3. Integration by Parts

This method reduces the problem of finding the integral to calculation of another simpler integral constructed by means the initial integrand. To get the formula of integration by parts, let us consider first two differentiable functions u and v and find the differential of their product:

$$d(u\cdot v)=du\cdot v+u\cdot dv.$$

After integration of the last equality we get:

$$\int d(u\cdot v)=\int du\cdot v+\int u\cdot dv,$$

$$u\cdot v=\int du\cdot v+\int u\cdot dv,$$

$$\boxed{\int u\cdot dv=u\cdot v-\int v\cdot du}.$$

The last formula is called *the formula of integration by parts*.

How to use it? First of all let us remember that $dv=v'dx,\ du=u'dx$ and therefore in that formula the initial integrand and the initial integral, namely

$$f(x)=u\cdot v' \quad \text{and} \quad \int f(x)dx=\int u\cdot v'dx,$$

are replaced by new integrand and new integral, namely

$$g(x)=u'\cdot v \quad \text{and} \quad \int g(x)dx=\int u'\cdot v\,dx.$$

Thus, to obtain new integral, one of two factors in $f(x)$ should be differentiated, another one should be integrated. This method is used to simplify the integral, therefore $f(x)$ *should be factorized in such way that after differentiating of one of factors the obtained integral becomes simpler.*

Example. Find $\int xe^x dx$. We should analyze which of two factors, x or e^x, gets simpler when ones finds the derivative of it? Since e^x is unchanged at differentiating, it is better to differentiate the polynomial to decrease its power and make the integral simpler. Let us consider both possible variants to check the validity of the assumption:

1) $\begin{cases} u=x \\ dv=e^x dx \end{cases} \Rightarrow \begin{cases} du=1\cdot dx \\ v=e^x \end{cases} \Rightarrow \int v\cdot du=\int e^x dx$

2) $\begin{cases} u = e^x \\ dv = xdx \end{cases} \Rightarrow \begin{cases} du = e^x \cdot dx \\ v = x^2/2 \end{cases} \Rightarrow \int v \cdot du = \frac{1}{2}\int x^2 e^x dx$

Variant one gives simpler integral. Let us apply the formula by integration by parts to find answer:

$$\int xe^x dx = \begin{bmatrix} u = x \\ dv = e^x dx \\ du = 1 \cdot dx \\ v = e^x \end{bmatrix} = \underbrace{x}_{u}\underbrace{e^x}_{v} - \int \underbrace{e^x}_{v}\underbrace{dx}_{du} = xe^x - e^x + C .$$

Note 1. Sometimes the integration by parts must be repeated to get the answer. For example, in the integral $\int x^2 e^x dx$ we have to differentiate x^2 twice to get constant and to obtain simpler integral:

$$\int x^2 e^x dx = \begin{bmatrix} u = x^2 \\ dv = e^x dx \\ du = 2x \cdot dx \\ v = e^x \end{bmatrix} = x^2 e^x - \int 2xe^x dx = \begin{bmatrix} u = -2x \\ dv = e^x dx \\ du = -2 \cdot dx \\ v = e^x \end{bmatrix} = x^2 e^x + (-2x)e^x - \int -2e^x dx =$$

$$= x^2 e^x - 2xe^x + 2e^x + C .$$

Note 2. As a rule this method is used in the case when the integrand contains a product of polynomial and either trigonometrical or transcendental functions. For such integrands the following recommendations could be given.

For the integral $\int P_n(x) f(x)\, dx$, *where* $P_n(x)$ *is a polynomial of the n-th order, use the method of integration by parts with the following choice of factors:*

$$f(x) = \begin{cases} \cos ax \\ \sin ax \\ a^{bx} \\ e^{bx} \end{cases} \Rightarrow \begin{cases} u = P_n(x) \\ dv = f(x)dx \end{cases} \qquad f(x) = \begin{cases} \arcsin ax \\ \arccos ax \\ \arctan ax \\ \operatorname{arccot} ax \\ \ln x \\ \log_a x \end{cases} \Rightarrow \begin{cases} u = f(x) \\ dv = P_n(x)dx \end{cases}$$

Examples.

1. $\int x\sin x dx = \begin{vmatrix} u = x, \ du = dx \\ dv = \sin x dx \\ v = -\cos x \end{vmatrix} = -x\cos x + \int \cos x dx = -x\cos x + \sin x + C.$

2. $\int x \arctan x dx = \begin{vmatrix} u = \arctan x; \ du = \dfrac{dx}{1+x^2} \\ dv = xdx; \quad v = \dfrac{x^2}{2} \end{vmatrix} = \arctan x \cdot \dfrac{x^2}{2} - \dfrac{1}{2}\int x^2 \dfrac{dx}{1+x^2} =$

$= \dfrac{x^2}{2}\operatorname{arctg} x - \dfrac{1}{2}\int \dfrac{(x^2+1)-1}{1+x^2}dx = \dfrac{x^2}{2}\arctan x - \dfrac{1}{2}\int dx + \dfrac{1}{2}\int \dfrac{dx}{1+x^2} = \dfrac{x^2}{2}\arctan x - \dfrac{x}{2} +$

$+\dfrac{1}{2}\arctan x + C.$

3. $\int \ln(x+1)dx = \begin{vmatrix} u = \ln(x+1); \ du = \dfrac{dx}{x+1} \\ dv = 1dx; \quad v = x+1 \end{vmatrix} = (x+1)\ln(x+1) - \int (x+1)\dfrac{dx}{x+1} =$

$= (x+1)\ln(x+1) - \int dx = (x+1)\ln(x+1) - x + C.$

c) $\int x\ln x dx = \begin{Vmatrix} \ln x = u, \quad du = \dfrac{1}{x}dx \\ xdx = dv, \ \ v = \dfrac{x^2}{2} \end{Vmatrix} = \dfrac{1}{2}x^2\ln x - \dfrac{1}{2}\int \dfrac{x^2}{x}dx = \dfrac{1}{2}x^2\ln x - \dfrac{1}{4}x^2 + C.$

b) $\int x\arcsin x dx = \begin{Vmatrix} \arcsin x = u, \ \ du = \dfrac{1}{\sqrt{1-x^2}}dx \\ xdx = dv, \quad v = \int xdx = \dfrac{1}{2}x^2 \end{Vmatrix} =$

$= \dfrac{1}{2}x^2 \cdot \arcsin x - \dfrac{1}{2}\int \dfrac{x^2dx}{\sqrt{1-x^2}} = \dfrac{1}{2}\left(x^2\arcsin x - I\right).$

Where I denotes integral $I = \int \dfrac{x^2dx}{\sqrt{1-x^2}}$.

To evaluate this integral we can use trigonometric substitution $x = \cos t$, then $dx = -\sin t dt$ and

$$I = -\int \frac{\cos^2 t \cdot \sin t}{\sin t}dt = -\int \frac{1+\cos 2t}{2}dt = -\left(\frac{1}{2}t + \frac{1}{4}\sin 2t\right) + C =$$

$$= \begin{Vmatrix} t = \arccos x \\ \sin 2t = 2\sin t \cdot \underbrace{\cos t}_{x} = 2\cdot\sqrt{1-\cos^2 t}\cdot\cos t = 2\cdot\sqrt{1-x^2} \end{Vmatrix} =$$

$$= -\frac{1}{2}\left(\arccos x + x\sqrt{1-x^2}\right) + C.$$

Finally,

$$\int x\arcsin x = \frac{1}{2}\left(x^2\arcsin x + \frac{1}{2}\arccos x + \frac{1}{2}x\sqrt{1-x^2}\right) + C.$$

Note 3. Sometimes the integration by parts helps to derive the equation for calculating the initial integral. Let us demonstrate this procedure by two examples:

1. $I=\int\sqrt{a^2+x^2}\,dx$ **.**

$$I=\int\sqrt{a^2+x^2}\,dx=\begin{bmatrix} u=\sqrt{a^2+x^2},\ du=\dfrac{xdx}{\sqrt{a^2+x^2}} \\ dv=dx, \qquad v=x \end{bmatrix}=x\sqrt{a^2+x^2}-$$

$$-\int\frac{x^2dx}{\sqrt{a^2+x^2}}=x\sqrt{a^2+x^2}-\int\frac{\left(x^2+a^2\right)-a^2}{\sqrt{a^2+x^2}}dx=x\sqrt{a^2+x^2}-$$

$$-\underbrace{\int\sqrt{a^2+x^2}\,dx}_{I}+a^2\int\frac{dx}{\sqrt{a^2+x^2}}=x\sqrt{a^2+x^2}-I+a^2\ln\left(x+\sqrt{x^2+a^2}\right)+2C.$$

Thus we have the equation

$$I=x\sqrt{a^2+x^2}-I+a^2\ln\left(x+\sqrt{x^2+a^2}\right)+2C;$$

$$2I=x\sqrt{a^2+x^2}+a^2\ln\left(x+\sqrt{x^2+a^2}\right)+2C;$$

$$I=\int\sqrt{a^2+x^2}\,dx=\frac{1}{2}\left(x\sqrt{a^2+x^2}+a^2\ln\left(x+\sqrt{x^2+a^2}\right)\right)+C.$$

2. $I=\int e^{ax}\cos nx\,dx$.

$$I=\int e^{ax}\cos nx\,dx=\begin{bmatrix} u=e^{ax}, & du=ae^{ax}dx \\ dv=\cos nx\,dx, & v=\dfrac{\sin nx}{n} \end{bmatrix}=e^{ax}\frac{\sin nx}{n}-\frac{a}{n}\int e^{ax}\sin nx\,dx=$$

$$=\begin{bmatrix} u=e^{ax}, & du=ae^{ax}dx \\ dv=\sin nx\,dx, & v=-\dfrac{\cos nx}{n} \end{bmatrix}=e^{ax}\frac{\sin nx}{n}-\frac{a}{n}\left(-e^{ax}\frac{\cos nx}{n}+\frac{a}{n}\underbrace{\int e^{ax}\cos nx\,dx}_{I}\right)=$$

$$=\frac{e^{ax}}{n}\left(\sin nx+\frac{a\cos nx}{n}\right)-\frac{a^2}{n^2}I+C.$$

$$I=\frac{e^{ax}}{n}\left(\sin nx+\frac{a\cos nx}{n}\right)-\frac{a^2}{n^2}I+C.$$

$$I\left(1+\frac{a^2}{n^2}\right)=\frac{e^{ax}}{n^2}\left(n\sin nx+a\cos nx\right)+C.$$

$$I=\frac{e^{ax}}{a^2+n^2}\left(n\sin nx+a\cos nx\right)+\underbrace{\frac{C}{a^2+n^2}n^2}_{C_0}.$$

$$I=\int e^{ax}\cos nx\,dx=\frac{e^{ax}}{a^2+n^2}\left(n\sin nx+a\cos nx\right)+C_0.$$

4. Integration of Rational Functions

4.1. Integration of Partial Fractions

I. $\dfrac{A}{(x-a)^k}$ or II. $\dfrac{Ax+B}{(x^2+px+q)^s}$, where $p^2-4q<0$.

(1) Integration of the partial fraction of the first type.

$$\int \frac{A}{(x-\alpha)^k}dx = A\int \frac{dx}{(x-\alpha)^k}, \quad k=1,2,3,\dots.$$

We can substitute $x-\alpha=t$, obtaining

$$\int \frac{dx}{(x-\alpha)^k} = \int \frac{dt}{t^k}$$

and

$$\int \frac{dt}{t^k} = \begin{cases} \ln|t|+C, & \text{if } k=1,, \\ \dfrac{t^{1-k}}{1-k}, & \text{if } k>1. \end{cases}$$

Thus,

$$\int \frac{A}{(x-\alpha)^k}dx = \begin{cases} A\ln|x-\alpha|+C, & \text{if } k=1, \\ \dfrac{A}{(1-k)(x-\alpha)^{k-1}}, & \text{if } k>1. \end{cases}$$

1. $\int \frac{dx}{x+5}dx = \left\|\begin{matrix} t=x+5 \\ dt=dx \end{matrix}\right\| = \int \frac{dt}{t} = \ln|t|+C = \ln|x+5|+C$

2. $\int \frac{dx}{2x-7}dx = \left\|\begin{matrix} t=2x-7 \\ dt=2dx, \\ dx=dt/2 \end{matrix}\right\| = \frac{1}{2}\int \frac{dt}{t} = \frac{1}{2}\ln|t|+C = \frac{1}{2}\ln|2x-7|+C$

3. $\int \frac{dx}{(x-3)^5}dx = \left\|\begin{matrix} t=x-3 \\ dt=dx \end{matrix}\right\| = \int \frac{dt}{t^5} = -\frac{1}{4t^4}+C = -\frac{1}{4(x-3)^4}+C$

(2) Integrals Involving Quadratic Polynomials

Let the integral be given

$$I=\int\frac{dx}{ax^2+bx+c}.$$

Let us first transform the trinomial in the denominator by representing it in the form of the sum or the difference of squares. It may be done as follows

$$ax^2+bx+c=a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)=a\left[x^2+\frac{b}{a}x+\frac{b^2}{4a^2}-\frac{b^2}{4a^2}+\frac{c}{a}\right]=$$

$$=a\left[\left(x+\frac{b}{2a}\right)^2+\underbrace{\frac{c}{a}-\frac{b^2}{4a^2}}_{\pm k^2}\right].$$

Then let us change the variable of integration:

$$x+\frac{b}{2a}=u;\ dx=du,\ I=\frac{1}{a}\int\frac{du}{u^2\pm k^2}.$$

These are tabular integrals.

Example 1. Calculate the integral.

$$\int\frac{dx}{2x^2+8x+20}=\frac{1}{2}\int\frac{dx}{x^2+4x+10}=\left\|\begin{matrix}x^2+4x+10=(x+2)^2+6\\ x+2=u; \quad dx=du\end{matrix}\right\|=\frac{1}{2}\int\frac{du}{u^2+6}=$$

$$=\frac{1}{2}\cdot\frac{1}{\sqrt{6}}\arctan\frac{x+2}{\sqrt{6}}+C.$$

Example 2. $\int\frac{x+3}{x^2-2x+5}dx=\int\frac{x+3}{(x-1)^2+4}dx=\left\|\begin{matrix}x-1=u\\ dx=du\\ x=u+1\end{matrix}\right\|=\int\frac{u+4}{u^2+4}du=$

$$=\int\frac{udu}{u^2+4}+4\int\frac{du}{u^2+4}=\frac{1}{2}\int\frac{d\left(u^2+4\right)}{u^2+4}+\frac{4}{2}\arctan\frac{u}{2}=\frac{1}{2}\ln\left(u^2+4\right)+2\arctan\frac{u}{2}+C=$$

$$=\frac{1}{2}\ln\left(x^2-2x+5\right)+2\arctan\frac{x-1}{2}+C.$$

Example 3. $\int\frac{5x+3}{\sqrt{x^2+4x+10}}dx=\int\frac{5x+3}{\sqrt{(x+2)^2+6}}dx=\left\|\begin{matrix}x+2=u\\ dx=du\\ x=u-2\end{matrix}\right\|=$

$$=\int\frac{5u-10+3}{\sqrt{u^2+6}}du=5\int\frac{udu}{\sqrt{u^2+6}}-7\int\frac{du}{\sqrt{u^2+6}}=\frac{5}{2}\int\frac{d\left(u^2+6\right)}{\sqrt{u^2+6}}-7\ln\left|u+\sqrt{u^2+6}\right|=$$

$$=5\sqrt{x^2+4x+10}-7\ln\left|x+2+\sqrt{x^2+4x+10}+C\right|.$$

2. Find $\int \frac{dx}{\sqrt{9x^2+6x+5}}$.

$$9x^2+6x+5=\underbrace{9x^2}_{A^2}+\underbrace{6x}_{2AB}+5=\left[A=3x,\ B=\frac{6x}{2A}=\frac{6x}{6x}=1\right]=\underbrace{9x^2}_{A^2}+\underbrace{6x}_{2AB}+\underbrace{1^2}_{B^2}-1^2+5=$$

$$=(3x+1)^2-1+5=(3x+1)^2+4.$$

$$\int \frac{dx}{\sqrt{9x^2+6x+5}}=\int \frac{dx}{\sqrt{(3x+1)^2+4}}=[\text{Table, №17}]=\frac{\ln\left|3x+1+\sqrt{(3x+1)^2+4}\right|}{3}+C.$$

3. Find $\int \frac{dx}{\sqrt{7+12x-4x^2}}$.

Since $-4x^2\neq A^2$, we are going to take the sign $''-''$ out the expression $7+12x-4x^2$.

$$-\left(\underbrace{4x^2}_{A^2}-\underbrace{12x}_{2AB}-7\right)=\left[A=2x,\ B=\frac{12x}{2A}=\frac{12x}{4x}=3\right]=-\left(\underbrace{4x^2}_{A^2}-\underbrace{12x}_{2AB}+\underbrace{3^2}_{B^2}-3^2-7\right)=$$

$$=-\left((2x-3)^2-9-7\right)=16-(2x-3)^2.$$

$$\int \frac{dx}{\sqrt{7+12x-4x^2}}=\int \frac{dx}{\sqrt{16-(2x-3)^2}}=[\text{Table, №16}]=\frac{\arcsin\frac{2x-3}{4}}{2}+C.$$

Note. Integrals $\int \frac{dx}{x\sqrt{ax^2+bx+c}}$ and $\int \frac{dx}{(cx+d)\sqrt{ax^2+bx+c}}$ could be reduced to the integrals of the considered in this section type by means of substitutions $x=\frac{1}{t}$ $cx+d=\frac{1}{t}$, correspondingly.

Example. $\int \frac{dx}{x\sqrt{2x^2+2x+1}}=\left[\begin{array}{c} x=\frac{1}{t} \\ dx=\frac{-1}{t^2}dt \end{array}\right]=\int \frac{\frac{-1}{t^2}dt}{\frac{1}{t}\sqrt{\frac{2}{t^2}+\frac{2}{t}+1}}=\int \frac{-dt}{\sqrt{2+2t+t^2}}=$

$$\int \frac{-dt}{\sqrt{1+(1+t)^2}} = -\ln\left|1+t+\sqrt{1+(1+t)^2}\right| + C = -\ln\left|1+\frac{1}{x}+\sqrt{1+\left(1+\frac{1}{x}\right)^2}\right| + C.$$

4.2. Integration of $\int \frac{Cx+D}{ax^2+bx+c}dx$ and $\int \frac{Cx+D}{\sqrt{ax^2+bx+c}}dx$.

It is enough to allocate a full square and use a substitution $A \pm B = t$ in the integral.

Example. $I = \int \frac{4x+1}{25x^2-10x+2}dx.$

Let us allocate the complete square in the denominator.

$$25x^2-10x+2 = 25x^2-10x+1^2-1^2+2 = (5x-1)^2+1.$$

Then

$$I = \int \frac{4x+1}{(5x-1)^2+1}dx = \left[\begin{array}{c} 5x-1=t \\ x=\frac{t+1}{5};\ dx=\frac{dt}{5} \end{array}\right] = \int \frac{4\left(\frac{t+1}{5}\right)+1}{t^2+1}\frac{dt}{5} = \frac{4}{25}\int \frac{tdt}{t^2+1} + \frac{9}{25}\int \frac{dt}{t^2+1} =$$

$$= \frac{2}{25}\int \frac{2tdt}{t^2+1} + \frac{9}{25}\int \frac{dt}{t^2+1} = \frac{2}{25}\int \frac{d\left(t^2+1\right)}{t^2+1} + \frac{9}{25}\int \frac{dt}{t^2+1} = \frac{2}{25}\ln\left|t^2+1\right| + \frac{9}{25}\arctan t + C =$$

$$= \frac{2}{25}\ln\left|(5x-1)^2+1\right| + \frac{9}{25}\arctan(5x-1) + C.$$

Note. Integrals $\int \frac{dx}{x^2\sqrt{ax^2+bx+c}}$ and $\int \frac{dx}{(cx+d)^2\sqrt{ax^2+bx+c}}$ could be reduced to the integrals of the considered in this section type by means of substitutions $x = \frac{1}{t}$ $cx+d = \frac{1}{t}$, correspondingly.

Example.

$$\int \frac{dx}{x^2\sqrt{x^2+1}} = \left[\begin{array}{c} x=\frac{1}{t} \\ dx=\frac{-1}{t^2}dt \end{array}\right] = \int \frac{\frac{-1}{t^2}dt}{\frac{1}{t^2}\sqrt{\frac{1}{t^2}+1}} = \int \frac{-tdt}{\sqrt{1+t^2}} = -\frac{1}{2}\int \frac{2tdt}{\sqrt{1+t^2}} =$$

$$= -\frac{1}{2}\int \frac{2tdt}{\sqrt{1+t^2}} = -\frac{1}{2}\int \frac{d(1+t^2)}{\sqrt{1+t^2}} = -\frac{1}{2}2\sqrt{1+t^2} + C = -\sqrt{1+\frac{1}{x^2}} + C = -\frac{\sqrt{x^2+1}}{x} + C.$$