The Evaluation of the Integrals $\int R(\sin x, \cos x)dx$.

$R(\sin x, \cos x)$ is called a rational function with respect to trigonometric functions $\sin x$ and $\cos x$.

Theorem: Integrals $\int R(\sin x, \cos x)dx$ by the substitution $\tan\frac{x}{2} = t$ are converted into the integrals of a rational function of the variable t. Indeed, since

$$\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{2t}{1+t^2},$$

$$\cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{1-t^2}{1+t^2},$$

and

$$x = 2\tan^{-1} t, \quad dx = \frac{2dt}{1+t^2},$$

it is clear that these substitutions will convert the integrand into a rational function of t:

$$\int R(\sin x, \cos x)dx = \int R\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)\frac{2dt}{1+t^2}.$$

Note: the substitution $\tan\frac{x}{2} = t$ is called a *universal trigonometric substitution.*

Example Find $\int \frac{dx}{\cos x}$.

Taking into account that $\cos x = \frac{1-t^2}{1+t^2}$ and $dx = \frac{2dt}{1+t^2}$, we can derive

$$\int \frac{dx}{\cos x} = \int \frac{\frac{2dt}{1+t^2}}{\frac{1-t^2}{1+t^2}}dt = 2\int \frac{dt}{1-t^2} = \ln\left|\frac{1+t}{1-t}\right| + C = \ln\left|\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right| + C.$$

There are various special cases of rational functions where simpler substitutions also work.

(a) For the integral $\int R(\sin x)\cos x\,dx$, we substitute $\sin x = t$ and $\cos x\,dx = dt$.

(b) For the integral $\int R(\cos x)\sin x\,dx$, we substitute $\cos x = t$ and $\sin x\,dx = -dt$.

(c) For the integral $\int R(\tan x)dx$, we substitute $\tan x = t$, $dx = \frac{dt}{1+t^2}$, and then

$$\int R(\tan x)dx = \int R(t)\frac{dt}{1+t^2}.$$

(d) If the integrand $R(\sin x, \cos x)$ involves only even powers of $\sin x$ and $\cos x$, then we substitute $\tan x = t$, and

$$\cos^2 x = \frac{1}{1+\tan^2 x} = \frac{1}{1+t^2},$$

$$\sin^2 x = \frac{\tan^2 x}{1+\tan^2 x} = \frac{t^2}{1+t^2}, \quad dx = \frac{dt}{1+t^2},$$

and the original integrand is transformed into a rational function.

Example: Find $I = \int \frac{dx}{2-\sin^2 x}$.

By substituting $\tan x = t$, we obtain

$$I = \int \frac{dt}{\left(2 - \frac{t^2}{1+t^2}\right)(1+t^2)} = \int \frac{dt}{2+t^2} = \frac{1}{\sqrt{2}} \tan^{-1} \frac{t}{\sqrt{2}} + C = \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{\tan x}{\sqrt{2}}\right) + C.$$

(e) $R(\sin x, \cos x) = \sin^m x \cos^n x$.

We treat two cases separately.

Case 1. At least one of the two numbers m and n is an odd positive integer (say $n = 2p+1$, p a non-negative integer). If so, the other may be any real number.

Case 2. Both m and n are nonnegative even integers, say $m = 2p$, $n = 2q$.

In the first case we split off one $\cos x$ factor and use the identity $\cos^2 x = 1 - \sin^2 x$to express the remaining factor $\cos^{n-1} x$ in terms of $\sin x$, as follows:

$$\int \sin^m x \cos^{2p+1} x\, dx = \int \sin^m x \cos^{2p} x \cos x\, dx$$
$$= \int \sin^m x (1-\sin^2 x)^p \cos x\, dx.$$

Then the substitution $\sin x = t$, $\cos x\, dx = dt$ yields

$$\int \sin^m x \cos^n x\, dx = \int t^m (1-t^2)^p dt.$$

Observe that the factor $(1-t^2)^p$ of the integrand is a polynomial in t, and so its product with t^m is easy to integrate.

Before treating the second case, we give a concrete example of the first case.

Example. Find $\int \frac{\cos^3}{\sin^4 x} dx$.

The substitution $\sin x = t$, $\cos x\, dx = dt$ yields

$$\int \frac{\cos^3 x}{\sin^4 x} dx = \int \frac{\cos^2 x \cos x\, dx}{\sin^4 x} = \int \frac{(1-\sin^2 x)\cos x\, dx}{\sin^4 x}$$
$$= \int \frac{(1-t^2)dt}{t^4} = -\frac{1}{3t^3} + \frac{1}{t} + c = -\frac{1}{3\sin^3 x} + \frac{1}{\sin x} + C.$$

In Case 2, we use the half-angle formulas of elementary trigonometry,

$$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x \text{ and } \cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x, \qquad (*)$$

to rewrite the even powers of $\sin x$ and $\cos x$ as follows:

$$\int \sin^{2p} x \cos^{2q} x\, dx = \int \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right)^p \left(\frac{1}{2} + \frac{1}{2}\cos 2x\right)^q dx.$$

By applying (*) repeatedly to the resulting powers of $\cos 2x$—if necessary—we eventually are reduced to integrals involving only odd powers, and we have seen how to handle these in Case 1.

Example. Evaluate $\int \sin^4 x\,dx$.

With the use of (8.30), we find that

$$\int \sin^4 x\,dx = \frac{1}{2^2}\int (1-\cos 2x)^2 dx = \frac{1}{4}\int (1-2\cos 2x+\cos^2 2x)dx$$
$$= \frac{1}{4}\left[x-\sin 2x+\frac{1}{2}\int (1+\cos 4x)dx\right] = \frac{1}{4}\left[\frac{3}{2}x-\sin 2x+\frac{\sin 4x}{8}\right]+C.$$

Example . Find $\int \frac{\sin^2 x}{\cos^4 x}\,dx$.

We use the substitution $\tan x = t$, and obtain

$$\int \frac{\sin^2 x}{\cos^4 x dx} = \int \frac{\sin^2 x(\sin^2 x+\cos^2 x)}{\cos^4 x dx} = \int \tan^2 x(1+\tan^2 x)dx$$
$$= \int t^2(1+t^2)\frac{dt}{1+t^2} = \int t^2 dt = \frac{t^3}{3}+c = \frac{tg^3 x}{3}+C.$$

(f) For the integral $\int \tan^m x \sec^n x\,dx$, the procedure breaks up into two cases.

Case 1. m is an odd positive integer.

Case 2. n is an even positive integer.

In Case 1, we use the substitution $t=\sec x$ and so split off the factor $\sec x \tan x$ to obtain $\sec x \tan x\,dx$, the differential of $\sec x$. Then we use the identity $\tan^2 x = \sec^2 x - 1$ to convert the remaining even power of $\tan x$ into powers of $\sec x$.

In Case 2, we use the substitution $t=\tan x$ and then split off $\sec^2 x$ to obtain the differential of $\tan x$. Use of the identity $\sec^2 x = 1+\tan^2 x$ to convert the remaining even power of $\sec x$ to powers of $\tan x$ completes the process.

Example Find $\int \tan^3 x \sec^3 x\,dx$.

We are in case 1, and so

$$\int \tan^3 x \sec^3 x\,dx = \int (\sec^2 x - 1)\sec^2 x \cdot \sec x \cdot \tan x\,dx = \int (t^4 - t^2)dt$$
$$= \frac{1}{5}t^5 - \frac{1}{3}t^3 + C = \frac{1}{5}\sec^5 x - \frac{1}{3}\sec^3 x + C.$$

(g) $R(\sin x, \cos x)$ has one of the forms $\cos mx \cos nx$, $\sin mx \cos nx$, or $\sin mx \sin nx$ $(m \neq n)$.

In this case, we use the representations for products of trigonometric functions obtained from the addition formulas:

$$\cos mx \cos nx = \frac{1}{2}[\cos(m+n)x + \cos(m-n)x],$$
$$\sin mx \cos nx = \frac{1}{2}[\sin(m+n)x + \sin(m-n)x],$$

and

$$\sin mx \sin nx = \frac{1}{2}[\cos(m-n)x - \cos(m+n)x].$$

Example. Find $\int \sin 5x \sin 3x\,dx$.

$$\int \sin 5x \sin 3x\,dx = \frac{1}{2}\int [\cos 2x - \cos 8x]dx = -\frac{\sin 8x}{16} + \frac{\sin 2x}{4} + C.$$

The Integration of Irrational Algebraic Functions

1. The evaluation of $\int R\left[x, \left(\frac{ax+b}{cx+d}\right)^{\frac{p}{q}}\right] dx$.

The substitution $\left(\frac{ax+b}{cx+d}\right) = t^q$ yields $\left(\frac{ax+b}{cx+d}\right)^{\frac{p}{q}} = t^p$ and $x = \frac{t^q d - b}{a - ct^q}$ and the integral $\int R\left[x, \left(\frac{ax+b}{cx+d}\right)^{\frac{p}{q}}\right] dx$ is then converted into an integral of rational functions.

Example. Find $I = \int \sqrt{\frac{1+x}{1-x}}\, dx$.

The substitution $\sqrt{\frac{1+x}{1-x}} = t$ yields

$$x = \frac{t^2-1}{t^2+1}, \quad dx = \frac{4t\,dt}{(t^2+1)^2}.$$

Thus,

$$I = \int t \frac{4t\,dt}{(t^2+1)^2} = 2\int t \frac{2t}{(t^2+1)^2}\,dt = -\frac{2t}{t^2+1} + 2\int \frac{dt}{t^2+1}$$
$$= -\frac{2t}{t^2+1} + 2\tan^{-1} t + C$$

Thus, in term of x, we obtain

$$I = -\sqrt{1-x^2} + 2\arctan\sqrt{\frac{1+x}{1-x}} + C.$$

More generally, for evaluating the integral

$$\int R\left[x, \left(\frac{ax+b}{cx+d}\right)^{\frac{m}{n}}, \ldots, \left(\frac{ax+b}{cx+d}\right)^{\frac{p}{q}}\right] dx,$$

we use the substitution $\frac{ax+b}{cx+d} = t^k$ to arrive at an integral of rational functions, where k is the least common multiple of the denominators of the fractions $\frac{m}{n}, \ldots, \frac{p}{q}$.

2. The Evaluation of the Integral $\int R\left(x, \sqrt{ax^2+bx+c}\right)dx$.

We use the trigonometric substitutions. Suppose $a \neq 0$ and $b^2 - 4ac \neq 0$ for a quadratic polynomial ax^2+bx+c. Note that if $a = 0$, then we have a known integral of algebraic functions. On the other hand, if $b^2-4ac = 0$, then $ax^2+bx+c = a\left(x+\frac{b}{2a}\right)^2$, and for $a > 0$ we are dealing with an integral of rational functions (if the discriminant is zero and $a < 0$, then the square root $\sqrt{ax^2+bx+c}$is not defined for any x). Thus if the discriminant is nonzero and $a \neq 0$, then we have

$$ax^2+bx+c = a\left(x+\frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right).$$

Denoting

$$x + \frac{b}{2a} = t, \;\; |a| = m^2 \;\text{ and }\; \left|c - \frac{b^2}{4a}\right| = n^2,$$

the square root $\sqrt{ax^2+bx+c}$ is converted into $\sqrt{m^2t^2 \pm n^2}$ or $\sqrt{n^2-m^2t^2}$.

Finally, we have the integrals

$$\int R\left(t, \sqrt{m^2t^2 + n^2}\right) dt, \tag{1}$$

$$\int R\left(t, \sqrt{m^2t^2 - n^2}\right) dt, \tag{2}$$

and

$$\int R\left(t, \sqrt{n^2 - m^2t^2}\right) dt. \tag{3}$$

Thus by substituting $t = \frac{}{m}\tan z$, $t = \frac{}{m}\sec z$, $t = \frac{}{m}\sin z$ in (1)–(3), respectively, we obtain a trigonometric integral of the form $\int R_1(\sin z, \cos z)$.

Example. Find $I = \int \frac{dx}{\sqrt{(4-x^2)^3}}$.

This integral has the form (3) and so we make the substitution $x = 2\sin z$. Then, since $dx = 2\cos z\,dz$, we can write

$$\begin{aligned} I &= \int \frac{2\cos z\,dz}{\sqrt{(4-4\sin^2 z)^3}} = \int \frac{2\cos z\,dz}{8\cos^3 z} = \frac{1}{4}\int \frac{dz}{\cos^2 z} \\ &= \frac{1}{4}\tan z + C = \frac{1}{4}\frac{\sin z}{\sqrt{1-\sin^2 z}} + C = \frac{x}{4\sqrt{4-x^2}} + C. \end{aligned}$$

Example. Evaluate $\int \frac{\sqrt{x^2-25}}{x}\,dx$.

The integral has the form (2). We make the substitution $x = 5\sec z$ so that $dx = 5\sec z\tan z\,dz$. Therefore,

$$\begin{aligned} \int \frac{\sqrt{x^2-25}}{x}\,dx &= \int \frac{\sqrt{25\sec^2 z - 25}}{5\sec z}(5\sec z\tan z)dz \\ &= 5\int \tan^2 z\,dz = 5\int(\sec^2 z - 1)dz = 5\tan z - 5z + C. \end{aligned}$$

But

$$z = \sec^{-1}\frac{x}{5} \text{ and } \tan z = \frac{\sqrt{x^2-25}}{5},$$

so that

$$\int \frac{\sqrt{x^2-25}}{x}\,dx = \sqrt{x^2-25} - 5\sec^{-1}\frac{x}{5} + C.$$

The Evaluation of the Integral $\int x^m(a+bx^n)^p dx$.

We assume that m, n, and p are rational numbers, and that a and b are real. We consider the following three cases.

(a) If p is an integer, then the integral has the form $\int R(x^m, x^n)dx$, where R is rational function. We then use the substitution $x = t^k$, where k is the least common multiple of m and n.

(b) If $\dfrac{m+1}{n}$ is an integer, then the substitution $t = x^n$ yields

$$\int x^m(a+bx^n)^p dx = \frac{1}{n}\int t^q(a+bt)^p dt,$$

where $q = \dfrac{m+1}{n} - 1$ is an integer. Let r be the denominator of p. Then using the substitution $u = \sqrt[r]{a+bt}$, the integral is converted into the integral of rational functions of u.

(c) If $\dfrac{m+1}{n} + p$ is an integer, then by substituting $t = \sqrt[r]{ax^{-n}+b}$, where r is the denominator of p, the original integral is converted into the integral of a rational function of t.

Example. Find $I = \int x^3(1+x^2)^{-1/2}\,dx$.

Here, $\dfrac{m+1}{n} = \dfrac{3+1}{2} = 2$ and $p = -\dfrac{1}{2}$. Then we use the substitution $1+x^2 = u^2$. Thus,

$$I = \int \frac{(u^2-1)u\,du}{u^2} = \int (u^2-1)du = \frac{u^3}{3} - u + C$$
$$= \frac{u(u^2-3)}{3} + C = \frac{1}{3}\sqrt{1+x^2}(x^2-2) + C.$$

Example. Find $I = \int x^{-3}(x^4+2)^{1/2}dx$.

Here, $\dfrac{m+1}{n} = \dfrac{-3+1}{4} = -\dfrac{1}{2}$ is not an integer. But $\dfrac{m+1}{n} + p = -\dfrac{1}{2} + \dfrac{1}{2} = 0$ is an integer. Then the substitution $x^4 + 2 = u^2x^4$ implies

$$x^4 = \frac{2}{u^2-1}, \quad dx = \frac{-u\,du}{x^3(u^2-1)^2}.$$

Thus,

$$I = \int \frac{ux^2}{x^3}\frac{-u\,du}{x^3(u^2-1)} = \int \frac{-u^2\,du}{x^4(u^2-1)^2} = \int \frac{-u^2(u^2-1)}{2(u^2-1)^2}du$$
$$= -\frac{1}{2}\int\left(1+\frac{1}{u^2-1}\right)du = -\frac{1}{2}u - \frac{1}{4}\ln\frac{u-1}{u+1} + C.$$
$$= -\frac{\sqrt{x^4+2}}{2x^2} - \frac{1}{4}\ln\frac{\sqrt{x^2+2}-x^2}{\sqrt{x^4+4}+x^2} + C.$$