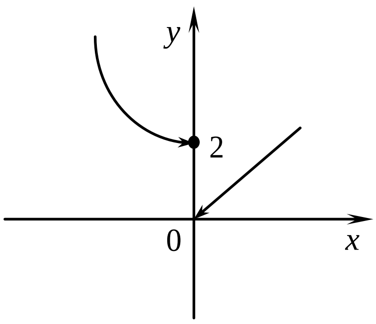

Fig. 3.3

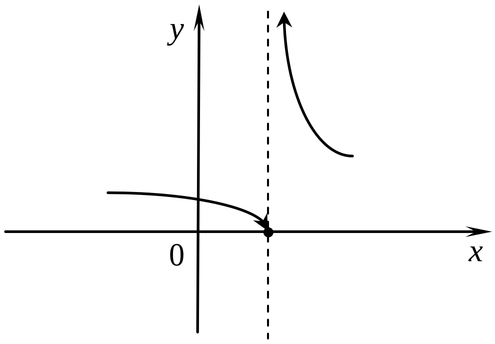

Fig. 3.4

one-sided limits:

$$\lim_{x \to +0} \left(x^2 + 2\right) = 2, \quad \lim_{x \to -0} (-2x) = 0.$$

You can see that one-sided limits are not equal to each other. Therefore the function has a break of the I-st kind of the type “jump” (Fig. 3.3).

Example 4. Investigate the function $f(x) = 3^{\frac{1}{x-1}}$ for continuity.

Solution.

1. An exponential function is continuous over range of definition, but the given function is undetermined at the point $x_0 = 1$.

2. Let us calculate $\lim\limits_{x\to 1-0} 3^{\frac{1}{x-1}} = 0$, $\lim\limits_{x\to 1+0} 3^{\frac{1}{x-1}} = \infty$.

3. The point $x_0 = 1$ is the point of discontinuity of the II-nd kind, as one of the one-sided limits tends to infinity.

Example 5. Research the function $f(x) = \dfrac{1}{x}$ for continuity (Fig. 3.5).

Solution. The given function is not defined at the point $x = 0$. Let us calculate the one-sided limits

$$\lim_{x\to -0} f(x) = -\infty,\ \lim_{x\to +0} f(x) = +\infty.$$

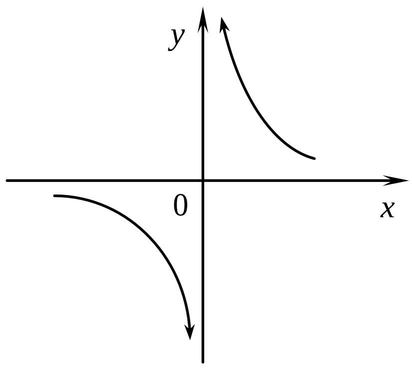

Fig. 3.5

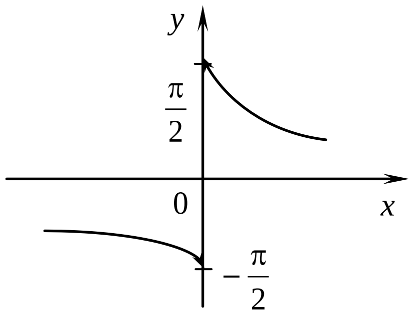

Fig. 3.6

Since these limits are infinite then the function has a break at the point $x = 0$ of the II-nd kind or infinite break.

Example 6. Investigate the function $y = \arctan\dfrac{1}{x}$ for continuity.

Solution. The function is not defined at the point $x = 0$. Let us calculate the one-sided limits

$$\lim_{\substack{x\to+0\\x\to-0}} \arctan\frac{1}{x} = \begin{cases} \dfrac{\pi}{2} \\ -\dfrac{\pi}{2} \end{cases}.$$

So the right and the left limits exist, but they do not equal to each other. This discontinuity point of the I-st kind of type “jump” (Fig. 3.6).

Example 7. Let function $f(x)=\dfrac{x}{\ln(x-2)}$ be given. The function is not defined at two points $x_1=3$ and $x_2=2$. At that for point $x_2=2$ it is necessary to find the right limit only due to the function definition $(x-2>0,\ x>2)$. Let us calculate the one-sided limits.

$$\lim_{x\to3+0} f(x)=+\infty,\quad \lim_{x\to3-0} f(x)=-\infty,\quad \lim_{x\to2+0} f(x)=0,$$

So, points $x_1=3$ and $x_2=2$ are discontinuity hjints of the II-nd kind (Fig. 3.7).

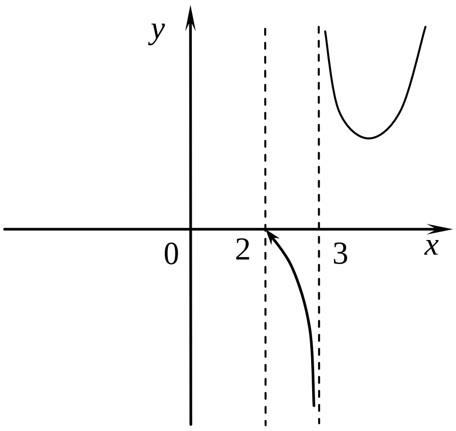

Fig. 3.7

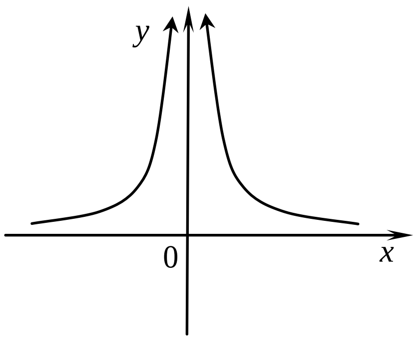

Fig. 3.8

Example 8. Let us consider the function $f(x)=\dfrac{1}{x^2}$. Here

$$\lim_{x\to-0} f(x)=\lim_{x\to+0} f(x)=+\infty.$$

In the given case we deal with break of the II-nd kind as well (Fig. 3.8).

Example 9. Let us investigate the function $f(x)=\sin\frac{1}{x}$ for continuity.

Solution. The function vanishes at conditions $\frac{1}{x}=\pm\pi,\pm 2\pi,\pm 3\pi,\ldots$, that is, at points $x=\pm\frac{1}{\pi},\pm\frac{1}{2\pi},\pm\frac{1}{3\pi},\ldots$. Consequently the function makes vibrations at the some neighborhood of the point $x=0$. It means that the function has nigher the right nor the left limits. In this case it is said that the limits do not exist on principle. But there exists limit as $x\to\pm\infty$. Indeed $\lim\limits_{x\to\pm\infty}\sin\frac{1}{x}=0$. The plot of this function is shown on Fig. 3.9.

Example 10. Let us consider the function $y=\arctan\left(\frac{1}{x-1}+\frac{1}{2-x}\right)$. Here are two points of discontinuity: $x=1$ and $x=2$. Let us investigate these points:

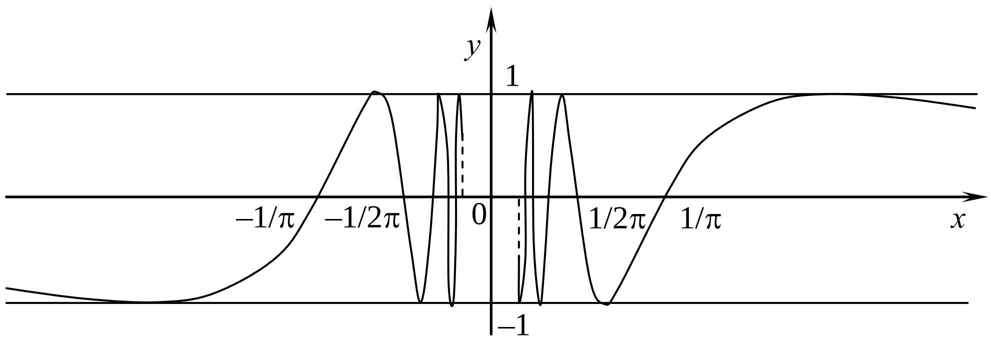

Fig. 3.9

$$(x\to 1-0)\Rightarrow(x-1\to -0)\Rightarrow\left(\frac{1}{x-1}\to-\infty\right)\Rightarrow$$

$$\Rightarrow\left(\frac{1}{x-1}+\frac{1}{2-x}\to-\infty\right)\Rightarrow\left(y\to-\frac{\pi}{2}+0\right);$$

$$(x\to 1+0)\Rightarrow(x-1\to +0)\Rightarrow\left(\frac{1}{x-1}\to+\infty\right)\Rightarrow$$

$$\Rightarrow\left(\frac{1}{x-1}+\frac{1}{2-x}\to+\infty\right)\Rightarrow\left(y\to\frac{\pi}{2}-0\right);$$

$$(x\to 2-0)\Rightarrow(2-x\to+0)\Rightarrow\left(\frac{1}{2-x}\to+\infty\right)\Rightarrow$$

$$\Rightarrow\left(\frac{1}{x-1}+\frac{1}{2-x}\to+\infty\right)\Rightarrow\left(y\to\frac{\pi}{2}-0\right);$$

$$(x\to 2+0)\Rightarrow(2-x\to-0)\Rightarrow\left(\frac{1}{2-x}\to-\infty\right)\Rightarrow$$

$$\Rightarrow\left(\frac{1}{x-1}+\frac{1}{2-x}\to-\infty\right)\Rightarrow\left(y\to-\frac{\pi}{2}+0\right).$$

So the given function has two break points of the I-st kind of type “jump”.

To get the general form of a graph let us investigate the given function on infinity. If

$$(x\to\pm\infty)\Rightarrow(x-1\to\pm\infty)\Leftrightarrow\left(\frac{1}{x-1}+\frac{1}{2-x}=\frac{1}{(x-1)(x-2)}\right)\Rightarrow$$

$$\Rightarrow(y\to-0).$$

So the graph of the given function is presented in Fig. 3.10.

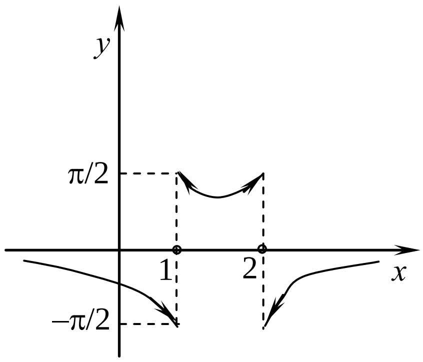

Fig. 3.10

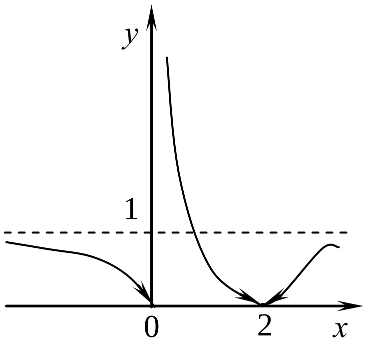

Fig. 3.11

Example 11. Let us consider the function

$$y = 3^{\frac{1-x}{x(x-2)^2}}.$$

The function has two break points: $x = 0$ and $x = 2$. Let us investigate the one-sided limits at these points. We get

$$(x \to -0) \Rightarrow \left(x(x-2)^2 \to -0\right) \Rightarrow \left(\frac{1-x}{x(x-2)^2} \to -\infty\right) \Rightarrow (y \to +0);$$

$$(x \to +0) \Rightarrow \left(x(x-2)^2 \to +0\right) \Rightarrow \left(\frac{1-x}{x(x-2)^2} \to +\infty\right) \Rightarrow (y \to +\infty);$$

$$(x \to 2 \pm 0) \Rightarrow \left(x(x-2)^2 \to +0\right) \Rightarrow \left(\frac{1-x}{x(x-2)^2} \to -\infty\right) \Rightarrow (y \to +0);$$

$$(x \to +\infty) \Rightarrow \left(x(x-2)^2 \to +\infty\right) \Rightarrow \left(\frac{1-x}{x(x-2)^2} \to -0\right) \Rightarrow (y \to 1-0);$$

$$(x \to -\infty) \Rightarrow \left(x(x-2)^2 \to -\infty\right) \Rightarrow \left(\frac{1-x}{x(x-2)^2} \to -0\right) \Rightarrow$$
$$\Rightarrow (y \to 1-0).$$

So the graph of this function is in Fig.°3.11. At the point $x = 0$ there is a break of the II-nd kind as infinite jump, at the point $x = 2$ there is a removable break (the I-st kind).

3.9 Corollaries of the Second Remarkable Limit

1. Let us consider the following limit

$$\lim_{x \to 0} \frac{\ln(1+x)}{x} = \lim_{x \to 0} \frac{1}{x} \ln(1+x) = \lim_{x \to 0} \ln(1+x)^{1/x} = \ln \lim_{x \to 0} (1+x)^{1/x} = \ln e = 1$$

so

$$\lim_{x \to 0} \frac{\ln(1+x)}{x} = 1, \tag{3.4}$$

whence it follows that $\ln(1+x) \sim x$, as $x \to 0$.

2. Let us show that $a^x - 1 \sim x \ln a$, where $a > 0, a \neq 1$. Indeed let calculate the limit $A = \lim\limits_{x\to 0} \dfrac{a^x - 1}{x \ln a}$. Since the value $a^x - 1 \to 0$ as $x \to 0$, then due to previous consequence we get $a^x - 1 \sim \ln\left[1 + \left(a^x - 1\right)\right]$, so

$$A = \lim_{x\to 0} \frac{\ln\left[1 + \left(a^x - 1\right)\right]}{x \ln a} = \lim_{x\to 0} \frac{\ln a^x}{x \ln a} = \lim_{x\to 0} \frac{x \ln a}{x \ln a} = 1.$$

Consequently $a^x - 1 \sim x \ln a$, as $x \to 0$.

In the particular case

$$\lim_{x\to 0} \frac{e^x - 1}{x} = 1. \tag{3.5}$$

It means that $e^x - 1 \sim x$ as $x \to 0$.

3. Show that sh$x \sim x$, as $x \to 0$.

Indeed

$$\lim_{x\to 0} \frac{\operatorname{sh} x}{x} = \lim_{x\to 0} \frac{e^x - e^{-x}}{2x} = \lim_{x\to 0} \frac{e^{-x}\left(e^{2x} - 1\right)}{2x} =$$

$$= \lim_{x\to 0} \frac{e^{-x}\, 2x}{2x} = 1.$$

In similar way it is possible to show that $\operatorname{th} x \sim x$, $\operatorname{Arsh} x \sim x$, $\operatorname{ch} x - 1 \sim \dfrac{x^2}{2}$ as $x \to 0$.

4. Let us consider the following limit

$$A = \lim_{x\to 0} \frac{(1+x)^\alpha - 1}{x},$$

where α is an arbitrary number, but $\alpha \neq 1$. It is obvious that $\left(\left((1+x)^\alpha - 1\right) \to 0\right)$, as $x \to 0$, then

$$A = \lim_{x\to 0} \frac{\ln\left\{1 + \left[(1+x)^\alpha - 1\right]\right\}}{x} = \lim_{x\to 0} \frac{\ln(1+x)^\alpha}{x} = \lim_{x\to 0} \frac{\alpha \ln(1+x)}{x}.$$

Finally we obtain that

$$\lim_{x\to 0}\frac{(1+x)^{\alpha}-1}{x}=\alpha. \qquad (3.6)$$

Obtained formulas (3.4) – (3.6) are useful for calculation limits.

While investigating indetermination of the kind 1^{∞} the following theorem is very useful.

Theorem. Let us assume that the values $\alpha\to 0, y\to\infty$ and $\beta\sim\alpha,\ z\sim y$ at same behavior of the argument. If there exists $\lim(1+\alpha)^{y}$, then $\lim(1+\beta)^{z}$ exists as well and these limits are equal to each other, that is,

$$\lim(1+\alpha)^{y}=\lim(1+\beta)^{z}. \qquad (3.7)$$

■ Indeed

$$\lim(1+\alpha)^{y}=\lim e^{y\ln(1+\alpha)}=e^{\lim\, y\ln(1+\alpha)}=e^{\lim\, y\alpha}=e^{\lim\, z\beta}=$$

$$=e^{\lim\, z\ln(1+\beta)}=\lim(1+\beta)^{z}.\square$$

It should note that we used the fact about continuity of the exponent and logarithm functions and rule of limit passing under sign of the continuous functions.

Table 1

Equivalent infinitesimals, obtained by the second remarkable limit

$a^{x}-1\sim x\ln a,\ x\to 0$	$e^{x}-1\sim x,\ x\to 0$
$\log_a(1+x)\sim\dfrac{x}{\ln a},\ x\to 0$	$\ln(1+x)\sim x,\ x\to 0$
$(1+x)^{\alpha}-1\sim\alpha x,\ x\to 0$	$(1+x)^{\frac{1}{\alpha}}-1\sim\dfrac{x}{\alpha},\ x\to 0$
$sh\sim x,\ x\to 0$	$\mathrm{th}\,x\sim x,\ x\to 0$
$\mathrm{ch}x-1\sim\dfrac{x^{2}}{2},\ x\to 0$	

Table 2

Equivalent infinitesimals, obtained by the first remarkable limit

$\sin x \sim x,\ x \to 0$	$\tan x \sim x,\ x \to 0$
$\arcsin x \sim x,\ x \to 0$	$\arctan x \sim x,\ x \to 0$
$1-\cos x \sim \dfrac{x^2}{2},\ x \to 0$	

Example 1.

$$\lim_{x\to\infty}\left(\frac{2x-1}{2x+3}\right)^x = \lim_{x\to\infty}\left[\frac{(2x+3)-4}{2x+3}\right]^x = \lim_{x\to\infty}\left(1-\frac{4}{2x+3}\right)^x =$$

$$= \lim_{x\to\infty}\left(1-\frac{2}{x}\right)^x = \frac{1}{e^2}.$$

Example 2.

$$\lim_{x\to 0}(1+\tan 3x)^{\cot x} = \lim_{x\to 0}(1+\tan 3x)^{\frac{1}{\tan x}} = \lim_{x\to 0}(1+3x)^{\frac{1}{x}} = e^3.$$

Example 3.

$$\lim_{x\to 0}(x\sin 2x+\cos x)^{\frac{1}{x^2}} = \lim_{x\to 0}\{1+[x\sin 2x-(1-\cos x)]\}^{\frac{1}{x^2}} =$$

$$= \lim_{x\to 0}\left(1+2x^2-\frac{x^2}{2}\right)^{\frac{1}{x^2}} = \lim_{x\to 0}\left(1+\frac{3}{2}x^2\right)^{\frac{1}{x^2}} = e^{\frac{3}{2}}.$$

Example 4. Calculate the following limit A.

$$A = \lim_{x\to 0}\left(\frac{2}{\pi}\arccos x\right)^{\frac{1}{x}}.$$

Then

$$A = \lim_{x\to 0}\left[1+\left(\frac{2}{\pi}\arccos x-1\right)\right]^{\frac{1}{x}} = \lim_{x\to 0}\left[1+\frac{2}{\pi}\left(\arccos x-\frac{\pi}{2}\right)\right]^{\frac{1}{x}} =$$

$$=\lim_{x\to 0}\left[1-\frac{2}{\pi}\left(\frac{\pi}{2}-\arccos x\right)\right]^{\frac{1}{x}}=\lim_{x\to 0}\left[1-\frac{2}{\pi}\sin\left(\frac{\pi}{2}-\arccos x\right)\right]^{\frac{1}{x}}=$$

$$=\lim_{x\to 0}\left[1-\frac{2}{\pi}\cos(\arccos x)\right]^{\frac{1}{x}}=\lim_{x\to 0}\left(1-\frac{2}{\pi}x\right)^{\frac{1}{x}}=e^{-\frac{2}{\pi}}.$$

Example 5.

$$\lim_{x\to 2}\frac{\ln(2x-3)}{5^x-25}=\lim_{x\to 2}\frac{\ln[1+(2x-4)]}{25\left(5^{x-2}-1\right)}=\lim_{x\to 2}\frac{\ln 2x-4}{25(x-2)\ln 5}=$$

$$=\lim_{x\to 2}\frac{2(x-2)}{25\ln 5(x-2)}=\frac{2}{25\ln 5}.$$

3.10. Limit of Power-exponential Function

Let us consider the function $y=u(x)^{v(x)}$ which is called a power-exponential one. Suppose that

$$\lim_{x\to x_0}u(x)=a,\ \lim_{x\to x_0}v(x)=b,$$

$$\lim_{x\to x_0}[u(x)]^{v(x)}=\lim_{x\to x_0}e^{v(x)\ln u(x)}=e^{\lim\limits_{x\to x_0}v(x)\ln\lim\limits_{x\to x_0}u(x)}=e^{b\ln a}=a^b$$

At calculating $\lim\limits_{x\to x_0}v(x)\ln u(x)$ then there are possible the following cases:

a) $b=0,\ a=\infty$,
b) $b=0,\ a=0$,
c) $b=\infty,\ a=1$.

These cases are indetermination forms of the kinds $(\infty)^0$, $(0)^0$, $(1)^\infty$ and they required specified investigation.

3.11. Lemma about Contracting Segments

Let us consider system of nested segments

$$[a,b]\supset[a_1,b_1]\supset[a_2,b_2]\supset\cdots\supset[a_n,b_n]\supset\cdots$$

such that the length of the segment $[a_n,b_n]$ approaches zero as $n\to\infty$, i.e.

$$\lim_{n\to\infty}(b_n-a_n)=0.$$

This system is called system of contracting segments.

Lemma. If a system is the system of contracting segments then there exists the only point that belongs to all segments of this system.

■ Let us consider the numerical sequence $\{a_n\}$. It is clear that it is monotone increasing and is bounded from above as $a_1<a_2<\cdots<a_n<\cdots<b$. Therefore this sequence is convergent, i.e. there exists $\lim\limits_{n\to\infty}a_n=c_1$. Analogously the sequence $\{b_n\}$ is monotone decreasing and is bounded from below as $b_1>b_2>\cdots>b_n>\cdots>a$. Hence this sequence has $\lim\limits_{n\to\infty}b_n=c_2$ as well.

Since

$$\lim_{n\to\infty}(a_n-b_n)=0\Rightarrow\lim_{n\to\infty}a_n=\lim_{n\to\infty}b_n\Rightarrow c_1=c_2=c.\square$$

3.12. Lemma By Boltsano-Weierstrass

Lemma. Any bounded sequence contains convergent subsequence.

■ Let the sequence $x_1,x_2,...,x_n,....$ be bounded. Then it is possible to indicate such segment $[a,b]$ that $x_n\in[a,b]\ \forall n$. Let us divide the segment $[a,b]$ by half. At least one half contains the infinite set of points x_n, of the given sequence. Let us denote this part by $[a_1,b_1]$. If each of parts of the segment $[a,b]$ contains infinite set of points x_n then we can take any of them.

In similar way let us divide the segment $[a_1,b_1]$ by two parts and denote as $[a_2, b_2]$ such half, which contains infinite set of points x_n and so on. On k–th step we obtain segment $[a_k, b_k]$, in which infinite set of points are contained and length of k–th segment approaches zero as $k \to \infty$, because $|b_k - a_k| = \dfrac{b-a}{2^k}$.

So we constructed the system of contracting segments $[a,b], [a_1,b_1], [a_2,b_2], ..., [a_k,b_k], ...$. Due to lemma about contracting segments there exists the only point c, which belongs to all these segments. Required subsequence $x_{n_1}, x_{n_2}, ... x_{n_k}, ...$ may be obtained, for example, in such way. As the first term x_{n_1} let us take the first of numbers x_n containing in $[a_1, b_1]$. As x_{n_2} we can take any number x_n, which belongs to the segment $[a_2, b_2]$, but such that $n_2 > n_1$. As x_{n_3} we can take any number x_n from the segment $[a_3, b_3]$, but such that $n_3 > n_2$.

Since $\forall n_k$ corresponding terms satisfy inequality $a_k \le x_{n_k} \le b_k$ and $\lim\limits_{k\to\infty} a_k = \lim\limits_{k\to\infty} b_k = c$, then by virtue of the first sign of the existence limit we obtain

$$\lim_{k\to\infty} x_{n_k} = c.$$

Which is what had to be proved.□

3.13. Properties of the Functions, Continuous on Closed Interval

Theorem 1. (The first theorem by Boltsano-Cauchy**).** If the function $f(x)$ is continuous on the closed interval $[a,b]$ and takes on values of different signs at the end points of this interval, then there will be at least one point $x = \xi$ lying between the points a and b, at which the function vanishes: $f(\xi)=0$, $\xi\in(a,b)$.

■ For definiteness let us assume that $f(a)<0$, $f(b)>0$ (Fig. 3.12). Let us divide the segment $[a,b]$ by half and denote the middle of the segment by $c=\dfrac{a+b}{2}$. Then the following two cases are possible:

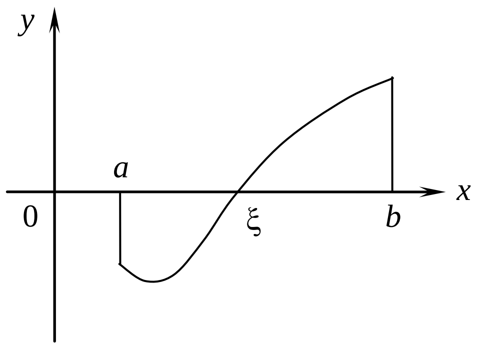

Fig. 3.12

1. $f(c)=0$, i.e. $c=\xi$, and the theorem proven.

2. $f(c)\neq 0$. Then either $f(c)<0$ or $f(c)>0$. Let us assume that $f(c)>0$. In this case we will put $c=b_1, a=a_1$; we get new segment $[a_1,b_1]$ such that $f(a_1)<0$, $f(b_1)>0$. In similar way let us divide by half the segment $[a_1,b_1]$ and denote as $[a_2,b_2]$ the part on which the inequalities $f(a_2)<0$, $f(b_2)>0$ are valid and so on. On each following step we will get $f(a_n)<0$, $f(b_n)>0$.

As result we obtain the system of nested segments $[a,b],[a_1,b_1],\ldots,[a_n,b_n],\ldots$, such that $\lim\limits_{n\to\infty}(b_n-a_n)=0$ and at the same time $f(a_n)<0, f(b_n)>0, \forall n$.

But $\lim\limits_{n\to\infty} a_n=\xi,$, $\lim\limits_{n\to\infty} b_n=\xi$ and due to continuity of the function $f(x)$ at the point ξ, we have that

$$\lim_{n\to\infty} f(a_n)=f\left(\lim_{n\to\infty} a_n\right)=f(\xi)\le 0;$$

$$\lim_{n\to\infty} f(b_n)=f\left(\lim_{n\to\infty} b_n\right)=f(\xi)\ge 0.$$

Since $(f(\xi)\le 0)\wedge(f(\xi)\ge 0)=(f(\xi)=0)$, it means that the point $x=\xi$ at which the given function is equal to zero exists. □.

This theorem has a simple geometrical meaning. The graph of the continuous function $y = f(x)$ joining the points $M_1(a, f(a))$ and $M_2(b, f(b))$, where $f(a) < 0$, $f(b) > 0$, cuts the x-axis in at least one point (Fig. 3.13).

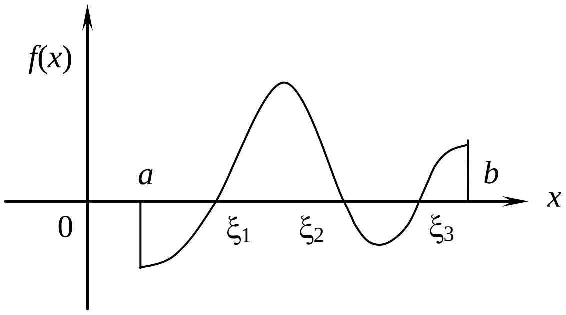

Fig. 3.13

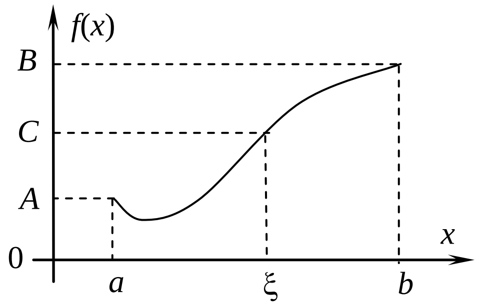

Fig. 3.14

Theorem 2. (The second theorem by Boltsano-Cauchy**).** If the function $f(x)$ is continuous on the closed interval $[a,b]$ and takes on unequal values $f(a) = A$, $f(b) = B$, at the end points of this interval, then no matter what the number C, lying between numbers A and B, there will be at least one a point $\xi \in (a,b)$, such that $f(\xi) = C$ (Fig. 3.14).

■ For definiteness let us assume that $A < B$. Then we can choose any number C, which lies between A and B, i.e. $A < C < B$. Let us introduce an auxiliary function $\varphi(x) = f(x) - C$. Obviously it is continuous on segment $[a,b]$ like function $f(x)$. Except of this it takes on values of the different signs at the end points. Indeed

$$\varphi(a) = f(a) - C = A - C < 0,$$
$$\varphi(b) = f(b) - C = B - C > 0.$$

By the first theorem of Boltsano-Cauchy there exists at least one point $\xi \in (a,b)$, such that $\varphi(\xi) = 0$, whence it follows that $f(\xi) - C = 0$. It means that $f(\xi) = C$, which is proof of the theorem. □

Theorem 3. (The first theorem by Weierstrass). If a function is continuous on the closed interval, then it is bounded on this interval**.**

■ Let us assume contrary, namely, we suppose that function $f(x)$, which is continuous on the closed interval, $[a,b]$ is not bounded (for example, from above). Divide the closed interval by half and take such part on which the function is not bounded and denote it by $[a_1,b_1]$. In similar way let us divide by half the segment $[a_1,b_1]$ and the part on which the function is not bounded denote as $[a_2,b_2]$. On each following step we will get new segment $[a_n,b_n]$, on which the function is not bounded. As result we get the sequence

$$[a,b]\supset[a_1,b_1]\supset[a_2,b_2]\supset\ldots\supset[a_n,b_n]\supset\ldots$$

of nested segments such that the length of a segment $[a_n,b_n]$ approaches zero, i.e. $\lim\limits_{n\to\infty}(b_n-a_n)=0$ as $n\to\infty$ or $\lim\limits_{n\to\infty}a_n=\lim\limits_{n\to\infty}b_n=c$.

The value $f(c)$ is finite number for the function which is continuous at this point.

It means that $\forall\varepsilon>0,\exists\ |\delta>0,\forall\left(|x-c|<\delta\Rightarrow\underset{x}{|}f(x)-f(c)|\right)<\varepsilon$.

Put $\varepsilon=1$. Then

$$|f(x)|=|f(x)-f(c)+f(c)|\le\left(|f(x)-f(c)|+|f(c)|\right)<\varepsilon+|f(c)|=$$
$$=\left(1+|f(c)|\right)=M\ .$$

From this it follows that the function is bounded $\forall x$ satisfying inequality $|x-c|<\delta$. Let us take such n that the segment $[a_n,b_n]$ is contained in $C_\delta(c)$. Then we obtain contradiction, by construction the segment $[a_n,b_n]$ on which the function is not bounded. But another hand we have shown, that it is bounded on this segment. Obtained contradiction proves the theorem.□

Note. From this theorem it follows that if the function $f(x)$ is continuous on the closed interval $[a,b]$, then there exist such finite numbers m and M, that all values of the function lie between these numbers, that is, $m \le f(x) \le M, \forall x,\ x \in [a,b]$ (Fig. 3.15).

The greatest of all possible numbers m is obviously $\inf\limits_{x\in[a,b]} f(x)$ and the smallest of all possible numbers M is $\sup\limits_{x\in[a,b]} f(x)$. We will mind further that namely these numbers m and M is under consideration. In connection with it there occurs the following question: are exist a points x_0 and x_1, that belong to segment $[a,b]$ and such that $f(x_0)=m,\ f(x_1)=M$? The answer gives the following theorem.

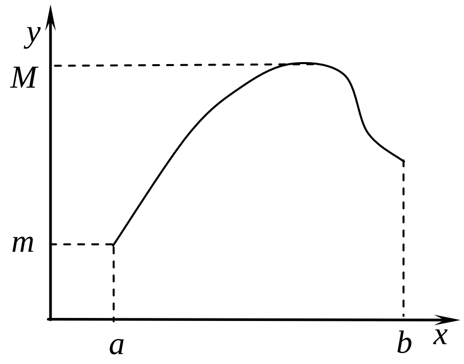

Fig. 3.15

Theorem 4. (The second theorem by Wierstrass). If the function $f(x)$ is continuous on the closed interval $[a,b]$, then it attains on this interval (at least one) the greatest value M and the smallest value m. It means that there are such points x_0 and x_1 on the interval $[a,b]$, that $f(x_0)=m,\ f(x_1)=M$.

■ Let us assume that such points are absent on the interval $[a,b]$. For example we will suppose that there is no point at which $f(x)=M$, $M=\sup\limits_{x\in[a,b]} f(x)$. Then $f(x)<M\ \forall x,\ \ x\in[a,b]$. Consider the auxiliary function $\varphi(x)=\dfrac{1}{M-f(x)}$. Obvious that it is continuous for all $x\in[a,b]$. Consequently by virtue of the theorem about boundless of the function continuous on the closed interval there exists such number $M_1>0$, that for all $x\in[a,b]$ the inequality $\varphi(x)\le M$ will be valid, i.e.

$$\frac{1}{M-f(x)}\le M_1,$$

whence

$$M - f(x) \geq \frac{1}{M_1}.$$

It means that

$$f(x) \leq M - \frac{1}{M_1}.$$

But from this it follows that $\sup\limits_{x\in[a,b]} f(x) < M$, which is contradiction for $\mathrm{M} = \sup\limits_{x\in[a,b.]} (f(x))$. Therefore we can account that the theorem is proven, that is there exists such point $x_1 \in [a,b]$, that $f(x_1) = M$.□

In the similar way it is possible to prove the existence of the point x_0 at which $f(x_0) = m$.

Note. If the function $f(x)$ is bounded on the closed interval, but is not continuous, then the supremum and unfimum might be not reached. The function $f(x) = \arctan\frac{1}{x}$ defined on the segment $[-2,2]$ is such example (Fig. 3.16). Indeed, $\sup\limits_{[-2,2]} \arctan\frac{1}{x} = \frac{\pi}{2}$, but there is no such point on the interval at which the function takes on this value.

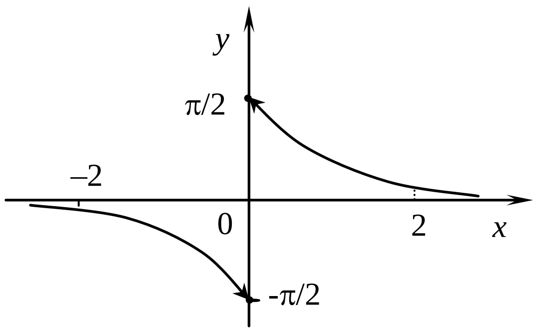

Fig. 3.16

3.14. Uniform Continuity of a Function

Let a function $f(x)$ be continuous on the interval X and point $x_0 \in X$ be an arbitrary point of this interval. Then for any $\varepsilon > 0$ there exists such number $\delta > 0$, that $(|x - x_0| < \delta) \Rightarrow (|f(x) - f(x_0)| < \varepsilon)$. Obvious that δ depends on both ε and point x_0. It means that for the same value ε and different points x_0 there might exist the different values δ. It is connected with the different velocity of changing function $f(x)$ on different parts of the interval. There is occurred the following question. Is there such value of δ, corresponding to given ε that for any two values of the arguments x_0 and x, satisfying inequality $|x - x_0| < \delta$ the inequality $|f(x) - f(x_0)| < \varepsilon$ holds true? That is if $x_0 \in X$ and $x \in X$ then

$$(|x - x_0| < \delta) \Rightarrow (|f(x) - f(x_0)| < \varepsilon). \qquad (3.32)$$

Another words the number δ is independent value on x_0, but it is dependent on ε.

Definition. If the function $f(x)$ is continuous one on the interval X and for any $\varepsilon > 0$ it is possible to indicate such number $\delta(\varepsilon) > 0$, that inequality $|f(x) - f(x_0)| < \varepsilon$ is fulfilled for any $x_0 \in X$ and $x \in X$, satisfying inequality $|x - x_0| < \delta$, then the function $f(x)$ is called *uniformly continuous on the interval X.*

Cantor's Theorem (without proof). If the function $f(x)$ is continuous on the closed interval $[a, b]$, then it is uniformly continuous on this interval.