Definite integral and its applications

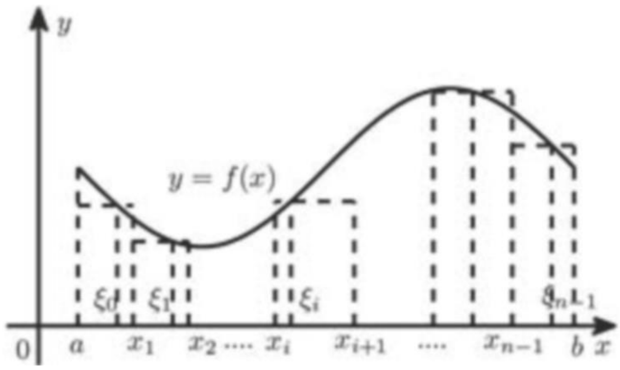
The most important result about integration is the fundamental theorem of calculus, which states that integration and differentiation are inverse operations in an appropriately understood sense. Among other things, this connection enables us to compute many integrals explicitly.

Integrability is a less restrictive condition on a function than differentiability. Roughly speaking, integration makes functions smoother, while differentiation makes functions rougher. For example, the indefinite integral of every continuous function exists and is differentiable, whereas the derivative of a continuous function need not exist (and generally doesn't).

The Riemann integral is the simplest integral to define, and it allows one to integrate every continuous function) as well as some not-too-badly discontinuous functions on a given definite interval $[a, b]$.

1.1 Definition and properties of definite integrals

The definition of the Riemann integral is motivated by the problem of defining and calculating the area of the region lying between the graph of a non-negative function $f(x)$ and the x-axis over a closed interval.



Let the continuous function $y = f(x)$ be determined in the interval $[a, b]$. Let's divide the interval [a,b] into n subintervals by n points $a = x_0 < x_1 < \cdots < x_{i-1} < x_i < \cdots < x_n = b$. Here,

Definition 1. A *partition* of $[a, b]$ is a set of points $P = \{x_0, x_1, \dots, x_n\}$ satisfying $a = x_0 < x_1 < \cdots < x_n = b$.

Definition 2. A partition P_2 of $[a, b]$ is said to be a *refinement* of P_1 if $P_1 \subset P_2$.

Definition 3. If P is any partition of $[a, b]$ and $\Delta x_i = x_{i+1} - x_i$ $(i = 0, 1, \dots, n - 1)$, then

$$\mu(P) = \max_{0 \le i \le n-1} \Delta x_i$$

is said to be the *step of the partition* P. It is clear that $n \geq \frac{b-a}{\mu(P)}$

Definition 4. Let *f*(*x*) be a function defined on the interval $[a,b]$. If *P* is a partition of $[a,b]$ and $\{\xi_0,\xi_1,\dots,\xi_{n-1}\}$, $\xi_i \in [x_i,x_{i+1}]$ $(i = 0,1,\dots,n-1)$ is a selection of points for *P*, then *the Riemann sum* for *f*(*x*) determined by *P* and the selection $\{\xi_0,\xi_1,\dots,\xi_{n-1}\}$, is $\sigma(x_i,\xi_i) = \sum_{i=0}^{n-1} f(\xi i)\Delta xi$.

We also say that *the Riemann sum* is associated with the partition *P*.

Other words: on each subinterval $[x_{i-1},x_i]$ we choose some point ξ_i and compose a sum $\sigma_n = \sum_{i=0}^{n-1} f(\xi_i)\Delta x_i$ which is called an integral sum of the function $y = f(x)$ in the interval [a,b].

Definition 5. Let *P* be a partition of $[a,b]$ and $\{\xi_0,\xi_1,\dots,\xi_{n-1}\}$ be a selection for *P*. We say that *I* is the limit of the Riemann sums $\sigma(x_i,\xi_i)$ as the step of the partition tends to zero, if for every $\varepsilon > 0$, there is a $\delta = \delta(\varepsilon)$ such that $\mu(P) < \delta$ implies $|I - \sigma| < \varepsilon$.

That is, there exists some final limit of the integral sum σ_n while $\mu = \max_{0\le i\le n-1}\{\Delta x_i\} \to 0$, not depending on the way of splitting the interval $[a,b]$ into elementary segments and choice of the points ξ_i.

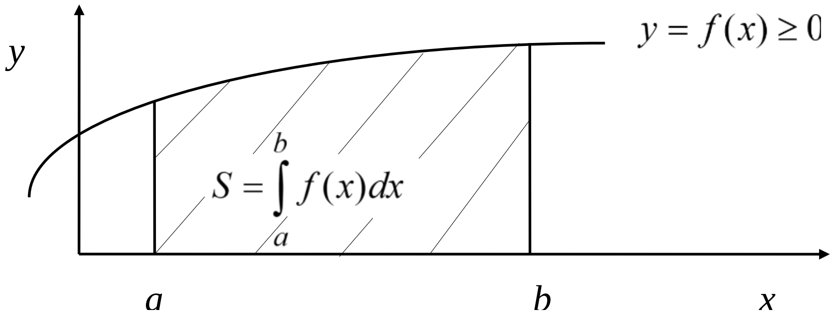
Definition 6. The limit *I* is called *the Riemann integral* of the function *f* over the interval from *a* to *b or **a definite integral*** of the function $f(x)$ in the interval $[a,b]$ and is designated as

$$\int_a^b f(x)dx = \lim_{\mu\to 0}\sum_{i=0}^{n-1} f(\xi_i)\Delta x_i$$

The numbers *a* and *b* are called the *lower limit* and the *upper limit*, respectively.

In this case we say that the function $f(x)$ is *integrable* on the interval [*a*,*b*].

The geometrical meaning of a definite integral: if $f(x) \ge 0$ for any $x \in [a,b]$, then $\int_a^b f(x)dx$ is numerically equal to area of the region (a curvilinear trapezoid) with the base $[a,b]$, restricted by the straight lines $x = a$, $x = b$ and the plot of the non-negative function $y = f(x)$ as seen in Figure.



Integrability of continuous and monotonic functions.

Theorem 1. A continuous function $f(x)$ on an interval $x \in [a, b]$ is Riemann integrable.

Theorem 2. A monotonic function $f(x)$ on an interval $x \in [a, b]$ is Riemann integrable.

Remark. Monotonic functions needn't be continuous, and they may be discontinuous at a countably infinite number of points.

Theorem 3. If the function *f* is Riemann integrable on $[a, b]$, then *f* is bounded on this interval.

Proof: On the contrary, suppose that *f* is integrable on [*a*,*b*] but unbounded on this interval. By Definition, for every $\varepsilon > 0$ there is a positive number $\delta = \delta(\varepsilon)$ such that $\mu(P) < \delta(\varepsilon)$ implies $|I - \sigma| < \varepsilon$. It follows that if $\mu(P) < \delta(\varepsilon)$, then the Riemann sum σ is bounded.

Let P be a partition such that $\mu(P) < \delta(\varepsilon)$. Since, by assumption, f is unbounded on [*a*,*b*], there exists a subinterval $[x_k, x_{k+1}]$ such that the function *f* is unbounded on $[x_k, x_{k+1}]$. Then it is possible, by selecting an appropriate $\xi_k \in [x_k, x_{k+1}]$, to arrange that $f(\xi_k)\Delta x_k$ is larger than any pre-assigned number. This implies that the Riemann sums are unbounded, which contradiction proves the theorem.

Remark. For the existence of the Riemann integral of function $f(x)$, the boundedness of *f* is necessary, but not sufficient.

Suppose a function *f* is bounded on the closed interval [*a*,*b*], and that $P = \{x_0, x_1, \ldots, x_n\}$ is any partition of [*a*,*b*]. We adopt the following notation:

$$M = \sup_{x\in[a,b]} f(x),\ M_i = \sup_{x\in[x_i, x_{i+1}]} f(x),$$

$$m = \inf_{x\in[a,b]} f(x),\ m_i = \inf_{x\in[x_i, x_{i+1}]} f(x),$$

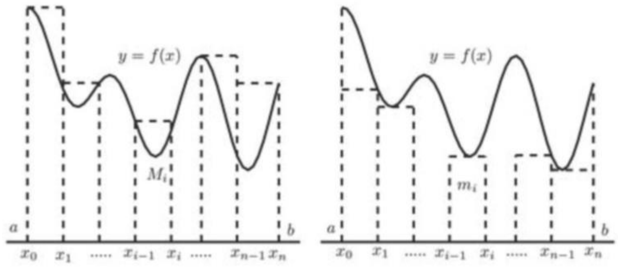
$$S(P) = \sum_{i=0}^{n-1} M_i \Delta x_i,\ s(P) = \sum_{i=0}^{n-1} m_i \Delta x_i$$

Definition 7. The *S*(*P*) and *s*(*P*) are called the *upper Riemann sum* and the *lower Riemann sum*, respectively, for a function *f*, associated with the partition *P* of [*a*,*b*].

Geometrically the upper and the lower Riemann sums of $y = f(x)$ are presented as follows:

for each $i = 1, \ldots, n$ consider the rectangle with base $[x_{i-1}, x_i]$, height M_i, and area $M_i \Delta x_i$. The union of these *n* rectangles contains the region $A = \{(x, y) : a \leq x \leq b, 0 \leq y \leq f(x)\}$ under the graph; it is a circumscribed rectangular polygon associated with the partition of [*a*,*b*]. Its area is *S*(*P*) in Figure.

Similarly, $s(P)$ is the area of an inscribed rectangular polygon, that is, the sum of the areas of the n rectangles each with base length Δx_i and height m_i as seen in Figure.



Lemma 1. Let $\sigma(x_i, \xi_i)$ be the Riemann sum for function $f(x)$ determined by an arbitrary partition P of $[a,b]$ and any selection $\{\xi_0, \xi_1, \ldots, \xi_{n-1}\}$ for the partition P. Then

$$s(P) \leq \sigma(x_i, \xi_i) \leq S(P).$$

Indeed, by the definition of M_i and m_i, the inequality $m_i \leq f(\xi_i) \leq M_i$ holds for any $\xi_i \in [x_i, x_{i+1}]$. Hence, by multiplying by Δx_i, $i = 0, \ldots, n-1$, and then summing these inequalities, we have the desired inequality.

Lemma 2. For any partition P of $[a,b]$,

$$S(P) = \sup_{\xi_i \in [x_i, x_{i+1}]} \sigma(x_i, \xi_i); \; s(P) = \inf_{\xi_i \in [x_i, x_{i+1}]} \sigma(x_i, \xi_i).$$

Lemma 3. If P_1 and P_2 are any partitions *of* $[a,b]$, then

$$s(P_1) \leq S(P_2),$$

i.e., a lower Riemann sum is never greater than any upper Riemann sum, regardless of the partition used.

Theorem 4. For the existence of the Riemann integral of a bounded function f defined on a closed interval $[a,b]$, it is necessary and sufficient that for every $\varepsilon > 0$, there exists a partition P of $[a,b]$ such that

$$S(P) - s(P) < \varepsilon.$$

Remark. Theorem can be reformulated as follows: For the existence of the Riemann integral of a bounded function f defined on $[a,b]$, it is necessary and sufficient that the upper and lower integrals are equal.

The properties of a definite integral.

1. $\int_a^b f(x)dx = -\int_b^a f(x)dx.$

2. $\int_a^a f(x)dx = 0$

3. If the functions $f(x)$ и $g(x)$ are integrable on $[a,b]$, then

a) $\int_a^b Cf(x)dx = C\int_a^b f(x)dx, \quad (C = const).$

b) $\int_a^b (f(x) \pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx.$

4. *Additivity* of an integral. If the function $f(x)$ is integrable on $[a,c]$ and $[c,b]$, where $c \in (a,b)$, then it is integrable on [*a,b*] and

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

5. If $a < b$ and integrable on $[a,b]$ function $f(x) \geq 0$, then

$$\int_a^b f(x)dx \geq 0,$$

and the equality to zero is possible if and only if $f(x) \equiv 0$ for any $x \in (a,b)$.

6. If $a < b$ and integrable on $[a,b]$ functions $f(x) \geq g(x)$, then

$$\int_a^b f(x)dx \geq \int_a^b g(x)dx.$$

7. If the function $f(x)$ is integrable on [*a,b*], then $|f(x)|$ is also integrable on [*a,b*] and the following inequality is valid:

$$\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx.$$

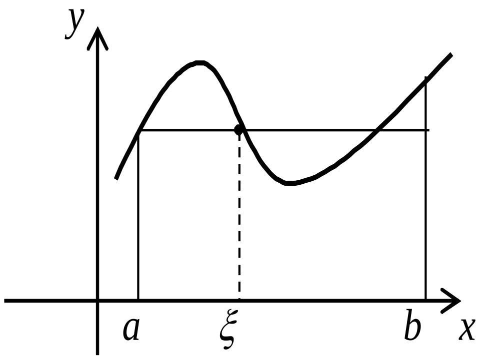
8. The theorem on *an estimate of a definite integral*. If integrable on [*a,b*] $m \leq f(x) \leq M$, where $m = \inf\limits_{[a,b]} f(x), \ M = \sup\limits_{[a,b]} f(x)$ then

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a).$$

9. **The Mean Value Theorem**. Suppose f is continuous on [*a,b*]. Then, for some point ξ of [*a,b*]

$$\int_a^b f(x)dx = f(\xi)(b-a).$$

Geometrical meaning of the theorem: let $f(x) \geq 0 \ \forall x \in [a,b]$, then there exists at least one point $\xi \in (a,b)$, that area of the curvilinear trapezoid, restricted above by the continuous curve $y = f(x)$ is equal to area of the rectangle with similar base and altitude, equal to $f(\xi)$ as presented in Figure.



10. **General Mean Value Theorem.** If the functions $f(x)$ and $\varphi(x)$ are continuous in [*a*,*b*], and $\varphi(x)$ doesn't change its sign in this interval, then

$$\int_a^b f(x)\varphi(x)dx = f(\xi)\int_a^b \varphi(x)dx \quad a < \xi < b$$

11. If the continuous function $f(x)$, $x \in [-l, l]$ is even, then

$$\int_{-l}^{l} f(x)dx = 2\int_0^l f(x)dx.$$

If the function $f(x)$ - odd, then $\int_{-l}^{l} f(x)dx = 0$.

1.2 The Newton–Leibniz formula. Calculation of definite integrals.

Consider a Riemann integrable function *f* on [*a*,*b*]. Then, for any $x \in [a,b]$, *f* is Riemann integrable on [*a*,*x*] also, so $\int_a^x f(t)dt$ is a well-defined function of *x*.

Theorem 5. If a function *f* is Riemann integrable on [*a*,*b*], then the function $F(x) = \int_a^x f(t)dt$ is differentiable at every point of continuity of *f*. Furthermore, $F'(x_0) = \left(\int_a^x f(t)dt\right)' = f(x_0)$ or $dF(x_0) = d\left(\int_a^x f(t)dt\right) = f(x_0)dx$. (If x_0 is either *a* or *b*, then $F'(x_0)$ is to be understood as the appropriate one-sided derivative.)

Corollary. Every function f which is continuous on $[a,b]$ has an antiderivative on this interval. The function $F(x) = \int_a^x f(t)dt$ is one of the antiderivatives.

Theorem 6. (The Fundamental Theorem of Calculus). Let f be a continuous function defined on $[a,b]$. If F is any antiderivative of f on $[a,b]$, then the following formula holds:

$$\boxed{\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a)}$$

The formula is called **the Newton-Leibnitz formula**, it establishes relation between definite and indefinite integrals and allows calculus of definite integrals as the difference of values at high and low limits of integrating.

Examples.

1. Find the integral $\int_{\pi/4}^{\pi/3} \sin x\,dx$.

Solution: $\int \sin x\,dx = -\cos x + C$, i.e. $\int_{\pi/4}^{\pi/3} \sin x\,dx = -\cos x|_{\frac{\pi}{4}}^{\frac{\pi}{3}} = -\left\{\cos\frac{\pi}{3} - \cos\frac{\pi}{4}\right\} = -\frac{1}{2} + \frac{\sqrt{3}}{2}$.

Remark. The definite integrals with variable upper (lower) bound can be used to define new functions that cannot be expressed in terms of finite combinations of the familiar elementary functions.

Method of substitution in a definite integral.

Let the function $f(x)$ be continuous in the interval $[a,b]$, and the function $x = \varphi(t)$ is monotone and has the continuous derivative in the interval $[\alpha,\beta]$, where $\varphi(\alpha) = a$ and $\varphi(\beta) = b$, then the following formula concerning the change (substitution) of variable in a definite integral is valid:

$$\int_a^b f(x)dx = \int_\alpha^\beta f(\varphi(t))\varphi'(t)dt\,.$$

While changing the integrating variable the values of the function $\varphi(t)$ shouldn't fall outside the limits of the interval $[a,b]$ while t being changing in $[\alpha,\beta]$. If the function $\varphi(t)$ is monotone in the interval $[\alpha,\beta]$, the mentioned condition is executed.

Examples.

1. $I = \int_0^a \sqrt{a^2 - x^2}\,dx$

*Solution***.** Let $x = a\,sint$. Let's define the new integrating limits for variable *t*. Let $x = 0$, i.e. *x* is equal to low limit of integrating in the initial integral. Then in state of *t* can be accepted any solution of the equation $a\sin t = 0$, for example $t = 0$. While calculating high limit for variable *t* in state of *x* we put high limit of integrating equal to *a* and solve the equation $a = a\,sint$, from here we obtain $sint = 1$, $t = \frac{\pi}{2} + 2\pi n,\ n \in z$, i.e. the equation has an infinite set of the solutions. Thus, taking the solution $t = \frac{\pi}{2}$ (at $n = 0$) while changing *t* from 0 to $\frac{\pi}{2}$ we obtain the variable *x* will change monotone from 0 to *a*. Thus

$$I = \begin{vmatrix} x = a\,sint, \\ dx = a\,cos\,tdt \\ x = 0,\ \ t = 0 \\ x = a,\ \ t = \pi/2 \end{vmatrix} = \int_0^{\pi/2} \sqrt{a^2 - a^2\,sin^2\,t}\,a\,cos\,tdt = a^2 \int_0^{\pi/2} cos^2\,tdt =$$

$$= a^2 \int_0^{\pi/2} \frac{1 + cos\,2t}{2}\,dt = \frac{a^2}{2}\left(t + \frac{sin\,2t}{2}\right)\Bigg|_0^{\pi/2} = \frac{a^2}{2}\left(\frac{\pi}{2} + \frac{sin\,\pi}{2}\right) = \frac{\pi a^2}{4}.$$

2. $I = \int_0^{-ln\,2} \sqrt{1 - e^{2x}}\,dx$.

*Solution***.** The function $\sqrt{1 - e^{2x}}$ is continuous and monotone in the interval [-*ln2,0*]. Considering $t = \sqrt{1 - e^{2x}}$, we find the integrating limits for variable *t*. At $x = 0$ we have $t = 0$; at $x = -ln\,2$ we find $t = \sqrt{1 - e^{-2ln\,2}} = \sqrt{1 - e^{-ln\,4}} = \frac{\sqrt{3}}{2}$. It's obvious the function inverse for *t* equal to $x = \frac{ln\left(1 - t^2\right)}{2}$ will be continuous and differentiable in the interval $0 < t < \frac{\sqrt{3}}{2}$. Then

$$I = \int_0^{-ln\,2} \sqrt{1 - e^{2x}}\,dx = \begin{Vmatrix} 1 - e^{2x} = t^2, -2e^{2x}dx = 2tdt, \\ dx = \frac{-tdt}{e^{2x}} = \frac{tdt}{t^2 - 1} \end{Vmatrix} =$$

$$I = \int_0^{\sqrt{3}/2} \frac{t^2 dt}{t^2 - 1} = \int_0^{\sqrt{3}/2} \frac{\left(t^2 - 1\right) + 1}{t^2 - 1}\,dt = \left(t + \frac{1}{2}\,ln\left|\frac{t - 1}{t + 1}\right|\right)\Bigg|_0^{\sqrt{3}/2} = \frac{\sqrt{3}}{2} + \frac{1}{2}\,ln\left|\frac{\frac{\sqrt{3}}{2} - 1}{\frac{\sqrt{3}}{2} + 1}\right| =$$

$$= \frac{\sqrt{3}}{2} + \frac{1}{2}\,ln\left(2 - \sqrt{3}\right)^2 = \frac{\sqrt{3}}{2} + ln\left(2 - \sqrt{3}\right).$$

3. While calculating the integral $I = \int_0^{2\pi} \frac{dx}{2 + \cos x}$, using the substitution $t = \tan\frac{x}{2}$, we find low limit of integrating equal to $t = \tan 0 = 0$, high limit equal to $t = \tan\pi = 0$. Then

$$I = 2\int_0^0 \frac{dt}{\left(1+t^2\right)\left(2 + \frac{1-t^2}{1+t^2}\right)} = 2\int_0^0 \frac{dt}{t^2+3} = 0,$$

that is impossible, i.e. the integrand $\frac{1}{2+\cos x} > 0$. The reason of this is that the function $\tan\frac{x}{2}$ in the point $x = \pi \in [0,\ 2\pi]$ has a break and therefore doesn't have the continuous derivative. The substitution $t = \tan\frac{x}{2}$ is inapplicable in the interval $[0,\ 2\pi]$. The following integral can be calculated as follows.

$$\int_0^{2\pi} \frac{dx}{2+\cos x} = \left\| \begin{matrix} x - \pi = t \\ 0 \to -\pi \\ 2\pi \to \pi \end{matrix} \right\| = \int_{-\pi}^{\pi} \frac{dt}{2 - \cos t} = 2\int_0^{\pi} \frac{dt}{1 + 2\sin^2\frac{t}{2}} = \int_0^{\pi} \frac{dt}{\sin^2\frac{t}{2}\left(\frac{1}{\sin^2\frac{t}{2}} + 1\right)} =$$

$$= -2\int_0^{\pi} \frac{d\left(\cot\frac{t}{2}\right)}{\frac{3}{2} + \frac{1}{2}\cot^2\frac{t}{2}} = -\frac{4}{\sqrt{3}}\arctan\frac{\cot\frac{t}{2}}{\sqrt{3}}\Bigg|_0^{\pi} = \frac{4}{\sqrt{3}}\frac{\pi}{2} = \frac{2\pi}{\sqrt{3}}.$$

The formula of integrating in parts

$$\boxed{\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du}.$$

Examples.

1. $I = \int_0^3 \arcsin\sqrt{\frac{x}{1+x}}\,dx$.

*Solution***.** Considering $u = \arcsin\sqrt{\frac{x}{1+x}}$, then

$$du = \frac{1}{\sqrt{1-\left(\frac{x}{1+x}\right)^2}}\sqrt{\frac{1+x}{x}}\frac{dx}{(1+x)^2} = \frac{dx}{2\sqrt{x}(1+x)}$$ and $dv = dx, x = v$.

$$I = x\,arcsin\sqrt{\frac{x}{1+x}}\Bigg|_0^3 - \frac{1}{2}\int_0^3 \frac{xdx}{\sqrt{x}(1+x)} = 3\,arcsin\frac{\sqrt{3}}{2} - \int_0^3 \frac{\left(\sqrt{x}\right)^2 d\left(\sqrt{x}\right)}{1+\left(\sqrt{x}\right)^2} = 3\frac{\pi}{3} -$$

$$-\int_0^3 \frac{\left(\sqrt{x}\right)^2 + 1}{1+\left(\sqrt{x}\right)^2} d\left(\sqrt{x}\right) + \int_0^3 \frac{d\left(\sqrt{x}\right)}{1+\left(\sqrt{x}\right)^2} = \pi - \sqrt{x}\Big|_0^3 + arctg\sqrt{x}\Big|_0^3 = \pi - \sqrt{3} + arctg\sqrt{3} =$$

$$= \frac{4}{3}\pi - \sqrt{3}.$$