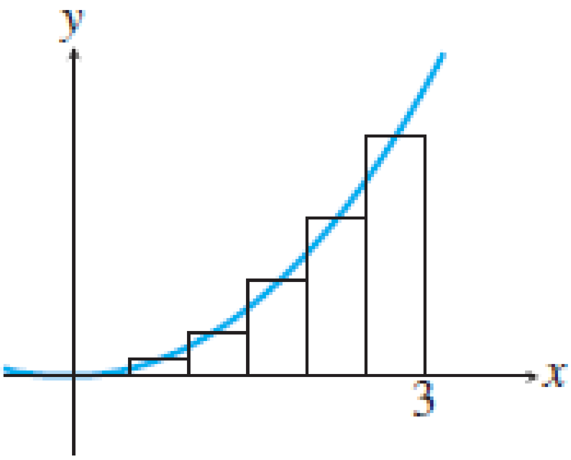
2. Applications of the Definite Integral

Here, we define the concepts of area, length, and volume: these geometric concepts are given by analytic definitions using the concept of the definite integral which we have developed earlier.

2.1 Calculation of the Area by the Definite Integral

In Introduction to Definite Integral, we developed the concept of the definite integral to calculate the area below a curve on a given interval. For this we used the area formula for a rectangle and find the area of an arbitrary polygonal figure. The reason is that any polygon can be divided into non-overlapping rectangles. This approach to area goes back several thousand years, to the ancient civilizations of Egypt and Sumeria.

Figure shows what an approximation looks like graphically when we approximate the area under the curve $y = x^2$ from $x = 0$ to $x = 3$ with six subintervals. We evaluated the function at the midpoint of each subinterval.



With 1000 subintervals, a summing procedure gives the area equal to 8.99999775. The exact area can be computed as a defined integral,

$$\int_0^3 x^2\,dx = \left.\frac{x^3}{3}\right|_0^3 = \frac{3^3}{3} - \frac{0^3}{3} = 9$$

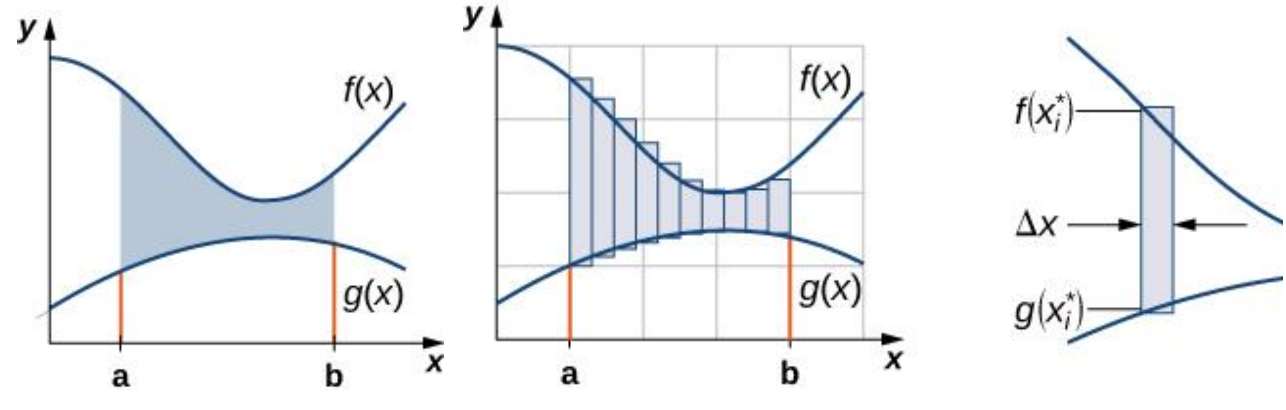
In this lecture, we expand that idea to calculate the area of more complex regions. We start by finding the area between two curves that are functions of x, beginning with the simple case in which one function value is always greater than the other. We then look at cases when the graphs of the functions cross. Last, we consider how to calculate the area between two curves that are functions of y.

2.1.1 Areas between Two Curves

Just as definite integrals stated during the previous lecture can be used to find the area under a curve, they can also be used to find the area between two curves.

Let $f(x)$ and $g(x)$ be continuous functions over an interval $[a, b]$ such that

$f(x) \geq g(x)$ on $[a, b]$. We want to find the area between the graphs of the functions, as shown in Figure 1a,



(a) (b) (c)

Figure 1. The area between the graphs of two functions, $f(x)$ and $g(x)$, on the interval $[a, b]$

As we did before, we are going to partition the interval on the x-axis and approximate the area between the graphs of the functions with rectangles. So, for $i = 0,1,2, \dots, n$, let $P = x_i$ be a regular partition of $[a, b]$. Then, for $i = 1,2, \dots, n$, choose a point $x_i^* \in [x_{i-1}, x_i]$, and on each interval $[x_{i-1}, x_i]$ construct a rectangle that extends vertically from $g(x_i^*)$ to $f(x_i^*)$. Figure 1b shows the rectangles when x_i^* is selected to be the left endpoint of the interval and $n = 10$. Figure 1c shows a representative rectangle in detail.

The height of each individual rectangle is $f(x_i^*) - g(x_i^*)$ and the width of each rectangle is Δx. Adding the areas of all the rectangles, we see that the area between the curves is approximated by

$$A = \sum_{i=0}^{n-1} \left[f\left(x_i^*\right) - g\left(x_i^*\right) \right] \Delta x_i$$

This is a Riemann sum, so we take the limit as $n \to \infty$ and we get that

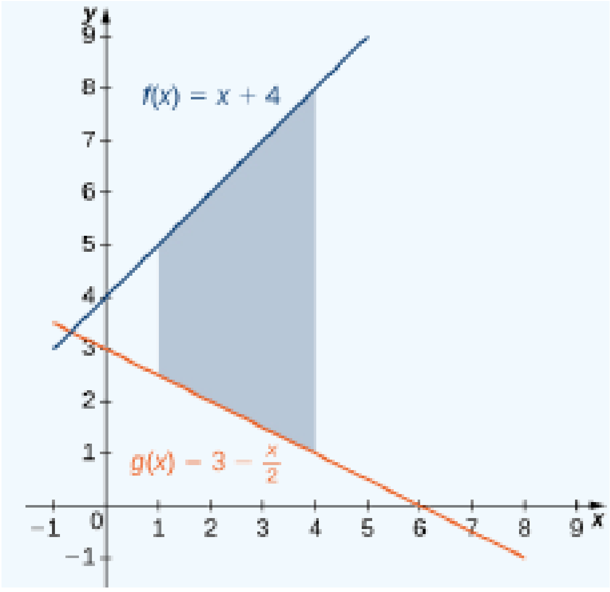
$$A = \underset{n \to \infty}{Lim} \sum_{i=0}^{n-1} \left[f\left(x_i^*\right) - g\left(x_i^*\right) \right] \Delta x_i = \int_a^b \left[f(x) - g(x) \right] dx$$

These findings are summarized in the following rule: Let $f(x)$ and $g(x)$ be continuous functions such that $f(x) \geq g(x)$ over an interval $[a, b]$. Let R denote the region bounded above by the graph of $f(x)$, below by the graph of $g(x)$, and on the left and right by the lines $x = a$ and $x = b$, respectively. Then, the area of R is given by

$$A = \int_a^b \left[f(x) - g(x) \right] \, \mathrm{d}x$$

***Example* 1.** If the region is bounded above by the graph of the function $f(x) = x + 4$ and below by the graph of the function $g(x) = 3 - \frac{x}{2}$ over the interval $[1,4]$, find the area of the region.

Solution: The region is depicted in the following figure.



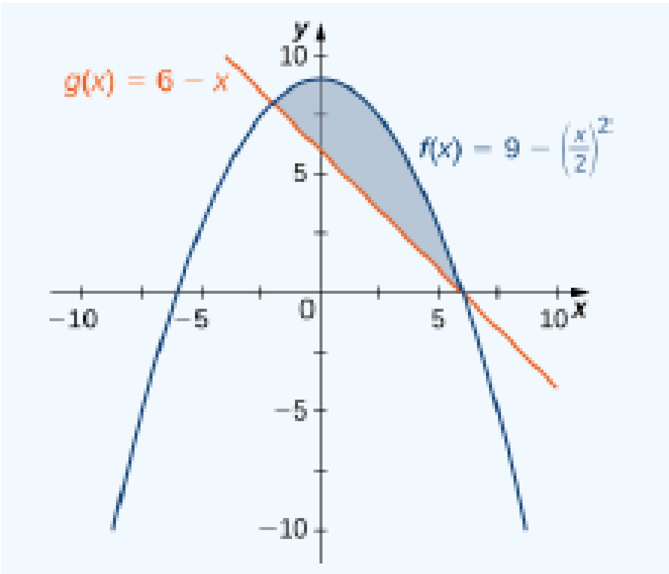
We have

$$A = \int_a^b [f(x) - g(x)]\,dx = \int_1^4 \left[(x+4) - \left(3 - \frac{x}{2}\right)\right] dx =$$

$$\int_1^4 \left[\frac{3x}{2} + 1\right] dx = \left[\frac{3x^2}{4} + x\right] |_1^4 = \left(16 - \frac{7}{4}\right) = \frac{57}{4}$$

***Example* 2.** If the region is bounded above by the graph of the function $f(x) = 9 - \frac{x^2}{4}$ and below by the graph of the function $g(x) = 6 - x$, find the area of the region.

Solution: The region is depicted in the following figure.



We first need to compute where the graphs of the functions intersect. Setting $f(x) = g(x)$, we get

$$f(x) = g(x), \qquad i.e. \qquad 9 - \left(\frac{x}{2}\right)^2 = 6 - x \rightarrow$$

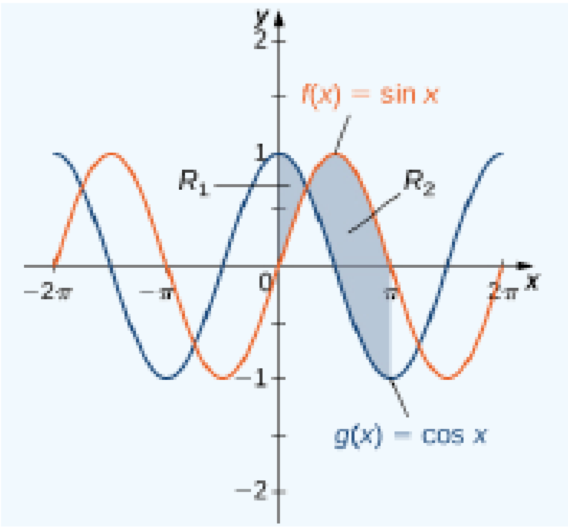
$$9-\frac{x^2}{4}=6-x \ \rightarrow 36-x^2=24-4x \rightarrow x^2-4x-12=0$$
$$(x-6)(x+2)=0.$$
The graphs of the functions intersect when $x=6$ or $x=-2$, so we want to integrate from -2 to 6. Since $f(x) \geq g(x)$ for $-2 \leq x \leq 6$, we obtain
$$A=\int_a^b [f(x)-g(x)]\,dx=\int_{-2}^{6}[9-(\frac{x}{2})^2-(6-x)]\,dx$$
$$=\int_{-2}^{6}[3-\frac{x^2}{4}+x]\,dx=[3x-\frac{x^3}{12}+\frac{x^2}{2}]|_{-2}^{6}=\frac{64}{3}$$

2.1.2 Areas of Compound Regions

So far, we have required $f(x) \geq g(x)$ over the entire interval of interest, but what if we want to look at regions bounded by the graphs of functions that cross one another? In that case, we modify the process we just developed by using the absolute value function.
$$A=\int_a^b |f(x)-g(x)|dx.$$
In practice, applying this theorem requires us to break up the interval $[a,b]$ and evaluate several integrals, depending on which of the function values is greater over a given part of the interval. We study this process in the following example:

***Example* 3.** If the region between the graphs of the functions $f(x)=sinx$ and $g(x)=cosx$ over the interval $[0,\pi]$ exists, find the area of the region.

Solution: The region is depicted in the following figure.



The graphs of the functions intersect at $x=\pi/4$. For $x \in [0,\pi/4]$, $\cos x \geq \sin x$,
$$|f(x)-g(x)|=|\sin x-\cos x|=\cos x-\sin x.$$
On the other hand, for

$$x \in [\pi/4, \pi], \sin x \geq \cos x, \text{ so}$$

$$|f(x) - g(x)| = |\sin x - \cos x| = \sin x - \cos x.$$

Then,

$$A = \int_a^b |f(x) - g(x)| dx$$

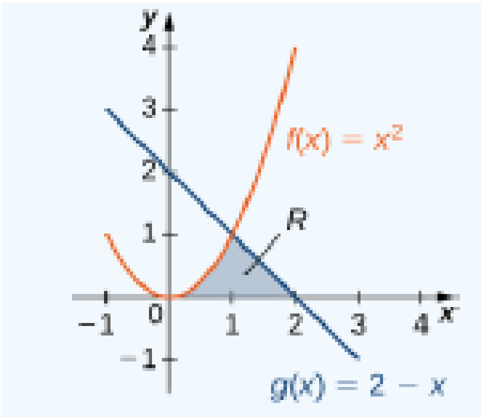
$$= \int_0^{\pi} |\sin x - \cos x| dx = \int_0^{\pi/4} (\cos x - \sin x)\, dx + \int_{\pi/4}^{\pi} (\sin x - \cos x)\, dx$$

$$= [\sin x + \cos x]|_0^{\pi/4} + [-\cos x - \sin x]|_{\pi/4}^{\pi}$$

$$= (\sqrt{2} - 1) + (1 + \sqrt{2}) = 2\sqrt{2}.$$

So, $A = 2\sqrt{2}$, *units*^{2}

***Example* 4.** Consider the region depicted in Figure. Find the area of the region



Solution: As with Example 3 , we need to divide the interval into two pieces. The graphs of the functions intersect at $x = 1$ (set $f(x) = g(x)$ and solve for x), so we evaluate two separate integrals: one over the interval $[0,1]$ and one over the interval $[1,2]$.

Over the interval $[0,1]$, the region is bounded above by $f(x) = x^2$ and below by the *x*-axis, so we have

$$A_1 = \int_0^1 x^2\, dx = \frac{x^3}{3} |_0^1 = \frac{1}{3}.$$

Over the interval $[1,2]$, the region is bounded above by $g(x) = 2 - x$ and below by the *x*-axis, so we have

$$A_2 = \int_1^2 (2 - x)\, dx = \left[2x - \frac{x^2}{2}\right] |_1^2 = \frac{1}{2}.$$

Adding these areas together, we obtain

$$A = A_1 + A_2 = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}.$$

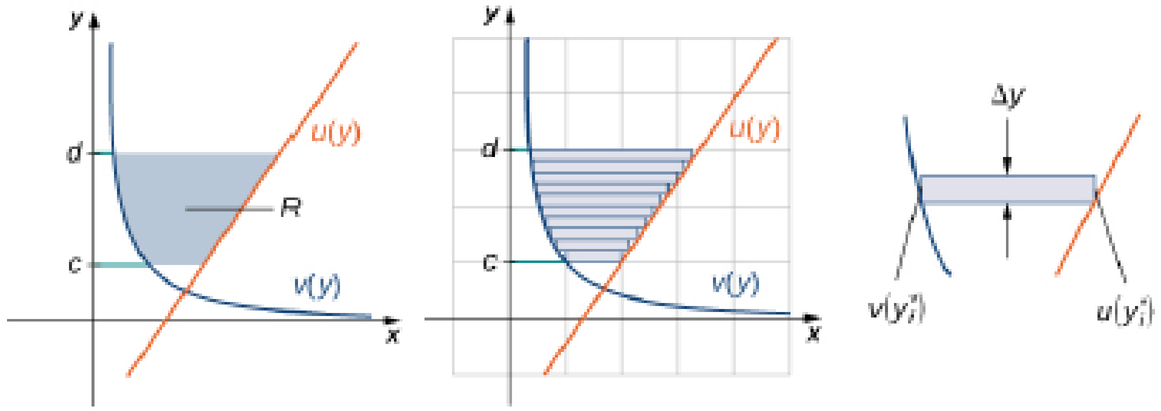
The area of the region is $5/6$ *units*^{2}

2.1.3 Areas of Regions Defined with Respect to y

In Example 4, we had to evaluate two separate integrals to calculate the area of the region. However, there is another approach that requires only one integral. What if we treat the curves as functions of y, instead of as functions of x?

Review Figure. Note that the left graph, shown in red, is represented by the function $y = f(x) = x^2$. We could just as easily solve this for x and represent the curve by the function $x = v(y) = \sqrt{y}$. (Note that $x = -\sqrt{y}$ is also a valid representation of the function $= f(x) = x^2$ as a function of y. However, based on the graph, it is clear we are interested in the *positive square root*). Similarly, the right graph is represented by the function $y = g(x) = 2 - x$, but could just as easily be represented by the function $x = u(y) = 2 - y$. When the graphs are represented as functions of y, we see the region is bounded on the left by the graph of one function and on the right by the graph of the other function. Therefore, if we integrate with respect to y, we need to evaluate one integral only. Let's develop a formula for this type of integration:

Let $u(y)$ and $v(y)$ be continuous functions over an interval $[c, d]$ such that $u(y) \geq v(y)$ for all $y \in [c, d]$. We want to find the area between the graphs of the functions, as shown in Figure



This time, we are going to partition the interval on the y-axis and use horizontal rectangles to approximate the area between the functions. Therefore, the area between the curves can be found as a Riemann sum, so we take the limit as $n \to \infty$, obtaining

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} [u(y_i^*) - v(y_i^*)]\,\Delta y = \int_c^d [u(y) - v(y)]\,dy$$

These findings are summarized in the following rule: Let $u(y)$ and $v(y)$ be continuous functions such that $u(y) \geq v(y)$ for all $y \in [c, d]$. Let R denote the region bounded on the right by the graph of $u(y)$, on the left by the graph of $v(y)$, and above and below by the lines $y = d$ and $y = c$, respectively. Then, the area of R is given by

$$A = \int_c^d [u(y) - v(y)]\,dy.$$

***Example* 5.** Let's revisit Example 4 , only this time let's integrate with respect to y.

Solution: We must first express the graphs as functions of y. As we saw at the beginning of this section, the curve on the left can be represented by the function $x = v(y) = \sqrt{y}$, and the curve on the right can be represented by the function $x = u(y) = 2 - y$.

Now we have to determine the limits of integration. The region is bounded below by the x-axis, so the lower limit of integration is $y = 0$. The upper limit of integration is determined by the point where the two graphs intersect, which is the point $(1,1)$, so the upper limit of integration is $y = 1$. Thus, we have $[c, d] = [0,1]$.

Calculating the area of the region, we get

$$A = \int_c^d [u(y) - v(y)]\, dy =$$

$$\int_0^1 [(2 - y) - \sqrt{y}]\, dy = [2y - \frac{y^2}{2} - \frac{2}{3} y^{3/2}] \,|_0^1 = \frac{5}{6}$$

2.1.4 Areas of Regions Bounded by Curves Defined with Respect to Parameter

Suppose we have parametric equations for a curve,

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases}, t \in [t_1, t_2],\ (x(t_1) = a, x(t_2) = b).$$

Assume that those equations define a function $y = f(x)$ on $[a, b]$. Then, by substituting $x = x(t)$ in formula for defining an area of a function above the *x*-axis so that $dx = x'(t)dt$, and $y = f[x(t)] = y(t)$, we obtain

$$A = \int_{t1}^{t2} y(t)x'(t)dt,\ (t_1 \le t \le t_2)$$

or

$$A = \int_{t1}^{t2} x(t)y'(t)dt,\ (t_1 \le t \le t_2).$$

***Example* 6.** Find area of the figure, restricted by the curves, given in parametric form:

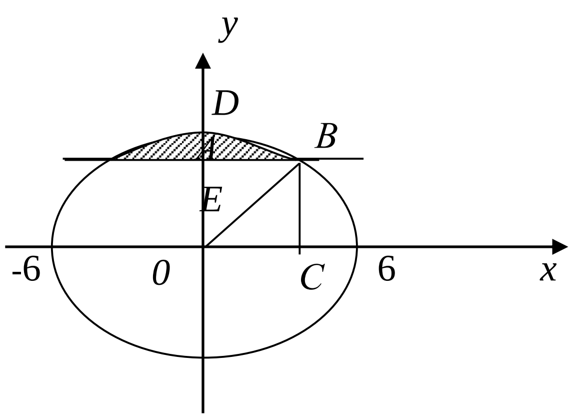
$$\begin{cases} x = 6\,cos\,t \\ y = 4\,sin t \end{cases} \text{ and } y = 2\sqrt{3}\left(y \ge 2\sqrt{3}\right).$$

Solution: Let’s find the point of intersection of the ellipse and the curve $2\sqrt{3} = 4\,sin t; \Rightarrow\ sin t = \frac{\sqrt{3}}{2} \Rightarrow t = \frac{\pi}{3} \Rightarrow x = 6\,cos\frac{\pi}{3} = 3.$

The figure represents a segment of the ellipse, given in parametric form. Due to the symmetry of the given figure we calculate half of the area

$$\frac{A}{2} = A_1 - A_2 = \int_{t1}^{t2} y(t)x'(t)dt - A_2,$$

where A_2 - area of the rectangle *OEBC*. Obviously $A_2 = 3 \cdot 2\sqrt{3} = 6\sqrt{3}$



For the curvilinear trapezoid *ODBC* the parameter t is changing from $\frac{\pi}{3}$ to $\frac{\pi}{2}$. As x decreases while t is changing in the mentioned interval we are to change the sign of the integral on the inverse.

$$A_1 = -\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 4\sin t \cdot 6(-\sin t)dt = 24\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (\sin t)^2\, dt = 24\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1-\cos 2t}{2}\, dt = 12\left(t - \frac{\sin 2t}{2} \right)\Bigg|_{\frac{\pi}{3}}^{\frac{\pi}{2}} =$$

$$= 12\left(\frac{\pi}{2} - \frac{\pi}{3} + \frac{1}{2}\sin\frac{2\pi}{3} \right) = 12\left(\frac{\pi}{6} + \frac{\sqrt{3}}{4} \right) = 2\pi + 3\sqrt{3}\,.$$

Conclusion: $S = 2\left(2\pi + 3\sqrt{3} - 6\sqrt{3}\right) = 4\pi - 6\sqrt{3}$.

***Example* 7.** Calculate area of the figure, restricted by the abscissa axis and arc of the cycloid

$$\begin{cases} x = a(t - sin\,t), \\ y = a(1 - cos\,t). \end{cases} \quad (0 \le t \le 2\pi).$$

Solution:

$$A = \int_0^{2\pi} a^2(1-\cos t)^2\, dt = a^2\int_0^{2\pi}\left(1 - 2\cos t + \cos^2 t\right)dt = a^2\int_0^{2\pi}\left(1 - 2\cos t + \frac{1+\cos 2t}{2} \right)dt =$$

$$= a^2\left(\frac{3}{2}t - 2\sin t + \frac{\sin 2t}{2} \right)\Bigg|_0^{2\pi} = a^2\left(\frac{3}{2}2\pi - 2\sin 2\pi + \frac{\sin 4\pi}{2} \right) = 3\pi a^2 \ (sq.unit).$$

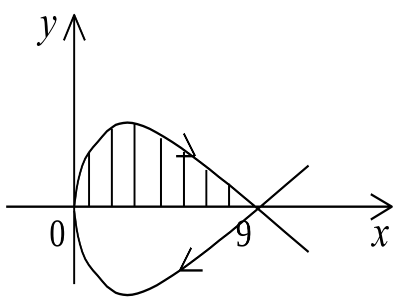
***Example* 8.** Calculate area of the figure, restricted by a closed loop of the curve:

$$\begin{cases} x = 3t^2 \\ y = 3t - t^3 \end{cases}.$$

Solution: As $x = x(t)$ is even, and $y = y(t)$ is odd, then $x = x(y)$ is even, and therefore the plot is symmetric concerning the axis *OX*.

$$y = t\left(3 - t^2\right), \quad y = 0\,.$$

At $t_1 = 0 \Rightarrow x = 0$. $t_{2,3} = \pm\sqrt{3} \ \Rightarrow x = 9$.



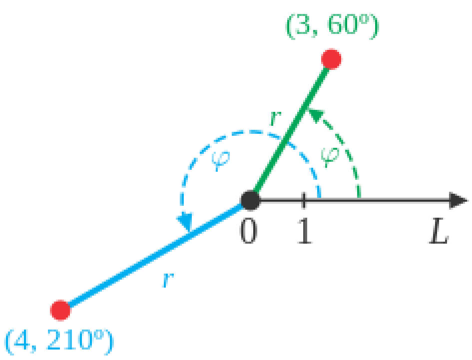
By virtue of the symmetry let's calculate the areas and then double the result

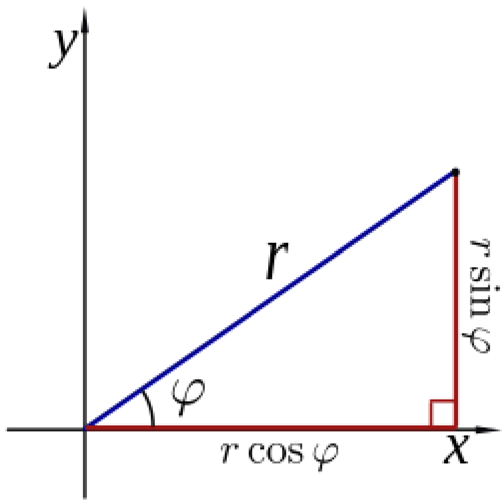
$$A = 2\int_0^{\sqrt{3}}\left(3t - t^3\right)6t\,dt = 12\int_0^{\sqrt{3}}\left(3t^2 - t^4\right)dt = 12\left(t^3 - \frac{t^5}{5}\right)\Bigg|_0^{\sqrt{3}} = 12\left(3\sqrt{3} - \frac{9\sqrt{3}}{5}\right) =$$

$$= \frac{12\cdot 6\sqrt{3}}{5} = \frac{72\sqrt{3}}{5}.$$

2.1.5 Areas of Regions Bounded by Curves Defined in Polar Coordinate System

Notation: the ***polar coordinate system*** is a two-dimensional coordinate system in which each point on a plane is determined by a distance from a reference point and an angle from a reference direction. The reference point (analogous to the origin of a Cartesian coordinate system) is called the pole, and the ray from the pole in the reference direction is the polar axis. The distance from the pole is called the radial coordinate, radial distance or simply radius, and the angle is called the angular coordinate, polar angle, or azimuth. The radial coordinate is often denoted by r or ρ, and the angular coordinate by φ, θ, or t. Angles in polar notation are generally expressed in either degrees or radians (2π rad being equal to 360°).





Converting between polar and Cartesian coordinates: The polar coordinates r and φ can be converted to the Cartesian coordinates x and y by using the trigonometric functions sine and cosine:

$$\cos\theta = \frac{x}{r} \to x = \mathrm{r}\cos\theta$$

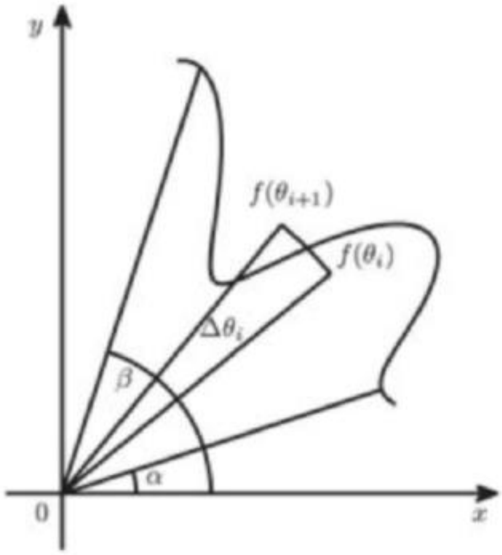
$$\sin\theta = \frac{y}{r} \to y = \mathrm{r}\sin\theta$$

$$r^2 = x^2 + y^2$$

$$\tan\theta = \frac{y}{x}$$

Suppose that the region R shown in Figure is bounded by the two radial lines $\theta = \alpha$ and $\theta = \beta$ and by the curve $\rho = f(\theta), \theta \in [\alpha, \beta]$. To approximate the area

A of R, we begin with a partition $T = \{\theta_0, \theta_1, \dots, \theta_n\}$ of the interval $[\alpha, \beta]$, $\alpha = \theta_0 < \theta_1 < \cdots < \theta_n = \beta$.



Let $\xi_i \in [\theta_i, \theta_{i+1}]$ $(i = 0, 1, \dots, n-1)$ be some selection of points for T, and $\Delta\theta_i = \theta_{i+1} - \theta_i$. It is not hard to see that the area of the sector bounded by the lines $\theta = \theta_i, \theta = \theta_{i+1}$ and the curve $\rho = f(\theta)$, is approximately equal to the area of the circular sector $1/2\,[f(\xi i)]^2 \Delta\theta_i$ with radius $f(\xi i)$ and bounded by the same angle (Figure).

We add the areas of these sectors for $i = 0, 1, \dots, n-1$, and thereby find that

$$A \approx \frac{1}{2}\sum_{i=0}^{n-1}[f(\xi i)]^2\,\Delta\theta i$$

This sum is a Riemann sum for the integral $\frac{1}{2}\int_{\alpha}^{\beta} f^2(\theta)\,d\theta$.

We conclude that the area A of the region R bounded by lines $\theta = \alpha$ and $\theta = \beta$ and by the curve $\rho = f(\theta)$, $\theta \in [\alpha, \beta]$, is the limit of the sum as the step μ(T)→0, that is

$$\lim_{\mu(T)\to 0}\frac{1}{2}\sum_{i=0}^{n-1}[f(\xi i)]^2\Delta\theta_i = \frac{1}{2}\int_{\alpha}^{\beta} f^2(\theta)\,d\theta$$

Thus, the area of the figure, given in the polar coordinates $\theta = \alpha$, $\theta = \beta$ and the curve $\rho = \rho(\theta)$, is calculated as follows:

$$A = \frac{1}{2}\int_{\alpha}^{\beta}\rho^2(\theta)d\theta .$$

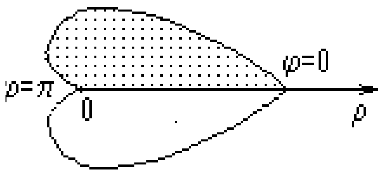
In the case of two curves $\rho_1 = f_1(\theta)$ and $\rho_2 = f_2(\theta)$ ($f_1(\theta) \geq f_2(\theta)$) for $\theta \in [\alpha, \beta]$. Then the area of the region bounded by these two curves and the rays $\theta = \alpha$ and $\theta = \beta$ may be subtracting the area bounded by the inner curve from that bounded by the outer curve. That is, the area between the curves is given by

$$A = \frac{1}{2}\int_{\alpha}^{\beta}\left[\rho_1^2(\theta) - \rho_2^2(\theta)\right]d\theta .$$

***Example* 9.** Calculate area of the figure, restricted by the cardioid $\rho = a(1+\cos\varphi)$.

Solution: Area in polar coordinates is calculated according to the formula:

$$A = \frac{1}{2}\int_{\alpha}^{\beta} \rho^2(\varphi)d\varphi.$$



By virtue of the symmetry we obtain:

$$A = \frac{1}{2}a^2 \cdot 2\int_0^{\pi}(1+\cos\varphi)^2 d\varphi =$$

$$= a^2\int_0^{\pi}(1+2\cos\varphi+\cos^2\varphi)d\varphi =$$

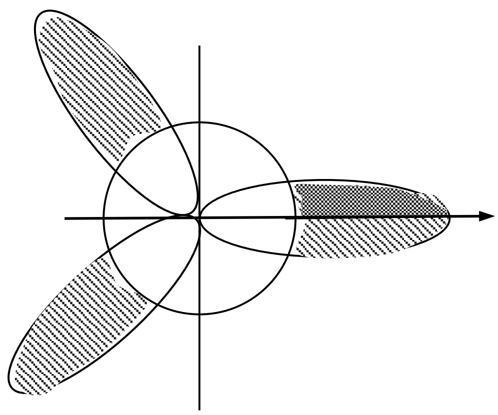
$$= a^2\int_0^{\pi}\left(1+2\cos\varphi+\frac{1+\cos 2\varphi}{2}\right)d\varphi = a^2\left(\frac{3}{2}\varphi+2\sin\varphi+\frac{\sin 2\varphi}{4}\right)\Bigg|_0^{\pi} = \frac{3}{2}\pi a^2.$$

***Example* 10**. Find area of the figure, restricted by the curve $r = 2a\cos 3\varphi$ lying outside the circle $r = a$.

Solution: $r = 2a\cos 3\varphi$ is a three-lamellar rose, i.e.

$$r \ge 0 \quad \cos 3\varphi \ge 0 \quad \Rightarrow$$

$$-\frac{\pi}{2}+2k\pi \le 3\varphi \le \frac{\pi}{2}+2k\pi \Rightarrow -\frac{\pi}{6}+\frac{2k\pi}{3} \le \varphi \le \frac{\pi}{6}+\frac{2k\pi}{3}$$



There are 3 lobes with axes

$$\varphi_1 = 0,\ \ \varphi_2 = \frac{2\pi}{3}\ \ \varphi_3 = \frac{4\pi}{3}\ \ r_{\max} = 2a$$

By virtue of the symmetry calculate $\frac{1}{6}$ part of the necessary area. Let’s find the points of intersections of the curves

$$\begin{cases} r = 2a\cos 3\varphi \\ r = a \end{cases} \Rightarrow \cos 3\varphi = \frac{1}{2} \Rightarrow \varphi = \pm\frac{\pi}{9}+\frac{2k\pi}{3}$$

according to the formula $A = \frac{1}{2}\int_{\varphi 1}^{\varphi 2}\rho^2 d\varphi$

1) let’s find $\frac{1}{2}$ area of one of three lobes

$$A_1 = \frac{1}{2}\int_0^{\pi/9} 4a^2\cos^2 3\varphi d\varphi = 2a^2\int_0^{\pi/9}\frac{1+\cos 6\varphi}{2}d\varphi =$$

$$= a^2\left(\varphi+\frac{\sin 6\varphi}{6}\Bigg|_0^{\pi/9}\right) = a^2\left(\frac{\pi}{9}+\frac{\sqrt{3}}{12}\right).$$

2) $A_2 -$ area of the quadrant of the circle with radius a and angle equal to $\frac{\pi}{9}$.

$$A_2 = \frac{R^2\varphi}{2} = \frac{a^2\pi}{2\cdot 9} = \frac{\pi a^2}{18}.$$

Hence, $\frac{1}{6}A = A_1 - A_2$,

$$A = 6\left(\frac{\pi a^2}{9} + \frac{a^2\sqrt{3}}{12} - \frac{\pi a^2}{18}\right) = \frac{a^2}{6}\left(2\pi + 3\sqrt{3}\right).$$

Two additional examples.

***Example* 11**. Calculate area of the figure, restricted by the circle $x^2 + y^2 = 16$ and parabola $x^2 = 12(y-1)$.

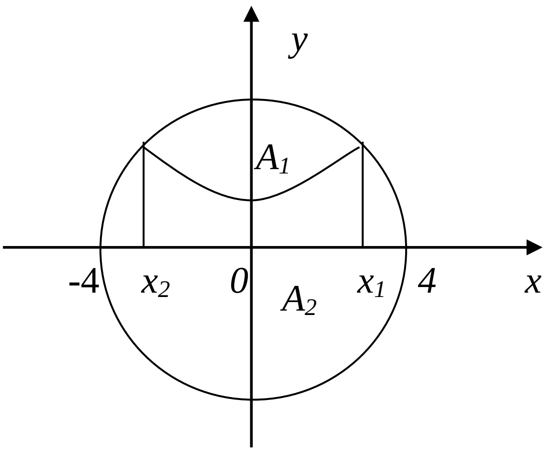
*Solution***:** Let's find the points of intersections of the given curves:

$$\begin{cases} x^2 + y^2 = 16 \\ x^2 = 12(y-1) \end{cases} \Rightarrow 16 - y^2 = 12y - 12, \Rightarrow$$

$$y^2 + 12y - 12 = 0$$

$y_1 = -14$ is unsuitable as $y \geq 0$.

$y_2 = 2 \Rightarrow x^2 = 12 \Rightarrow x_{1,2} = \pm 2\sqrt{3}$.



The given curves form two figures. Let's find area A_1, then A_2 can be calculated as a difference between areas of the circle and A_1.

$$A_1 = 2\left(\underbrace{\int_0^{2\sqrt{3}} \sqrt{16 - x^2}\,dx}_{=J_1} - \frac{1}{12}\int_0^{2\sqrt{3}} (x^2 + 12)dx\right)$$

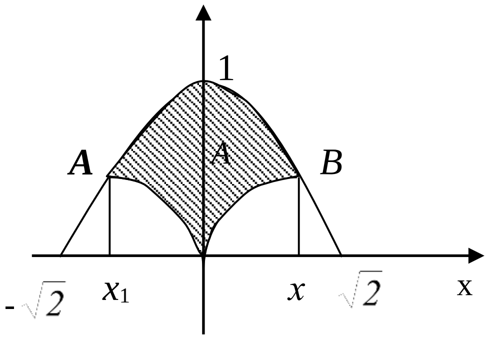
$$J_1 = \int_0^{2\sqrt{3}} \sqrt{16 - x^2}\,dx \left\| \begin{matrix} x = 4\sin t & dx = 4\cos t dt \\ \sqrt{16 - x^2} = 4\cos t & \end{matrix} \right\| =$$

$$= 16\int_0^{\pi/3} \cos^2 t dt = 8\int_0^{\pi/3} (1 + \cos 2t)dt = 8\left(t + \frac{1}{2}\sin 2t\right)\Big|_0^{\pi/3} = 8\left(\frac{\pi}{3} + \frac{\sqrt{3}}{4}\right).$$

Then,

$$A_2 = 16\pi - 8\left(\frac{\pi}{3} + \frac{\sqrt{3}}{4}\right) = \frac{40\pi}{3} - 2\sqrt{3}.$$

***Example* 12.** Calculate area A of the figure, restricted by the curves $y = 2 - x^2$ and $y^2 = x^3$.



Solution. Let's solve the set of equations $\begin{cases} y = 2 - x^2 \\ y^2 = x^3 \end{cases}$, and find the limits of integrating $x_1 = -1$ and $x_2 = 1$. Due to the formula $S = \int_a^b (y_1 - y)dx$, we get the following:

$$S = \int_{-1}^{1} \left(2 - x^2 - x^{2/3}\right)dx = \left(2x - \frac{x^3}{3} - \frac{3}{5}x^{5/3}\right)\Bigg|_{-1}^{1} = 2\frac{2}{15}.$$