4. Integration of Rational Fractions

Rational fractions form the class of integrands that always could be integrated in the elementary functions.

Definition. Rational function is a ratio of two polynomials.

In general form the rational fraction could be written as

$$R(x)=\frac{P_n(x)}{Q_m(x)}=\frac{a_0x^n+a_1x^{n-1}+\ldots+a_{n-1}x+a_n}{b_0x^m+b_1x^{m-1}+\ldots+b_{m-1}x+b_m}.$$

Here $P_n(x)$ and $Q_m(x)$ are polynomials of the *n*-th and *m*-th degree, correspondingly.

Rational fraction is called proper if $n<m$, and improper in other case. Any improper fraction could be presented as sum of a polynomial and a proper rational fraction. This presentation could be obtained by dividing the numerator by denominator.

It is clear that if $R(x)$ is not proper, then by "long division" of $P(x)$ into $Q(x)$ we find that

$$R(x)=\frac{Q(x)}{P(x)}=F(x)+\frac{Q_1(x)}{P(x)},$$

where $F(x)$ is a polynomial and $\frac{Q_1(x)}{P(x)}$ is a proper fraction.

Example. $R(x)=\frac{x^4+2x^3-3x^2+1}{x^2-x}$.

$$\begin{array}{l|l}
\;\;x^4+2x^3+3x^2+1 & x^2-x \\ \hline
-\underline{x^4-x^3} & x^2+3x+6 \\
\quad\;\; 3x^3+3x^2 & \\
\quad -\underline{3x^3-3x^2} & \\
\qquad\quad 6x^2+1 & \\
\qquad -\underline{6x^2-6x} & \\
\qquad\qquad 6x+1 \;\; (\text{degree is less than } 2) &
\end{array}$$

So, x^2+3x+6 is a quotient, $6x+1$ is a remainder. Thus,

$$R(x)=\frac{x^4+2x^3-3x^2+1}{x^2-x}=x^2+3x+6+\frac{6x+1}{x^2-x}.$$

Recall that a rational function $R(x)$ is one that can be expressed as a quotient of two polynomials $Q(x)$ and $P(x)$. That is, $R(x)=\frac{Q(x)}{P(x)}$. Without loss of generality, we consider the case where $Q(x)$ and $P(x)$ have no common roots.

Definition 8.3. If the degree of the numerator $Q(x)$ is less than that of the denominator $P(x)$, we say that the rational fraction $R(x)$ is a *proper fraction*.

Lemma 8.1. *Let $R(x)$ be a proper fraction with real coefficients Suppose that the real number α is a repeated root of $P(x)$ of multiplicity k,*

$$P(x) = (x-\alpha)^k \varphi(x), \quad \varphi(x) \neq 0. \tag{8.6}$$

Then $R(x)$ has the representation

$$R(x) = \frac{Q(x)}{P(x)} = \frac{A}{(x-\alpha)^k} + \frac{\psi(x)}{(x-\alpha)^{k-\lambda}\varphi(x)}, \tag{8.7}$$

where $A = \dfrac{Q(\alpha)}{\varphi(\alpha)}$ is a real constant, λ $(\lambda \geqslant 1)$ is an integer, and $\psi(x)$ is the polynomial such that the last fraction in (8.7) is proper.

□ Since, $A = \dfrac{Q(\alpha)}{\varphi(\alpha)}$ is a finite real number. On the other hand, the number is a repeated root of the polynomial $\varphi(x) = Q(x) - A\varphi(x)$ with multiplicity $\lambda \geqslant 1$, because

$$\varphi(\alpha) = Q(\alpha) - A\varphi(\alpha) = Q(\alpha) - \frac{Q(\alpha)}{\varphi(\alpha)}\varphi(\alpha) = 0.$$

Hence, there exists a representation

$$\varphi(x) = (x-\alpha)^{\lambda}\psi(x), \quad \psi(\alpha) \neq 0,$$

where $\psi(x)$ is a polynomial.

Then we can write

$$\frac{Q(x)}{P(x)} - \frac{A}{(x-\alpha)^k} = \frac{Q(x) - A\varphi(x)}{(x-\alpha)^k\varphi(x)} = \frac{\varphi(x)}{(x-\alpha)^k\varphi(x)} = \frac{\psi(x)}{(x-\alpha)^{k-\lambda}\varphi(x)}. \tag{8.8}$$

This ends the proof of (8.7). Moreover, as the difference of two proper fractions, the right hand side of (8.8) is a proper fraction. ■

Lemma 8.2. *Suppose $R(x)$ is a proper fraction and $P(x)$ has a pair of complex conjugate roots $\alpha = a + ib$ and $\overline{\alpha} = a - ib$, of multiplicity l:*

$$P(x) = (x^2+px+q)^l\varphi(x), \quad \varphi(\alpha) \neq 0, \quad \varphi(\overline{\alpha}) \neq 0,$$

$$p = -2a, \quad q = a^2 + b^2.$$

Then there exists a representation:

$$\frac{Q(x)}{P(x)} = \frac{Mx+N}{(x^2+px+q)^l} + \frac{\psi(x)}{(x^2+px+q)^{l-\lambda}\varphi(x)}, \tag{8.9}$$

where M and N are real numbers, λ $(\lambda \geqslant 1)$ is integral, and $\psi(x)$ is the polynomial such that the last fraction in (8.9) is a proper fraction.

To obtain such a decomposition, we must first factor the denominator $P(x)$ into a product of linear factors of the form $(x-\alpha)$ and irreducible quadratic factors of the form x^2+px+q, $\dfrac{p^2}{4} - q < 0$. This is always possible in principle but may be quite difficult in practice.

Theorem 8.2. *Let $R(x) = \dfrac{Q(x)}{P(x)}$ be a proper fraction with real coefficients and*

$$P(x) = (x-\alpha_1)^{k_1}(x-\alpha_2)^{k_2}\cdots(x-\alpha_m)^{k_m}(x^2+p_1x+q_1)^{l_1}\cdots(x^2+p_rx+q_r)^{l_r}. \tag{8.13}$$

Then the partial-fraction decomposition of $R(x)$ has the form

$$\begin{aligned}\frac{Q(x)}{P(x)} &= \frac{A_{k_1}^{(1)}}{(x-\alpha_1)^{k_1}} + \frac{A_{k_1-1}^{(1)}}{(x-\alpha_1)^{k_1-1}} + \cdots + \frac{A_1^{(1)}}{x-\alpha_1} \\ &+ \cdots + \frac{A_{k_m}^{(m)}}{(x-\alpha_m)^{k_m}} + \frac{A_{k_m-1}^{(m)}}{(x-\alpha_m)^{k_m-1}} + \cdots + \frac{A_1^{(m)}}{x-\alpha_m} \\ &+ \frac{M_{l_1}^{(1)}x+N_{l_1}^{(1)}}{(x^2+p_1x+q_1)^{l_1}} + \frac{M_{l_1-1}^{(1)}x+N_{l_1-1}^{(1)}}{(x^2+p_1x+q_1)^{l_1-1}} + \cdots + \frac{M_1^{(1)}x+N_1^{(1)}}{x^2+p_1x+q_1} \\ &+ \cdots + \frac{M_{l_r}^{(r)}x+N_{l_r}^{(r)}}{(x^2+p_rx+q_r)^{l_r}} + \frac{M_{l_r-1}^{(r)}x+N_{l_r-1}^{(r)}}{(x^2+p_rx+q_r)^{l_r-1}} + \cdots + \frac{M_1^{(r)}x+N_1^{(r)}}{x^2+p_rx+q_r}, \qquad (8.14)\end{aligned}$$

where $A_1^{(1)}, A_2^{(1)}, \ldots, A_{k_m}^{(m)}, M_1^{(1)}, N_1^{(1)}, \ldots, M_{l_r}^{(r)}, N_{l_r}^{(r)}$ are real constants to be determined.

Example 8.17. Find the partial-fraction decomposition of the proper fraction

$$\frac{x^2+2}{(x+1)^3(x-2)}.$$

By formula (8.14),

$$\frac{x^2+2}{(x+1)^3(x-2)} = \frac{A}{(x+1)^3} + \frac{A_1}{(x+1)^2} + \frac{A_2}{x+1} + \frac{B}{x-2}.$$

To find the constants A, A_1, A_2, B, we multiply both sides of the equality by $(x+1)^3(x-2)$, to obtain

$$x^2+2 = A(x-2) + A_1(x+1)(x-2) + A_2(x+1)^2(x-2) + B(x+1)^3,$$

and hence,

$$x^2+2 = (A_2+B)x^3 + (A_1+3B)x^2 + (A-A_1-3A_2+3B)x + (-2A-2A_1-2A_2+B).$$

Equating coefficients of like terms gives

$$\begin{cases} 0 = A_2+B, \\ 1 = A_1+3B, \\ 0 = A-A_1-3A_2+3B, \\ 2 = -2A-2A_1-2A_2+B \end{cases}$$

and solving this system yields $A=-1, A_1 = \frac{1}{3}, A_2 = -\frac{2}{9}, B = \frac{2}{9}$.
Thus,

$$\frac{x^2+2}{(x+1)^3(x-2)} = -\frac{1}{(x+1)^3} + \frac{1}{3(x+1)^2} - \frac{2}{9(x+1)} + \frac{2}{9(x-2)}.$$

Example 8.18. Find the partial-fraction decomposition of the proper fraction $\dfrac{3x^4+2x^3+3x^2-1}{(x-2)(x^2+1)^2}$.

The quadratic polynomial x^2+1 is irreducible and so, by (8.14),

$$\frac{3x^4+2x^3+3x^2-1}{(x-2)(x^2+1)^2}=\frac{A}{x-2}+\frac{M_1x+N_1}{x^2+1}+\frac{M_2x+N_2}{(x^2+1)^2}.$$

To find the constants A, M_i, N_i $(i=1,2)$, we multiply both sides of this equation by the left-hand denominator $(x-2)(x^2+1)^2$ and find that

$$3x^4+2x^2+3x^2-1=A(x^4+2x^2+1)+(M_1x+N_1)(x^3-2x^2+x-2)+(M_2x+N_2)(x-2).$$

Again we remember that two polynomials are equal only if the coefficients of the corresponding powers of x are the same, and so we may conclude that

$$\begin{cases} A+M_1= & 3,\\ N_1-2M_1= & 2,\\ 2A+M_1-2N_1+M_2= & 3,\\ N_1-2M_1+N_2-2M_2= & 0,\\ A-2N_1-2N_2= & -1.\end{cases}$$

We solve this system and find that $A=3, M_1=0, N_1=2, M_2=1, N_2=0$.

Substituting these values into the previous partial-fraction decomposition, we have

$$\frac{3x^4+2x^3+3x^2-1}{(x-2)(x^2+1)^2}=\frac{3}{x-2}+\frac{2}{x^2+1}+\frac{x}{(x^2+1)^2}.$$

Example 8.19. Find the partial-fraction decomposition of $\dfrac{x+1}{x(x-1)(x-2)}$.

By Theorem 8.2,

$$\frac{x+1}{x(x-1)(x-2)}=\frac{A_1}{x-1}+\frac{A_2}{x}+\frac{A_3}{x-2}.$$

Thus

$$x+1=A_1x(x-2)+A_2(x-1)(x-2)+A_3x(x-1).$$

There is an alternative way of finding A_i $(i=1,2,3)$, one that is especially convenient in the case of non-repeated linear factors. Substitution of $x=1,\ 2$, and 0 into the last equation immediately gives $A_1=-2$, $A_3=\frac{3}{2}$, and $A_2=\frac{1}{2}$, respectively.

Above, we saw that by using long division, any rational function can be written as a sum of a polynomial and proper rational function. To see how to integrate an arbitrary rational function, we therefore only need to know how to integrate the partial-fraction decomposition of rational functions. By Theorem 8.2, it turns out that a proper fraction is the sum of partial fractions of the first and second types. We begin, then, with the integration of these.

Let $R(x)=\dfrac{P_n(x)}{Q_m(x)}$ ($n<m$) be irreducible proper rational fraction, i.e. polynomials $P_n(x)$ and $Q_m(x)$ do not have common factors. *The integration of such fractions is performed according to the following algorithm:*

1. Factorize the denominator $Q_m(x)$ into the simplest factors, namely the polynomials of the first order $(x-a)$ and the polynomials of the second order without real roots $\left(x^2+px+q\right)$ (here $D=p^2-4q<0$). It could be done due to the theorems of polynomial algebra.

2. Expand the fraction into the sum of the partial rational fractions due to the theorem:

Theorem. If $Q_m(x)=b_0(x-a)^{\alpha}(x-b)^{\beta}\ldots\left(x^2+px+q\right)\ldots\left(x\ \ +lx+s\right)$, *then proper irreducible rational fraction* $R(x)=\dfrac{P_n(x)}{Q_m(x)}$ *could be expand as*

$$R(x)=\frac{P_n(x)}{Q_m(x)}=\frac{A_0}{(x-a)^{\alpha}}+\frac{A_1}{(x-a)^{\alpha-1}}+\ldots+\frac{A_{\alpha-1}}{(x-a)}+\frac{B_0}{(x-b)^{\beta}}+\frac{B_1}{(x-b)^{\beta-1}}+\ldots+\frac{B_{\beta-1}}{(x-b)}+$$

$$+\ldots+\frac{M_0x+N_0}{\left(x^2+px+q\right)^{\mu}}+\frac{M_1x+N_1}{\left(x^2+px+q\right)^{\mu-1}}+\ldots+\frac{M_{\mu-1}x+N_{\mu-1}}{\left(x^2+px+q\right)}+\ldots+$$

$$+\frac{P_0x+Q_0}{\left(x^2+lx+s\right)^{\nu}}+\frac{P_1x+N_1}{\left(x^2+lx+s\right)^{\nu-1}}+\ldots+\frac{P_{\nu-1}x+N_{\nu-1}}{\left(x^2+lx+s\right)}.$$

3. Calculate integral of the obtained sum as sum of integrals.

Note. The coefficients of this expansion can be determined for the following reasons. Written equality is an identity. Therefore, if we reduce the sum of fractions to the right to the common denominator, we obtain the same polynomials in the numerators of the right and left fractions. Equating the coefficients at the same powers of *x*, we obtain a system of equations to determine the unknown coefficients.

In addition, the following remark can be used to determine the coefficients: since the polynomials in the numerators are equal, then their values are equal for any values of *x*. Given *x* specific values, we obtain the equations to determine the coefficients. Notice, that it is convenient to choose as values of *x* the root of common denominator. In practice, you can use both approaches at the same time to find the coefficients.

Summary

Indefinite integrals (antiderivatives) of rational functions can always be found by the following steps:

1. *Polynomial Division:* Divide the denominator into the numerator (if needed) to write the integrand as a polynomial plus a proper rational function.
2. *Partial Fraction Expansion:* Expand the proper rational function using partial fractions.
3. *Completing the Square:* If any terms involve quadratics, eliminate the linear term if needed by completing the square.
4. *Term by Term Integration:* Use elementary integral formulas and substitution.

2 *Partial Fraction Expansion*

The idea of partial fraction expansion is to take a proper rational function and express it as the sum of simpler rational functions. This is just the reverse of ordinary addition of rationals. For example, we know that

$$\frac{5}{x-3} - \frac{2}{x+1} = \frac{5(x+1) - 2(x-3)}{(x-3)(x+1)} = \frac{3x+11}{x^2-2x-3}.$$

What we want to do now is turn this around: that is, start with the right-hand side of this equation (a proper rational function) and somehow split it up to obtain the left-hand side (a sum of simpler rational functions). This can be accomplished step by step as follows.

Step 1: Factor the denominator. In the simplest case, we can factor the denominator into linear (degree one) factors. For instance, for the above example we have by inspection that

$$x^2 - 2x - 3 = (x-3)(x+1).$$

In cases which can't be factored readily, we can turn to the quadratic formula (for quadratics) or other root-finding methods for higher-degree polynomials, as studied in high school algebra. Sometimes a quadratic (degree two) factor cannot be further broken down (using real numbers): $x^2 + 4$ is such an *irreducible* quadratic. However, it can be shown that any polynomial with real coefficients is a product of linear and/or irreducible quadratic factors with real coefficients.

Step 2: Expand using undetermined coefficients A, B, C, ... This means writing out the rational function as a sum of terms involving the factors of the denominator. For the example at hand, this expansion takes the form

$$\frac{3x+11}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1} \tag{$*$}$$

where A and B are constants which are yet to be determined. Note: the specific form of the expansion depends on what type of factors the denominator has—more on this a bit later.

Step 3: Clear fractions. Multiply both sides by the denominator so no fractions remain. For the example at hand, we multiply both sides of equation (*) by $(x-3)(x+1)$ to obtain

$$(x-3)(x+1)\frac{3x+11}{(x-3)(x+1)} = (x-3)(x+1)\frac{A}{(x-3)} + (x-3)(x+1)\frac{B}{(x+1)}.$$

Cancelling the common factors reduces this to

$$3x + 11 = (x+1)A + (x-3)B. \tag{$**$}$$

Step 4: Solve for the coefficients A, B, C, ... There are two approaches here. The systematic approach is to gather together like powers of the variable and equate their coefficients, which gives a set of equations to solve for A, B, C, ... For the example at hand, we rewrite the equation (**) as

$$3x + 11 = (A + B)x + (A - 3B).$$

Since this must hold for all values of x, the coefficients of like powers of x on both sides must match. Thus, from the x terms we obtain

$$3 = A + B,$$

and from the constant terms we obtain

$$11 = A - 3B.$$

Solving these equations yields $A = 5$ and $B = -2$.

An alternate (and often easier) approach here is to simply plug specific values of x into the expansion (after clearing fractions) to obtain equations for A, B, C, ... By choosing the x values intelligently, the resulting equations usually can be made much simpler. For the example at hand, we can plug $x = 3$ into equation (**) to obtain

$$3(3) + 11 = (3 + 1)A + (3 - 3)B,$$

which simplifies immediately to $20 = 4A$ so $A = 5$. Likewise, plugging $x = -1$ into equation (**) gives

$$3(-1) + 11 = (-1 + 1)A + (-1 - 3)B,$$

which reduces to $8 = -4B$ so $B = -2$.

Step 5: Substitute A, B, C, ... into the expansion. At this point we're done: since the values of the coefficients are known, the expansion is known. For the example at hand, we substitute $A = 5$ and $B = -2$ (from step 4) into equation (*) to obtain

$$\frac{3x + 11}{(x - 3)(x + 1)} = \frac{5}{x - 3} - \frac{2}{x + 1},$$

which is the desired partial fraction expansion.

3 *Completing the Square*

In dealing with quadratic factors, it is often useful to rewrite them in a form which does not explicitly involve a linear (x) term. To illustrate this process of *completing the square*, consider the polynomial

$$x^2 - 6x + 8.$$

Take the coefficient of x, divide it by two, and square it to get $(-6/2)^2 = 3^2 = 9$. Add and subtract this number and factor the result to get

$$x^2 - 6x + (9 - 9) + 8 = (x^2 - 6x + 9) - 1 = (x - 3)^2 - 1.$$

Thus, we have rewritten the original quadratic in a form which lacks a linear term.

If the coefficient of the x^2 term isn't 1, we must factor it out before starting. The following example shows how to do this.

Example 7: Complete the square for $21 - 4s - s^2$.

Solution: We start by factoring out the coefficient -1 of s^2, and then add and subtract $(4/2)^2 = 4$:

$$\begin{aligned} 21 - 4s - s^2 = (-1)(s^2 + 4s - 21) &= (-1)(s^2 + 4s + 4 - 4 - 21) \\ &= (-1)[(s + 2)^2 - 25] \\ &= 5^2 - (s + 2)^2. \end{aligned}$$

Examples.

1. $$\int \frac{7x^2+9x-12}{x^3+2x^2-3x}dx = \int \frac{7x^2+9x-12}{x(x^2+2x-3)}dx = \left[\begin{array}{c} x^2+2x-3=0 \\ x_1=-3,\ x_2=1 \\ x^2+2x-3=(x+3)(x-1) \end{array} \right] = \int \frac{7x^2+9x-12}{x(x+3)(x-1)}dx.$$

$$\frac{7x^2+9x-12}{x(x+3)(x-1)} = \frac{A}{x} + \frac{B}{x+3} + \frac{C}{x-1} \Rightarrow 7x^2+9x-12 = A(x+3)(x-1) + Bx(x-1) + Cx(x+3).$$

Roots of denominator are: $x_1=-3,\ x_2=1,\ x_3=0$. Let us substitute them into the last equation:

$$x=-3: 7\cdot(-3)^2+9(-3)-12 = A\cdot 0 + B(-3)(-4) + C\cdot 0 \Rightarrow B=2$$

$$x=1\ : 7\cdot 1^2+9\cdot 1-12 = A\cdot 0 + B\cdot 0 + C\cdot 1(1+3) \qquad \Rightarrow C=1$$

$$x=0\ : 7\cdot 0^2+9\cdot 0-12 = A\cdot 3\cdot(-1) + B\cdot 0 + C\cdot 0 \qquad \Rightarrow A=4$$

$$\frac{7x^2+9x-12}{x(x+3)(x-1)} = \frac{4}{x} + \frac{2}{x+3} + \frac{1}{x-1};$$

$$\int \frac{7x^2+9x-12}{x(x+3)(x-1)}dx = \int\left(\frac{4}{x} + \frac{2}{x+3} + \frac{1}{x-1}\right)dx = 4\ln|x| + 2\ln|x+3| + \ln|x-1| + C.$$

2. $$\int \frac{3x^3-4x^2-x+1}{x^4-x^3}dx = \int \frac{3x^3-4x^2-x+1}{x^3(x-1)}dx.$$

$$\frac{3x^3-4x^2-x+1}{x^3(x-1)} = \frac{A}{x^3} + \frac{B}{x^2} + \frac{C}{x} + \frac{D}{x-1} \Rightarrow$$

$$3x^3-4x^2-x+1 = A(x-1) + Bx(x-1) + Cx^2(x-1) + Dx^3.$$

Roots of denominator are: $x_1=0,\ x_2=1$. Let us substitute them into the last equation:

$$x=0\ : 1 = A\cdot(-1) + B\cdot 0 + C\cdot 0 + D\cdot 0 \qquad \Rightarrow A=-1$$

$$x=1\ : -1 = A\cdot 0 + B\cdot 0 + C\cdot 0 + D\cdot 1 \qquad \Rightarrow D=-1$$

To determine the coefficients *B* and *C*, let us equate coefficients at x and x^3:

$$3x^3-4x^2-x+1 = Ax - A + Bx^2 - Bx + Cx^3 - Cx^2 + Dx^3$$

$$x^1\ : -1 = A - B \qquad \Rightarrow B = A+1 = 0$$

$$x^3\ : 3 = C + D \qquad \Rightarrow C = 3 - D = 4$$

$$\frac{3x^3-4x^2-x+1}{x^3(x-1)} = \frac{-1}{x^3} + \frac{4}{x} + \frac{-1}{x-1}.$$

Thus

$$\int \frac{3x^3-4x^2-x+1}{x^3(x-1)}dx = \int\left(\frac{-1}{x^3} + \frac{4}{x} + \frac{-1}{x-1}\right)dx = \frac{1}{2x^2} + 4\ln|x| - \ln|x-1| + C.$$

3. $\int \frac{dx}{x^3-1} = \int \frac{dx}{(x-1)\left(x^2+x+1\right)}.$

$$\frac{1}{(x-1)\left(x^2+x+1\right)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1} \;\Rightarrow\; 1 = A\left(x^2+x+1\right) + (Bx+C)(x-1).$$

To find unknowns we substitute the root of denominator $x=1$ in the last equation and also equate the coefficients at equal powers of *x* from different sides of equation:

$$x=1:\ 1 = A\cdot 3 \qquad \Rightarrow A = \frac{1}{3}$$

$$x^2 \quad : 0 = A + B \qquad \Rightarrow B = -A = -\frac{1}{3}$$ Нехай x=1, тоді 1=3A, A=1/3.

$$x^0 \quad : 1 = A - C \qquad \Rightarrow C = A - 1 = -\frac{2}{3}$$

So,

$$\int \frac{dx}{x^3-1} = \frac{1}{3}\int \frac{dx}{x-1} - \frac{1}{3}\int \frac{x+2}{x^2+x+1}dx = \frac{1}{3}\int \frac{dx}{x-1} - \frac{1}{3\cdot 2}\int \frac{2x+4}{x^2+x+1}dx =$$

$$= \frac{1}{3}\int \frac{dx}{x-1} - \frac{1}{3\cdot 2}\int \frac{2x+1}{x^2+x+1}dx - \frac{1}{3\cdot 2}\int \frac{3dx}{x^2+x+1} = \frac{1}{3}\int \frac{dx}{x-1} - \frac{1}{3\cdot 2}\int \frac{d(x^2+x+1)}{x^2+x+1} -$$

$$-\frac{1}{2}\int \frac{dx}{(x+1/2)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln\left(x^2+x+1\right) - \frac{1}{\sqrt{3}}\operatorname{arctg}\frac{2x+1}{\sqrt{3}} + C.$$

4. $\int \frac{x^2dx}{(x^2+1)(x^2+4)}.$

$$\frac{x^2}{(x^2+1)(x^2+4)} \equiv \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+4} = \frac{(Ax+B)(x^2+4)+(Cx+D)(x^2+1)}{(x^2+1)(x^2+4)};$$

$$x^2 \equiv (Ax+B)(x^2+4) + (Cx+D)(x^2+1).$$

There are no roots in the denominator, therefore we equate coefficients at the equal powers of *x* to find unknown coefficients:

$$x^3 \quad : 0 = A + C \qquad \Rightarrow 0 = 4A + C - (A+C) = 3A \Rightarrow A = C = 0$$

$$x^2 \quad : 1 = B + D \qquad \Rightarrow 1 = B + D - (4B + D) = -3B \Rightarrow B = -1/3$$

$$x^1 \quad : 0 = 4A + C$$

$$x^0 \quad : 0 = 4B + D \qquad \Rightarrow D = -4B = 4/3$$

$$\int \frac{x^2dx}{(x^2+1)(x^2+4)} = \int \frac{-\frac{1}{3}}{x^2+1}dx + \int \frac{\frac{4}{3}}{x^2+4} = -\frac{1}{3}\operatorname{arctg} x + \frac{2}{3}\operatorname{arctg}\frac{x}{2} + C.$$