# CHAPTER I. AN INDEFINITE INTEGRAL

## 1. Definition and Properties of an Indefinite Integral

### 1.1. Antiderivative of a function

Differential calculus deals with finding the derivative or differential of a given function. To each function $f(x)$ there was stated corresponding its derivative $f'$. For example if $f$ is defined by expression

$$f(x)=x^3-6x+12,$$

then $f'$ is defined by

$$f'=3x^2-6.$$

Finding a function by its derivative or differential is studied in integral calculus.

Suppose that a function $f$ be given and we want to know does there exist a function $F$ such that $f$ is the derivative of $F$? For example, $f$ is defined by formula

$$f=5x^4-2x.$$

It may be check up that given function $f$ is derivative of function $F$ defined by relation

$$F(x)=x^5-x^2+C,$$

where $C$ is an arbitrary constant. That is $F'(x)=f(x)$.

**Definition 1.** A function $F(x)$ is called an antiderivative of the function $f(x)$ on the interval $[a,b]$ if the equality

$$F'(x)=f(x) \qquad (1)$$

is fulfilled at all points of the given interval.

It is easy to see that if for the given function $f(x)$ an antiderivative exists, then it isn't single. Really, we could take the following functions as antiderivative in the foregoing example

$$F(x)=x^5-x^2+5,$$
$$F(x)=x^5-x^2+7,$$

or generally,

$$F(x)=x^5-x^2+C$$

(where $C$ is an arbitrary constant), since

$$F'(x)=\left(x^5-x^2+C\right)'=5x^4-2x=f(x).$$

On the other hand, it may be proved that family of functions $(F(x)+C)$ exhausts all antiderivatives of the given function. This is a consequence of the following theorem.



**Theorem.** If $F_1(x)$ and $F_2(x)$ are two antiderivatives of the same function $f(x)$ on the interval $[a,b]$ then the difference between them is a constant.

□ By virtue of the definition of an antiderivatives we have

$$\begin{cases} F_1'(x)=f(x) \\ F_2'(x)=f(x) \end{cases} \qquad (1.2)$$

for any value of $x$ on the interval $[a,b]$.

Let us put

$$F_1(x)-F_2(x)=\varphi(x).$$

Then by (1.2) we have

$$F_1'-F_2'=\varphi' \Rightarrow \varphi'\equiv 0 \text{ (identity)}$$

for any value of $x$ on the interval $[a,b]$.

But from $\varphi'=0$ it follows that $\varphi(x)$ is a constant: $\varphi(x)=C$. □

So if antiderivative $F(x)$ is known then any other antiderivative $F_1(x)$ may be expressed as

$$F_1(x)=F(x)+C.$$

It means that all antiderivatives for the given function are included in the following collection.

**Definition 2.** If the function $F(x)$ is an antiderivative of $f(x)$, then the expression $F(x)+C$ is called indefinite integral and is denoted by the symbol $\int f(x)dx$. Thus, by definition

$$\int f(x)dx=F(x)+C, \text{ if } F'(x)=f(x). \qquad (1.3)$$

Here the function $f(x)$ is called the integrand, $f(x)dx$ is the element of integration (or the expression under the integral sign), and $\int$ – is integral sign.

Thus, an ***indefinite integral*** is a ***family of functions*** $y=F(x)+C$.

***The finding an antiderivative*** of a given function $f(x)$ is ***called integration*** of the function $f(x)$.

From the geometrical point of view, an indefinite integral is a collection (family) of curves, each of them is obtained by translating one of the curve parallel to itself upwards and down wards (that is, along the $y$ – axis).

A natural question arises. ***Do antiderivatives exist for every function $f(x)$?*** ***The answer is no.*** Let us note, however, without proof, that if a function $f(x)$ is continuous on an interval $[a,b]$, then this function has an antiderivative (and hence there is also an indefinite integral).

## 1.2. Some Properties of an Indefinite Integral

1. The derivative of an indefinite integral is equal to the integrand.
Really,

$$\left(\int f(x)dx\right)' = \left(F(x)+C\right)' = F'(x) = f(x).$$

2. The differential of an indefinite integral is equal to the expression under the integral sign:

$$d\left(\int f(x)dx\right) = f(x)dx.$$

Indeed,

$$d\left(\int f(x)dx\right) = \left(\int f(x)dx\right)' \cdot dx = f(x)dx.$$

3. The indefinite integral of differential of function is equal to this function plus an arbitrary constant

$$\int dF(x) = F(x) + C.$$

The truth of this equation may easily be checked up by differentiation.

**Theorem 1.** The indefinite integral of algebraic sum of two or more functions is equal to the algebraic sum of integrals of these functions:

$$\int (f_1 \pm f_2)dx = \int f_1 dx \pm \int f_2 dx. \qquad (1.4)$$

To prove this let us find the derivatives of the left and right sides of this equation

$$\left(\int (f_1 \pm f_2)dx\right)' = f_1 \pm f_2,$$

$$\left(\int f_1 dx + \int f_2 dx\right)' = \left(\int f_1 dx\right)' + \left(\int f_2(x)dx\right)' = f_1 \pm f_2,$$

$$\Downarrow$$

$$\left(\int (f_1 \pm f_2)dx\right)' = \left(\int f_1 dx + \int f_2 dx\right)'.$$

Thus, the derivatives of the left and right sides of (1.4) are equal each to other: in other words, the derivative of any antiderivative on the left hand side is equal to the derivative of any function on the right hand side.

Therefore the difference of any two function on the left and right is constant. That is how we should understand equality (1.4).

**Theorem 2.** A constant factor may be taken outside the integral sign; that is, if $a = const$, then

$$\int af(x)dx = a\int f(x)dx. \qquad (1.5)$$

The proof of equality (1.5) is similar to proof of the theorem 1.

## 1.3. Table of Integrals

Table of integrals of the simplest functions follows directly from the



definition of an indefinite integral and from the table of derivatives. The truth of the formulas can easily be checked up by differentiation: by establishing that derivative of right hand side is equal to integrand.

***The table of the basic indefinite integrals***

1. $\int dx = x + C.$
2. $\int x^{\alpha} dx = \dfrac{x^{\alpha+1}}{\alpha+1} + C \ (\alpha \neq -1).$
3. $\int \dfrac{dx}{x} = \ln|x| + C \quad (x \neq 0).$
4. $\int a^x dx = \dfrac{a^x}{\ln a} + C \quad (0 < a \neq 1).$
5. $\int e^x dx = e^x + C.$
6. $\int \cos x dx = \sin x + C.$
7. $\int \sin x dx = -\cos x + C.$
8. $\int \dfrac{dx}{\cos^2 x} = \tan x + C.$
9. $\int \dfrac{dx}{\sin^2 x} = -\cot x + C.$
10. $\int \dfrac{dx}{\sqrt{a - x^2}} = \begin{cases} \arcsin\dfrac{x}{a} + C \\ -\arccos\dfrac{x}{a} + C \end{cases}$
11. $\int \dfrac{dx}{a + x^2} = \dfrac{1}{a}\arctan\dfrac{x}{a} + C$
12. $\int \dfrac{dx}{a^2 - x^2} = \dfrac{1}{2a}\ln\left|\dfrac{a+x}{a-x}\right| + C.$
13. $\int \dfrac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left|x + \sqrt{x^2 \pm a^2}\right| + C.$
14. $\int \tan x dx = -\ln|\cos x| + C.$
15. $\int \operatorname{ctan} x dx = \ln|\sin x| + C.$
16. $\int \dfrac{dx}{x^2 + a^2} = \dfrac{1}{a}\arctan\dfrac{x}{a} + C.$
17. $\int \dfrac{dx}{\sqrt{a^2 - x^2}} = \arcsin\dfrac{x}{a} + C.$
18. $\int \dfrac{dx}{\sin x} = \ln\left|\tan\dfrac{x}{2}\right| + C.$
19. $\int \dfrac{dx}{\cos x} = \ln\left|\tan\left(\dfrac{x}{2} + \dfrac{\pi}{4}\right)\right| + C.$
20. $\int \operatorname{sh} x\, dx = \operatorname{ch} x + C.$
21. $\int \operatorname{ch} x dx = \operatorname{sh} x + C.$
22. $\int \dfrac{dx}{\operatorname{ch}^2 x} = \operatorname{th} x + C.$
23. $\int \dfrac{dx}{\operatorname{sh}^2 x} = -\operatorname{cth} x + C.$

Consider some examples of calculation of an indefinite integrals by direct way.

1) $\int 3^x dx = \dfrac{3^x}{\ln 3} + C;$

2) $\int \frac{dx}{x^2+9} = \frac{1}{3}\arctan\frac{x}{3} + C$;

3) $\int \frac{dx}{25-x^2} = \frac{1}{10}\ln\left|\frac{5+x}{5-x}\right| + C$;

4) $\int \frac{dx}{\sqrt{4-x^2}} = \arcsin\frac{x}{2} + C$;

5) $\int \frac{dx}{\sqrt{x^2-16}} = \ln\left|x+\sqrt{x^2-16}\right| + C$;

6) $\int \frac{\cos 2x}{\cos^2 x \sin^2 x}dx = \int \frac{\cos^2 x - \sin^2 x}{\cos^2 x \cdot \sin^2 x}dx = \int \frac{dx}{\sin^2 x} - \int \frac{dx}{\cos^2 x} =$
$= -(\mathrm{ctg}x + \mathrm{tg}x) + C$.

### 1.4. Testing

1. What function is called as an antiderivative function?
2. How much antiderivatives may be existed for the same function?
3. How are connected two antiderivatives for the same function?
4. What is called an indefinite integral?
5. What derivative of indefinite integral is equal to?
6. What differential from indefinite integral is equal to?
7. Formulate the basic properties of an indefinite integral having the relation to algebraic operations.
8. How do you check correctness of found antiderivative?
9. Write down the table of basic indefinite integrals.

## 2. Methods of Integration

### 2.1. Direct Integration of the Function

While integrating the functions we seldom have on opportunity to use the basic formulas directly. As a rule, the integrand has to be transformed in order to reduce the integral to the tabular one. Consider several examples of such transformations:

1. $\int \frac{dx}{\cos^2 x \cdot \sin^2 x} = \left\| \begin{array}{l} \textit{since } \cos^2 x + \sin^2 x = 1 \\ \textit{then unity in numerator may be} \\ \textit{replaced by expression} \end{array} \right\| = \int \frac{\cos^2 x + \sin^2 x}{\cos^2 x \cdot \sin^2 x}dx =$



$= \|\textit{devide by members}\| = \int \frac{dx}{\sin^2 x} + \int \frac{dx}{\cos^2 x} = -\cot x + \tan x + C$.

2. $\int \tan^2 x dx = \left\| \begin{array}{l} \textit{adding and substracting} \\ \textit{one to the integrand we get} \end{array} \right\| = \int\left(\left(1+\tan^2 x\right) - 1\right)dx =$

$= \int\left(1+\tan^2 x\right) + \int dx = \left\| \begin{array}{l} \textit{use trigonometric formula} \\ 1+\tan^2 x = \frac{1}{\cos^2 x} \end{array} \right\| =$

$= \int \frac{dx}{\cos^2 x} + x = \tan x + x + C$.

3. $\int 2^{3x} \cdot e^{2x} dx = \int \left(2^3\right)^x \cdot \left(e^2\right)^x dx = \int \left(8e^2\right)^x dx = \frac{\left(8 \cdot e^2\right)^x}{\ln\left(8 \cdot e^2\right)} + C = \frac{8^x \cdot e^{2x}}{3\ln 2 + 2} + C$.

4. $\int \frac{1+2x^2}{\left(1+x^2\right)x^2}dx = \int \frac{1+x^2+x^2}{\left(1+x^2\right)x^2}dx = \int \frac{dx}{x^2} + \int \frac{dx}{1+x^2} = -\frac{1}{x} + \arctan x$

**Note 1.** An indefinite integral does not depend on designation of the integration variable, that is

$$\int f(x)dx = \int f(u)du = \int f(t)dt.$$

**Note 2.** Any formula of integration preserves it is form if a variable of integration is an independent or some differentiable function, that is:
if

$$\int f(x)dx = F(x) + C,$$

then

$$\int f(\varphi(x))d(\varphi(x)) = F(\varphi(x)) + C.$$

Indeed, since $\int f(x)dx = F(x) + C$, then $F'(u) = f(u)$. By virtue of preservation form of the differential of the first order we get $dF(u) = F'(u)du = f(u)du$. Let us consider

$$\int f(u)du = \int dF(u) = F(u) + C$$

or

$$\int f(\varphi(x))d(\varphi(x)) = F(\varphi(x)) + C.$$

**Note 3.** Let we know that

$$\int f(x)dx = F(x) + C,$$

then

$$\int f(ax+b)dx = \frac{1}{a}F(ax+b)+C.$$

Indeed, let us consider the integral $\int f(ax+b)dx$. Taking into account that $d(ax+b)=adx$ we can write argument of integrand under differential sign. Then we get

$$\int f(ax+b)dx = \frac{1}{a}\int f(ax+b)d(ax+b) = \frac{1}{a}F(ax+b)+C.$$

**Example.**

$$\int \sin^2 x dx = \int \frac{1-\cos 2x}{2}dx = \frac{1}{2}\int dx - \frac{1}{2}\int \cos 2x dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + C$$

You must always remember that

1) integral is not changed if you add or substract constant to function under differential sign, i.e. $\int f(x)dx = \int f(x)d(x+a)$

$$\int \frac{dx}{x+a} = \int \frac{d(x+a)}{x+a} = \ln|x+a|+C.$$

2) If you multiply function, standing under differential sign by a constant then indefinite integral will be divided on the same constant, hence the next equality

$$\int f(x)d(ax+c) = a\int f(x)dx \Rightarrow$$

$$\int f(x)dx = \frac{1}{a}\int f(x)d(ax+c)$$

is truth.

**Example**. Many of integrals may be calculated by transformation of expression under differential sign.

1. $\int \sin 2x \cos 5x dx = \left\| \sin\alpha\cos\beta = \frac{1}{2}(\sin(\alpha+\beta)x + \sin(\alpha-\beta)x) \right\| =$

$= \frac{1}{2}\int(\sin 7x - \sin 3x)dx = \frac{-1}{14}\cos 7x + \frac{1}{6}\cos 3x + C;$

2. $\int \frac{\cos^3 x}{\sin^4 x}dx = \int \frac{\cos^2 x \cdot d(\sin x)}{\sin^4 x} = \int \frac{(1-\sin^2 x)}{\sin^4 x}d\sin x = -\frac{1}{3\sin^3 x} + \frac{1}{\sin x} + C.$

3. $\int \frac{dx}{\sin x} = \int \frac{dx}{2\sin\frac{x}{2}\cos\frac{x}{2}} = \frac{1}{2}\int \frac{dx}{\cos^2\frac{x}{2}\cdot\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}} = \int \frac{d(\tan\frac{x}{2})}{\tan\frac{x}{2}} = \ln\left|\tan\frac{x}{2}\right| + C.$



4. $\int x^3\sqrt{1+x^2}dx = \int x^2\sqrt{1+x^2}xdx = \int \frac{1}{2}(x^2+1-1)\sqrt{1+x^2}d(x^2+1) =$

$= \frac{1}{2}\int(x^2+1)\sqrt{1+x^2}d(x^2+1) - \frac{1}{2}\int\sqrt{1+x^2}d(x^2+1) = \frac{1}{2}(x^2+1)^{5/2}\cdot\frac{2}{5} +$

$+ \frac{8}{8\cdot 3}(x^2+1)^{3/2} + C.$

5. $\int \frac{dx}{2\cos^2 x + \sin^2 x} = \int \frac{dx}{\cos^2 x(2+\tan^2 x)} = \int \frac{d(\tan x)}{2+\tan^2 x} =$

$= \frac{1}{\sqrt{2}}\arctan\frac{\tan x}{\sqrt{2}} + C.$

6. $\int \frac{dx}{x\sqrt{4-\ln^2 x}} = \int \frac{d(\ln x)}{\sqrt{4-\ln^2 x}} = \|u=\ln x\| = \int \frac{du}{\sqrt{4-u^2}} = \arcsin\left(\frac{\ln x}{2}\right) + C.$

7. $\int \frac{x^2 dx}{\sqrt{8-x^3}} = -\frac{1}{3}\int \frac{d(8-x^3)}{\sqrt{8-x^3}} = -\frac{2}{3}\sqrt{8-x^3} + C.$

## 2.2. Integrals of Function Containing a Quadratic Trinomial

Let the integral be given

$$I = \int \frac{dx}{ax^2+bx+c}.$$

Let us first transform the trinomial in the denominator by representing it in the form of the sum or the difference of squares. It may be done as follows

$$ax^2+bx+c = a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right) = a\left[x^2+\frac{b}{a}x+\frac{b^2}{4a^2}-\frac{b^2}{4a^2}+\frac{c}{a}\right] =$$

$$= a\left[\left(x+\frac{b}{2a}\right)^2 + \underbrace{\frac{c}{a}-\frac{b^2}{4a^2}}_{\pm k^2}\right].$$

Then let us change the variable of integration:

$$x+\frac{b}{2a}=u;\ dx=du,\ I=\frac{1}{a}\int\frac{du}{u^2\pm k^2}.$$

These are tabular integrals.

**Example 1.** Calculate the integral.

$$\int \frac{dx}{2x^2+8x+20} = \frac{1}{2}\int \frac{dx}{x^2+4x+10} = \left\| \begin{matrix} x^2+4x+10=(x+2)^2+6 \\ x+2=u; \quad dx=du \end{matrix} \right\| = \frac{1}{2}\int\frac{du}{u^2+6} =$$

$$=\frac{1}{2}\cdot\frac{1}{\sqrt{6}}\arctan\frac{x+2}{\sqrt{6}}+C.$$

**Example 2.** $\int\frac{x+3}{x^2-2x+5}dx=\int\frac{x+3}{(x-1)^2+4}dx=\left\|\begin{matrix}x-1=u\\ dx=du\\ x=u+1\end{matrix}\right\|=\int\frac{u+4}{u^2+4}du=$

$$=\int\frac{udu}{u^2+4}+4\int\frac{du}{u^2+4}=\frac{1}{2}\int\frac{d\left(u^2+4\right)}{u^2+4}+\frac{4}{2}\arctan\frac{u}{2}=\frac{1}{2}\ln\left(u^2+4\right)+2\arctan\frac{u}{2}+C=$$

$$=\frac{1}{2}\ln\left(x^2-2x+5\right)+2\arctan\frac{x-1}{2}+C.$$

**Example 3.** $\int\frac{5x+3}{\sqrt{x^2+4x+10}}dx=\int\frac{5x+3}{\sqrt{(x+2)^2+6}}dx=\left\|\begin{matrix}x+2=u\\ dx=du\\ x=u-2\end{matrix}\right\|=$

$$=\int\frac{5u-10+3}{\sqrt{u^2+6}}du=5\int\frac{udu}{\sqrt{u^2+6}}-7\int\frac{du}{\sqrt{u^2+6}}=\frac{5}{2}\int\frac{d\left(u^2+6\right)}{\sqrt{u^2+6}}-7\ln\left|u+\sqrt{u^2+6}\right|=$$

$$=5\sqrt{x^2+4x+10}-7\ln\left|x+2+\sqrt{x^2+4x+10}+C\right|.$$

### 2.3. Integration by Substitution (Change of Variable)

Let it be required to find the integral

$$\int f(x)dx.$$

Suppose that we cannot directly select the antiderivative of $f(x)$ but we know that it exists. Let us change the variable in the expression under the integral sign, putting

$$x=\varphi(t), \qquad (1.6)$$

where $\varphi(t)$ is a continuous monotone function with continuous derivative (it is known, that such function has inverse function). Then $dx=\varphi'(t)dt$. Let us prove that in this case the following equality

$$\int f(x)dx=\int f(\varphi(t))\varphi'(t)dt \qquad (1.7)$$

is valid.

Here it is assumed that after integration we'll substitute on the right side the expression of $t$ in terms of $x$ on the basis (1.6).

To establish the equality (1.7) should be treated in the sense indicated early, it is necessary to prove that their derivatives with respect to $x$ are equal each to other.

Find the derivative of the left side



$$\left(\int f(x)dx\right)'_x=f(x).$$

Let us differentiate the right side of (1.7) with respect to $x$ as a composite function, where $t$ is the intermediate argument. The dependence of $t$ from $x$ is expressed by (1.6). We thus have

$$\left(\int f(\varphi(t))\varphi'(t)dt\right)'_x=\left(\int f(\varphi(t))\varphi'(t)dt\right)'_t\frac{dt}{dx}=f(\varphi(t))\varphi'(t)\frac{1}{\varphi'(t)}=f(\varphi(t))=f(x)$$

So the derivatives with respect to $x$ of the right and left sides of (1.7) are equal each to other as it is required.

The function $x=\varphi(t)$ should be chosen so that one can evaluate the indefinite integral on the right side of (1.7).

**Note.** It is sometimes better to choose a change of the variable at the form of $t=\psi(x)$, but not at $x=\varphi(t)$. For example let it be required to calculate an integral of the form

$$\int\frac{\psi'(x)}{\psi(x)}dx.$$

Here it is convenient to put $\psi(x)=t$ then $\psi'(x)dx=dt$, and

$$\int\frac{\psi'(x)}{\psi(x)}dx=\int\frac{dt}{t}=\ln|t|+C=\ln|\psi(x)|+C.$$

**Example.**

1) $\int\tan xdx=\int\frac{\sin x}{\cos x}dx=-\int\frac{d(\cos x)}{\cos x}=-\ln|\cos x|+C.$

2) $\int\frac{2x+5}{x^2+5x+25}dx=\ln\left(x^2+5x+25\right)+C.$

3) $\int\frac{2x+3}{x^2+3x+10}dx=\ln\left|x^2+3x+10\right|+C;$

4) $\int\frac{e^x}{e^x+5}dx=\ln\left(e^x+5\right)+C;$

5) $\int\frac{dx}{\left(1+x^2\right)\text{arctg}x}=\ln|\text{arctg}x|+C;$

6) $\int\frac{\left(3x^2+5x\right)dx}{x^3+\frac{5}{2}x^2+10}=\ln\left|x^3+\frac{5}{2}x^2+10\right|+C.$

You have to remember that success of integration depends largely on how appropriate substitution simplifies the given integral.

**Example.**

1) $$\int\frac{\arctan x\sqrt{1+\arctan^2 x}}{1+x^2}dx=\left\|\begin{matrix}1+\arctan^2 x=t^2\\ \frac{2\arctan x}{1+x^2}dx=2tdt\end{matrix}\right\|=\int t^2dt=\frac{t^3}{3}+C=$$

$$=\frac{\left(\sqrt{1+\arctan^2 x}\right)^3}{3}+C.$$

2) $$\int\frac{\cos x}{\sqrt{e^{\sin x}-1}}dx=\left\|\begin{matrix}e^{\sin x}-1=t^2\\ e^{\sin x}\cos xdx=2tdt\\ \cos xdx=\frac{2tdt}{t^2+1}\end{matrix}\right\|=\int\frac{2tdt}{(t^2+1)t}=2\arctan\sqrt{e^{\sin x}-1}+C.$$

As you can see the method of substitution is one of the basic method for calculating indefinite integrals. It should be noted that even when we integrate by some other method we often resort to substitution in the intermediate stages of calculation.

Essentially, the study of methods of integration is reduced to finding out what kind of substitution has to be performed for a given element of integration.

Bellow some useful recommendations for integration some irrational functions are given.

Let integrand be expression like $R\left(x,\sqrt{x^2\pm a^2}\right)$, where $R$ – is sign of rational function of its arguments. What does "rational function" mean?

It means that operations of addition, multiplication and involution in integer power can be only carried out under arguments of this function.

While integrating the expressions like $R\left(x,\sqrt{x^2\pm a^2}\right)$ we use such substitutions that allow to get rid of irrationality

a) if $R\left(x,\sqrt{a^2+x^2}\right)$,

then you can use substitutions $x=a\tan t$ or $x=asht$;

b) if $R\left(x,\sqrt{a^2-x^2}\right.$,

then the substitutions $x=a\sin t$ or $x=a\cos t$ are convenient;

c) if $R\left(x,\sqrt{x^2-a^2}\right)$ there are convenient the following substitutions



$$x=\frac{a}{\cos t},\ x=\frac{a}{\sin t}\qquad\text{or}\qquad x=acht.$$

**For example.**

a) $$\int\sqrt{1-x^2}dx=\left\|\begin{matrix}x=\sin t\\ dx=\cos tdt\end{matrix}\right\|=$$

$$=\int\sqrt{1-\sin^2 t}\cos tdt=\int\cos^2 tdt=\int\frac{1+\cos 2t}{2}dt=\frac{1}{2}\left(t+\frac{1}{2}\sin 2t\right)=\left(\arcsin x+x\sqrt{1-x^2}\right)+C$$

b) $$\int\frac{dx}{x\cdot\sqrt{1+x^2}}=\left\|\begin{matrix}x=\tan t\\ dx=\frac{1}{\cos^2 t}dt\end{matrix}\right\|=\int\frac{\cos tdt}{\cos^2 t\sin t\sqrt{1+\tan^2 t}}=\int\frac{dt}{\sin t}=\ln\left|\tan\frac{t}{2}\right|+C=$$

$$=\ln\left|\tan(\arctan x/2)+C\right|.$$

**Note.** This integral may be calculated more simple.

$$\int\frac{dx}{x\sqrt{1+x^2}}=\int\frac{dx}{x^2\sqrt{1+\left(\frac{1}{x}\right)^2}}=-\int\frac{d\left(\frac{1}{x}\right)}{\sqrt{1+\left(\frac{1}{x}\right)^2}}=-\ln\left|\frac{1}{x}+\sqrt{1+\left(\frac{1}{x}\right)^2}\right|+C.$$

c) $$\int\frac{\sqrt{x^2-4}}{x}dx=\left\|\begin{matrix}x=\frac{2}{\cos t}\\ dx=\frac{2\sin t}{\cos^2 t}dt\end{matrix}\right\|=\int\frac{2\sqrt{1-\cos^2 t}\cos t2\sin t}{\cos t2\cos^2 t}dt=$$

$$=2\int\tan^2 dt=2\int\frac{1-\cos^2 t}{\cos^2 t}dt=2(\tan t-t)+C,$$

where $t=\arccos\frac{2}{x}$.

It should be noted that the last integrals could be calculated by another substitution.

### 2.4. Method of Integration by Parts

Let $u$ and $v$ be two differentiable functions of $x$. Then the differential of the product $u\cdot v$ is found as

$$d(uv)=udv+vdu.$$

Whence, by integration of the left and right sides, we obtain

$$uv = \int u dv + \int v du$$

or transposing one of the integrals on the left side, we get

$$\int u dv = uv - \int v du. \quad (1.8)$$

This formula is called ***the formula of integration by parts***.

**Note**. The formula (1.8) does not produce the final result but only transforms the problem from calculating $\int u dv$ to calculating the integral $\int v du$, which is simpler at the successful choice $u$ and $v$.

It should be mentioned there are no common methods for the right choice of $u$ and $dv$. Bellow some successful recommendations for certain cases are proposed.

As a rule this method is used in the case when the integrand contains a product of rational and transcendental functions and other methods are unsuitable. The typical integrals, which are calculated by method of integration by parts are following: $\int P_n(x)\cos\alpha x dx$, $\int P_n(x)\sin\alpha x dx$, $\int P_n(x)e^{\alpha x}dx$, $\int x^k arc\tan xdx$, $\int x^k \ln x dx$.

1. If the integrand looks like $P_n(x)\cos\alpha x$, $P_n(x)\sin\alpha x$, $P_n(x)e^{\alpha x}$, then the polynomial $P_n$ is selected as $u$.
2. If the integrand looks like $x^k \arctan x$, $x^k \ln x$, that is as product of logarithmic or inverse trigonometric functions by a polynomial then the $\arctan x$ (inverse trigonometric function) or $\ln x$ are selected as "$u$".

**Example 1.**

$$\int x\sin xdx = \left\| \begin{matrix} x = u \Rightarrow du = dx \\ \sin xdx = dv \Rightarrow v = -\cos x \end{matrix} \right\| = -x\cdot\cos x + \int \cos x du = -x\cos x + \sin x + C.$$

**Example 2.**

$$\int \arctan xdx = \left\| \begin{matrix} \arctan x = u \Rightarrow du = \frac{1}{l+x^2}dx \\ dx = dv \Rightarrow v = x \end{matrix} \right\|.$$

**Solution.** If we try to match $\arctan xdx$ with $udv$, we may take $u = \arctan x$ and $dv = dx$. To use (1.8) it is required to find $du = \frac{1}{1+x^2}dx$, $v = x$, then

$$I = xarc\tan x - \int \frac{x}{1+x^2}dx = xarc\tan x - \frac{1}{2}\int \frac{d(x^2+1)}{x^2+1} = xarc\tan x - \frac{1}{2}\ln(x^2+1) + C.$$

Sometimes an integration by parts must be repeated to obtain an answer, as in the following example.

**Example 3.**



$$\int x^2 e^x dx = \left\| \begin{matrix} let\ x^2 = u \Rightarrow 2xdx = du \\ e^x dx = dv \Rightarrow v = e^x \end{matrix} \right\| = x^2 e^x - 2\int xe^x dx =$$

The integral on the right is similar to the original integral, except that we have reduced the power of $x$ from 2 to 1.

If we could now reduce it from 1 to 0 we could see success ahead. In integral $\int xe^x dx$ we therefore put

$$\begin{matrix} u = x \\ du = dx \end{matrix} \quad \text{and} \quad \begin{matrix} dv = e^x dx \\ v = e^x \end{matrix}, \text{ so that.}$$

Then

$$I = x^2 e^x - 2(xe^x - e^x) + C = e^x(x^2 - 2x + 2) + C.$$

The integral in the following example occurs in electrical engineering problems

$$\int e^{2x}\cos bxdx.$$

Its evaluation requires two integration by parts. After that we get equation for the unknown integral.

**Solution.** Let $u = e^{ax}$ and $dv = \cos bxdx$. Then $du = ae^{2x}dx$ and $v = \frac{1}{b}\sin bx$.

$$I = \frac{1}{b}e^{ax}\sin bx - \frac{a}{b}\int e^{ax}\sin bxdx.$$

The second integral is like the first one except that it has $\sin bx$ in place of $\cos bx$. Apply integration by parts to it again, putting,

$$\begin{matrix} U = e^{ax} \\ dU = ae^{ax}dx \end{matrix} \quad \text{and} \quad \begin{matrix} dV = \sin bxdx \\ V = -\frac{1}{b}\cos bx \end{matrix}.$$

Then we get

$$I = \frac{1}{b}e^{ax}\sin bx - \frac{a}{b}\left(\frac{e^{ax}}{b}\cos bx + \frac{a}{b}\int e^{ax}\cos bxdx\right) =$$

$$= \frac{1}{b}e^{ax}\sin bx + \frac{a}{b}e^{ax}\cos bx - \frac{a^2}{b^2}I.$$

Now the unknown integral appears on the left with a coefficient "unity" and on the right with a coefficient "$-\frac{a^2}{b^2}$". Transposing this term to the left and dividing by the new coefficient

$$1 + \frac{a^2}{b^2} = \frac{b^2 + a^2}{b^2},$$

we obtain

$$\int e^{ax}\cos bx\,dx=\frac{e^{ax}}{a^2+b^2}[b\sin bx+a\cos bx]+C.$$

**Example:** a) $\int (x+5)\sin 3x\,dx$;

b) $\int x\arcsin x\,dx$;

c) $\int x\ln x\,dx$;

d) $\int \sin\ln x\,dx$, $\int e^{x}\cos x\,dx$.

**Solution.**

a) $$\int (x+5)\sin 3x\,dx=\left\|\begin{matrix} x+5=u, & du=dx \\ \sin 3x\,dx=dv, & v=\int\sin 3x\,dx=-\frac{1}{3}\cos 3x \end{matrix}\right\|=$$

$$=-\frac{1}{3}(x+5)\cdot\cos 3x+\frac{1}{3}\int\cos 3x\cdot dx=-\frac{1}{3}(x+5)\cdot\cos 3x+\frac{1}{9}\sin 3x+C.$$

b) $$\int x\arcsin x\,dx=\left\|\begin{matrix} \arcsin x=u, & du=\frac{1}{\sqrt{1-x^2}}dx \\ x\,dx=dv, & v=\int x\,dx=\frac{1}{2}x^2 \end{matrix}\right\|=$$

$$=\frac{1}{2}x^2\cdot\arcsin x-\frac{1}{2}\int\frac{x^2dx}{\sqrt{1-x^2}}=\frac{1}{2}\left(x^2\arcsin x-I\right).$$

Where $I$ denotes integral $I=\int\frac{x^2dx}{\sqrt{1-x^2}}$.

To evaluate this integral we can use trigonometric substitution $x=\cos t$, then $dx=-\sin t\,dt$ and

$$I=-\int\frac{\cos^2t\cdot\sin t}{\sin t}dt=-\int\frac{1+\cos 2t}{2}dt=-\left(\frac{1}{2}t+\frac{1}{4}\sin 2t\right)+C=$$

$$=\left\|\begin{matrix} t=\arccos x \\ \sin 2t=2\sin t\cdot\underbrace{\cos t}_{x}=2\cdot\sqrt{1-\cos^2t}\cdot\cos t=2\cdot\sqrt{1-x^2} \end{matrix}\right\|=$$

$$=-\frac{1}{2}\left(\arccos x+x\sqrt{1-x^2}\right)+C.$$

Finally,

$$\int x\arcsin x=\frac{1}{2}\left(x^2\arcsin x+\frac{1}{2}\arccos x+\frac{1}{2}x\sqrt{1-x^2}\right)+C.$$



c) $$\int x\ln x\,dx=\left\|\begin{matrix} \ln x=u, & du=\frac{1}{x}dx \\ x\,dx=dv, & v=\frac{x^2}{2} \end{matrix}\right\|=\frac{1}{2}x^2\ln x-\frac{1}{2}\int\frac{x^2}{x}dx=\frac{1}{2}x^2\ln x-\frac{1}{4}x^2+C.$$

d) $$\int\sin\ln x\,dx=\left\|\begin{matrix} \sin\ln x=u, & du=\frac{\cos\ln x}{x}dx \\ dx=dv, & v=x \end{matrix}\right\|=x\cdot\sin\ln x-\int x\cdot\frac{\cos\ln x}{x}dx=$$

$$=x\cdot\sin\ln x-\int\cos\ln x\,dx=\left\|\begin{matrix} \cos\ln x=u, & du=\frac{\sin\ln x}{x} \\ dx=dv, & v=x \end{matrix}\right\|=$$

$$=x\sin\ln x-\left(x\cdot\cos\ln x+\int x\cdot\frac{\sin\ln x}{x}dx\right)=x\left(\sin\ln x-\cos\ln x\right)+\int\sin\ln x\,dx.$$

On the right side we got the same integral which is on the left side. In other words we received the equation concerning integral. Transporting the unknown integral from right side to left one we get

$$\int\sin\ln x\,dx=\frac{1}{2}x\left(\sin\ln x-\cos\ln x\right)+C.$$

## 3. Complex Numbers and Operations With Them

### 3.1. Definition. Algebraic Form of the Complex Number

A pair of the real number $a$ and $b$, considered as union complex presented by expression $z=a+bi$ is called a complex number. There symbol $i$ is called imaginary unit, which is defined as

$i=\sqrt{-1}$ or $i^2=-1$

The number $a$ is called the ***real part*** and $b$ the ***imaginary part*** of the complex number $z=a+bi$. They are designated, respectively, as follows:

$$a=\operatorname{Re}z,\ b=\operatorname{Im}z.$$

If $a$=0, then the number $z=0+bi$ or just $z=bi$ is a pure imaginary. If $b$=0, then we have the real number $a+i0=a$ or just $z$=$a$. It said that the complex number transforms into real number. Consequently the set of real numbers is subset of all complex numbers. We agree upon the two following basic definitions.

1. Two complex numbers $z_1=a_1+b_1i$ and $z_2=a_2+b_2i$ are equal each to other if and only if $a_1=a_2$, $b_1=b_2$. In other words

$$z_1=z_2\Leftrightarrow \operatorname{Re}z_1=\operatorname{Re}z_2,$$
$$\operatorname{Im}z_1=\operatorname{Im}z_2.$$

2. A complex number z is equal to zero $z=a+bi$=0 if and only if $a$=$b$=0.