2. Applications of the Definite Integral

In the preceding lecture, we used definite integrals to find the area between two curves. In this section, we use definite integrals to find volumes of three-dimensional solids. We consider an approach—slicing—for finding these volumes.

2.2 Volume and the Slicing Method (the Method of Cross Sections)

Just as area is the numerical measure of a two-dimensional region, volume is the numerical measure of a three-dimensional solid. Most of us have computed volumes of solids by using basic geometric formulas:

$$V_{sphere} = \frac{4}{3}\pi r^3,$$

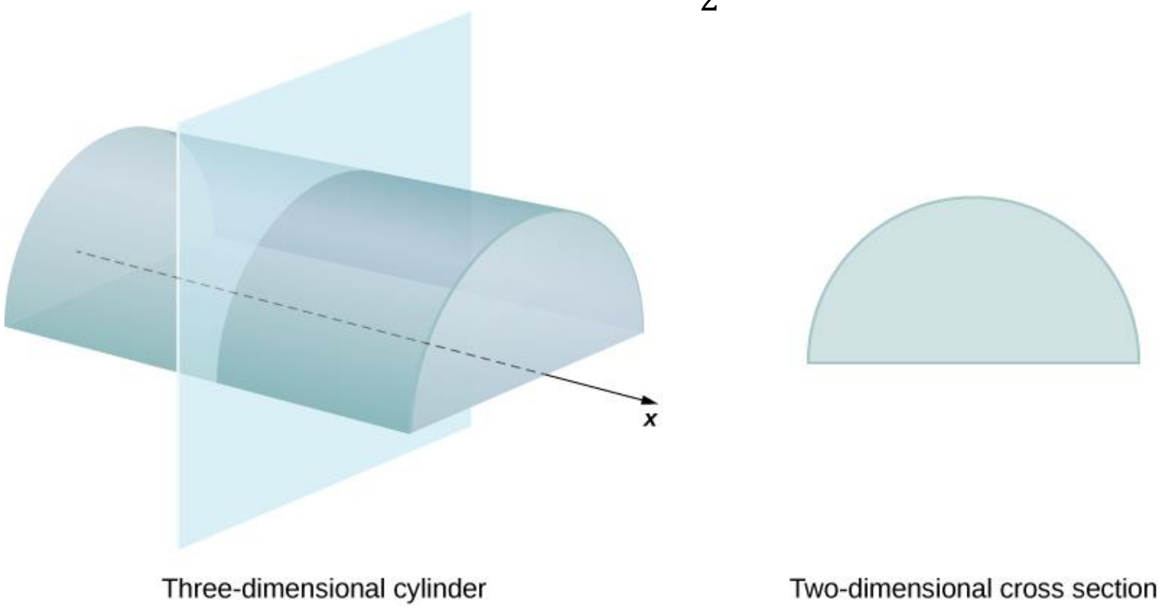
$$V_{cone} = \frac{1}{3}\pi r^2 h,$$

$$V_{pyramid} = \frac{1}{3}Ah$$

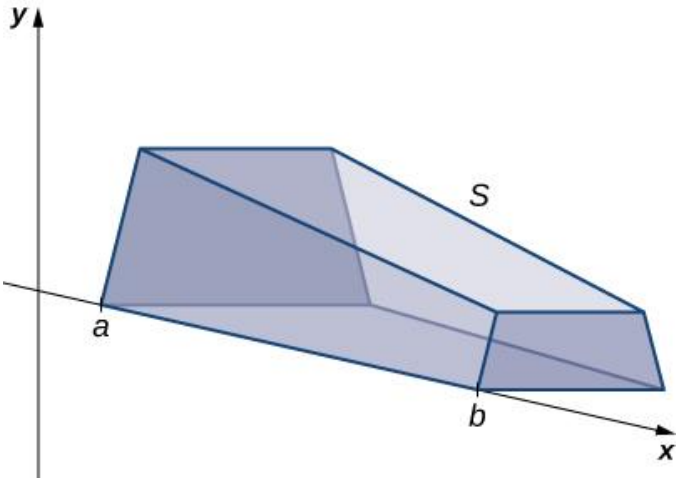
These formulas were derived using geometry alone, all these formulas can be obtained by using integration. Let's calculate the volume of a cylinder:

We define the *cross-section* of a solid to be the intersection of a plane with the solid. A cylinder is defined as any solid that can be generated by translating a plane region along a line perpendicular to the region, called the *axis of the cylinder*. Thus, all cross-sections perpendicular to the axis of a cylinder are identical. The solid shown in Figure 1 is an example of a cylinder with a noncircular base. To calculate the volume of a cylinder, then, we simply multiply the area of the cross-section by the height of the cylinder:

$$V = A \cdot h \Longrightarrow V = \frac{\pi r^2}{2} h.$$

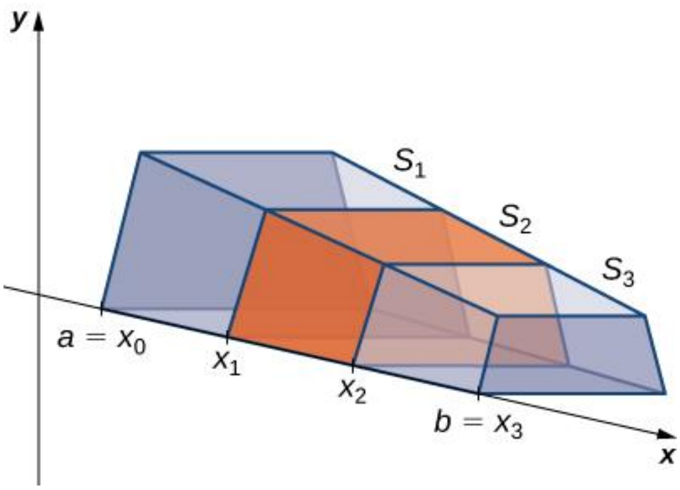


If a solid *does not have a constant cross-section* (and it is not one of the other basic solids), we may not have a formula for its volume. In this case, we can use a definite integral to calculate the volume of the solid. We do this by *slicing* the solid into pieces, estimating the volume of each slice, and then adding those estimated volumes together. The *slices* (*cross-sections*) should all be parallel to one another, and when we put all the slices together, we should get the whole solid. Consider, for example, the solid S shown in Figure 2 , extending along the x -*axis*.



Notation: We want to divide S into slices perpendicular to the x -axis. There may be times when we want to slice the solid in some other direction—say, with slices perpendicular to the y -axis. The decision of which way to slice the solid is very important. If we make the wrong choice, the computations can get quite messy.

For the purposes of this section, however, we use slices perpendicular to the x -axis. Because the cross-sectional area is not constant, we let $A(x)$ represent the area of the cross-section at point x. Now let $P = x_0, x_1 \dots, X_n$ be a regular partition of $[a, b]$, and for $i = 1,2, \dots n$, let Si represent the slice of S stretching from x_{i-1} to x_i . The following figure shows the sliced solid with $n = 3$.



Finally, for $i = 1,2, \dots n$, let x_i^* be an arbitrary point in $[x_{i-1}, x_i]$. Then the volume of slice S_i can be estimated by $V(S_i) \approx A(x_i^*)\Delta x$. Adding these approximations together, we see the volume of the entire solid S can be approximated by

$$V(S) \approx \sum_{i=1}^{n} A(x_i^*)\Delta x .$$

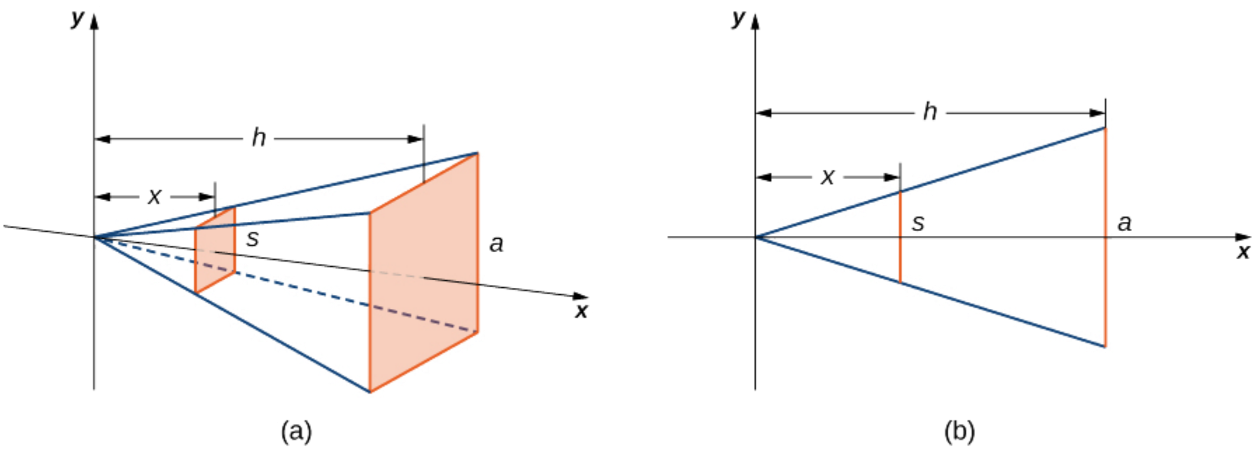
By now, we can recognize this as a Riemann sum, and our next step is to take the limit as $n \to \infty$. Then we have

$$V(S) = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\Delta x = \int_a^b A(x)dx .$$

***Example* 1**: deriving the formula for the volume of a pyramid. We know from geometry that the formula for the volume of a pyramid is $V = \frac{1}{3}Ah$. If the pyramid has a square base, this becomes $V = \frac{1}{3}a^2h$, where a denotes the length of one side of the base. We are going to use the slicing method to derive this formula.

Solution:

Let's apply the slicing method to a pyramid with a square base. To set up the integral, consider the pyramid shown in Figure 3, oriented along the x –axis:



First, we need to determine the shape of a cross-section of the pyramid. We know the base is a square, so the cross-sections are squares as well. Now we can determine a formula for the area of one of these cross-sectional squares. Looking at Figure 3 (b), and using a proportion, since these are similar triangles, we have

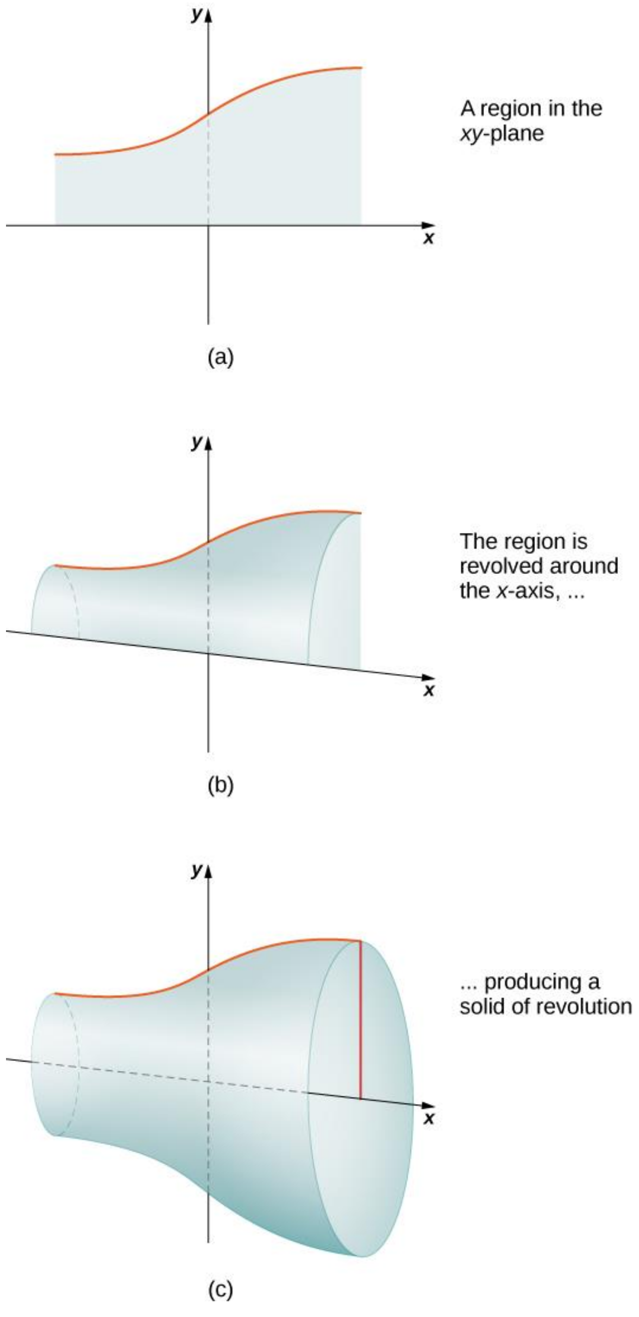
$$\frac{s}{a} = \frac{x}{h} \text{ or } s = \frac{ax}{h}.$$

Therefore, since the area of one of the cross-sectional squares is $A(x) = s^2 = (\frac{ax}{h})^2$, then we find the volume of the pyramid by integrating from 0 to h

$$V = \int_0^h A(x)\,dx = \int_0^h (\frac{ax}{h})^2\,dx = \frac{a^2}{h^2}\int_0^h x^2\,dx = [\frac{a^2}{h^2}(\frac{1}{3}x^3)]|_0^h = \frac{1}{3}a^2h.$$

2.3 Solids of Revolution

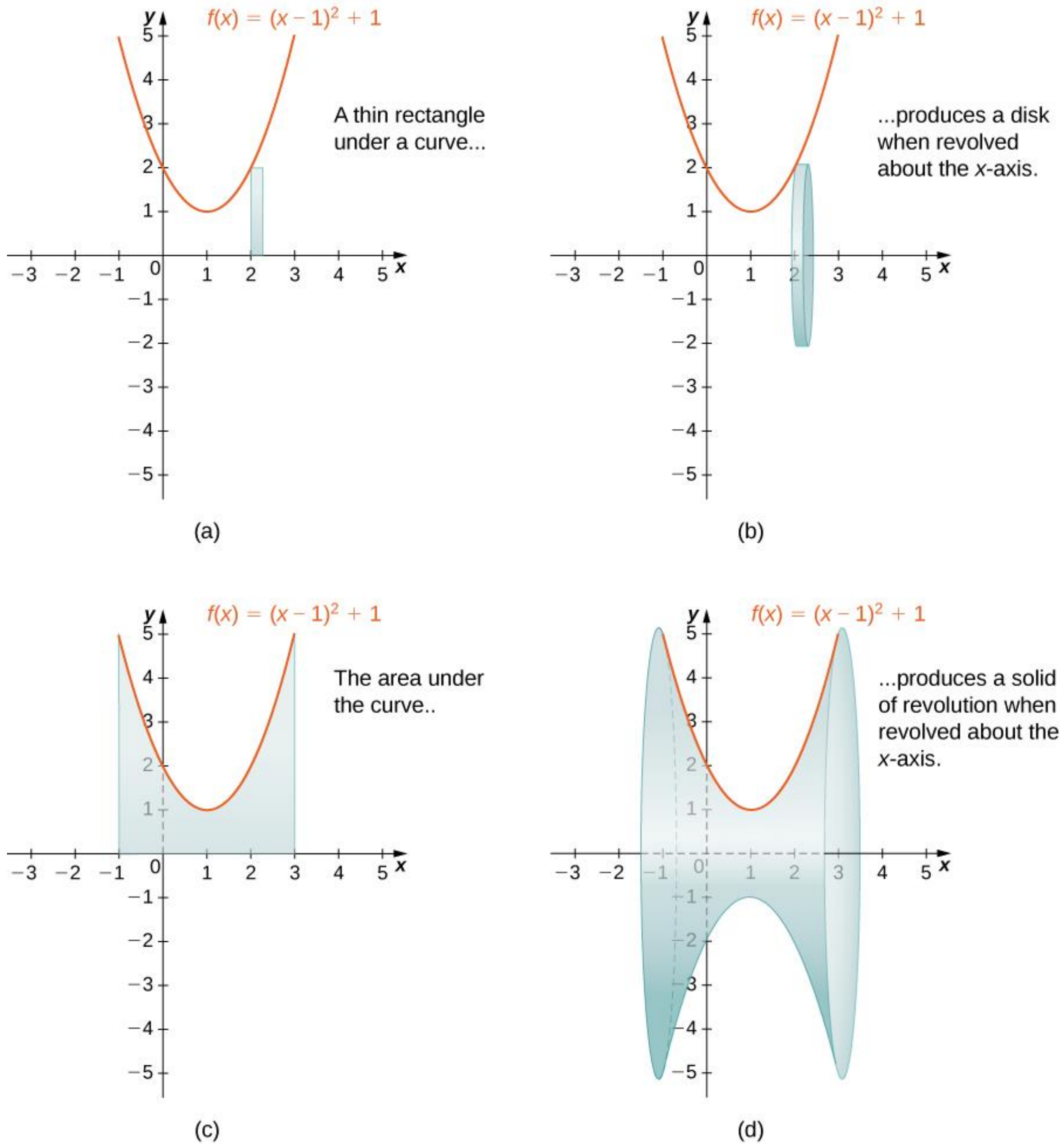
If a region in a plane is revolved around a line in that plane, the resulting solid is called a solid of revolution, as shown in the following Figure:



2.3.1 The Disk Method

The slicing method can be used to calculate the volume of a solid of revolution. When we use the slicing method with solids of revolution, it is often called the *disk method* because, for solids of revolution, the slices used to over approximate the volume of the solid are disks.

To see this, consider the solid of revolution generated by revolving the region between the graph of the function $f(x) = (x-1)^2 + 1$ and the x-axis over the interval $[-1,3]$ around the x-axis. The graph of the function and a representative disk are shown in Figure 4 (a) and (b). The region of revolution and the resulting solid are shown in Figure 4 (c) and (d):



We already used the formal Riemann sum development of the volume formula when we developed the slicing method. We know that $\int_a^b A(x)\,dx$. The only difference with the disk method is that we know the formula for the cross-sectional area ahead of time; it is the area of a circle. This gives the following rule:

Let $f(x)$ be continuous and nonnegative. Define R as the region bounded above by the graph of $f(x)$, below by the x-axis, on the left by the line $x = a$, and on the right by the line $x = b$. Then, the volume of the solid of revolution formed by revolving R around the x-axis is given by

$$V_{Ox} = \int_a^b \pi\,[f(x)]^2 dx.$$

***Example* 2**. The volume of the solid we have been studying (Figure 4) is given by

$$V = \int_a^b \pi\, [f(x)]^2 dx$$

$$= \int_{-1}^{3} \pi\, [(x-1)^2 + 1]^2 dx = \pi \int_{-1}^{3} [\,(x-1)^4 + 2(x-1)^2 + 1]^2 dx$$

$$= \pi[\frac{1}{5}(x-1)^5 + \frac{2}{3}(x-1)^3 + x]|_{-1}^{3}$$

$$= \pi[(\frac{32}{5} + \frac{16}{3} + 3) - (-\frac{32}{5} - \frac{16}{3} - 1)] = \frac{412\pi}{15} \text{ units}^3$$

So far, our examples have all concerned regions revolved around the x -axis, but we can generate a solid of revolution by revolving a plane region around any horizontal or vertical line. In the next example, we look at a solid of revolution that has been generated by revolving a region around the y -axis. The mechanics of the disk method are nearly the same as when the x -axis is the axis of revolution, but we express the function in terms of y and we integrate with respect to y as well. This is summarized in the following rule:

Let $g(y)$ be continuous and nonnegative. Define Q as the region bounded on the right by the graph of $g(y)$, on the left by the y -axis, below by the line $y = c$, and above by the line $y = d$. Then, the volume of the solid of revolution formed by revolving Q around the y -axis is given by

$$V_{oy} = \int_c^d \pi\, [g(y)]^2 dy.$$

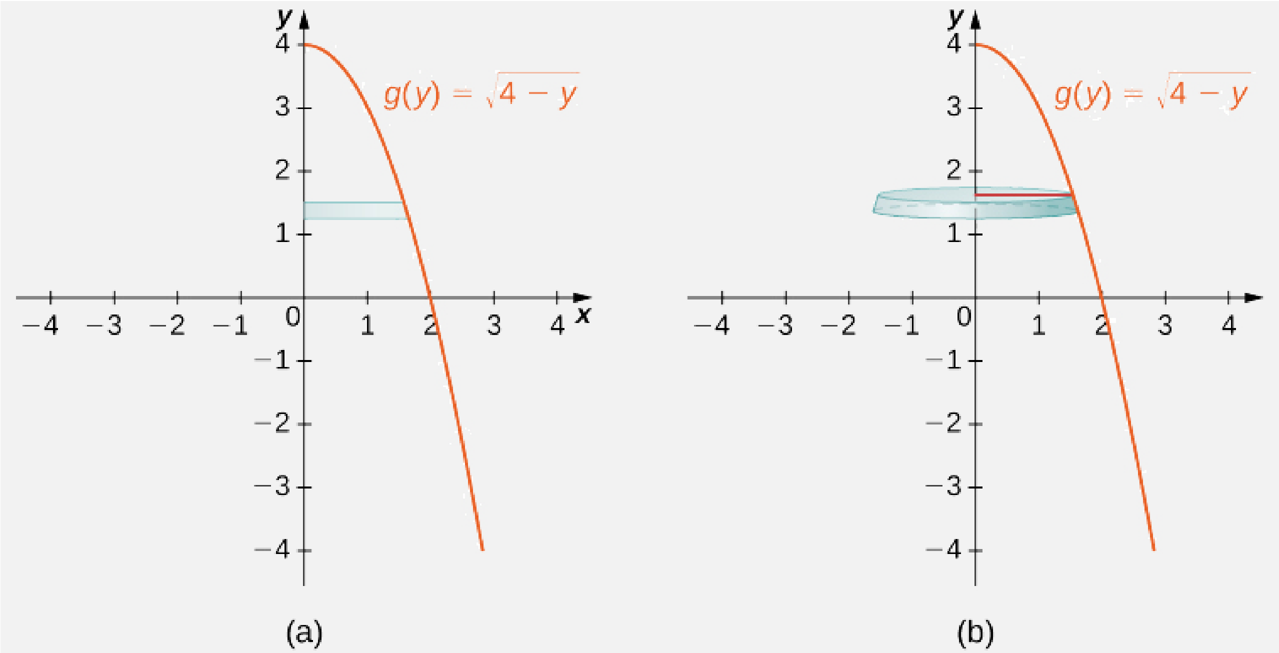
***Example* 3.** Let R be the region bounded by the graph of $g(y) = \sqrt{4-y}$ and the y-axis over the y-axis interval $[0,4]$. Use the disk method to find the volume of the solid of revolution generated by rotating R around the y -axis.

Solution:

Figure 5 shows the function and a representative disk that can be used to estimate the volume. Notice that since we are revolving the function around the y -axis, the disks are horizontal, rather than vertical.

To find the volume, we integrate with respect to y . We obtain

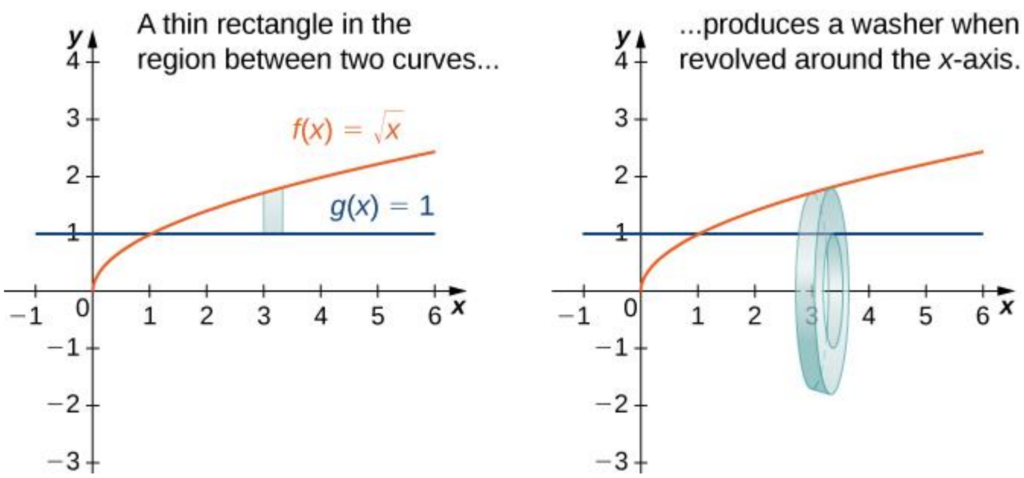
$$V = \int_c^d \pi\, [g(y)]^2 dy = \int_0^4 \pi\, [\sqrt{4-y}]^2 dy = \pi \int_0^4 (4-y)\, dy = \pi[4y - \frac{y^2}{2}]|_0^4 = 8\pi.$$



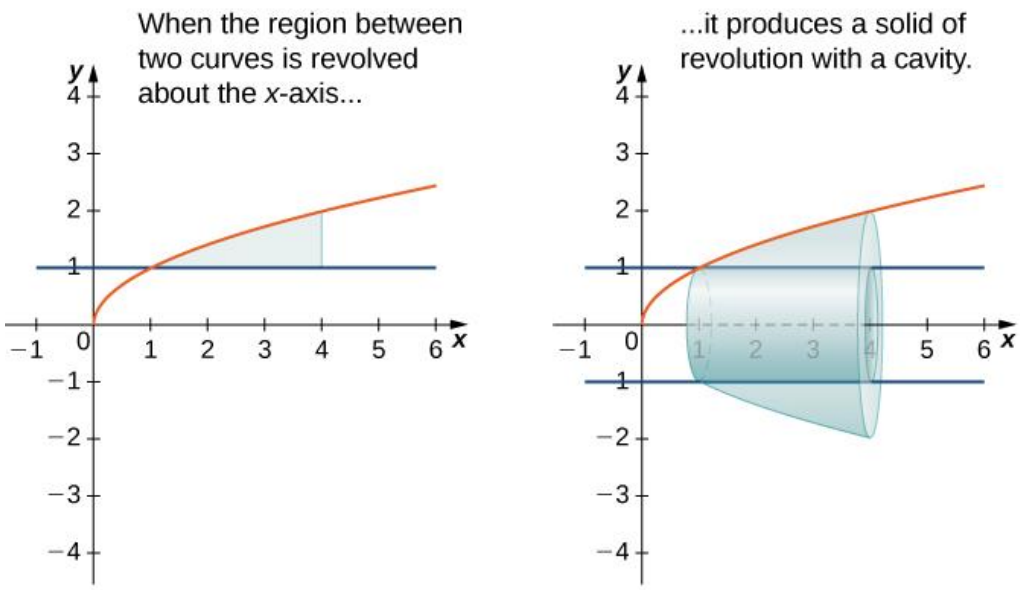
2.3.2 The Washer Method

Some solids of revolution have cavities in the middle; they are not solid all the way to the axis of revolution. Sometimes, this is just a result of the way the region of revolution is shaped with respect to the axis of revolution. In other cases, cavities arise when the region of revolution is defined as the region between the graphs of two functions. A third way this can happen is when an axis of revolution other than the x -axis or y -axis is selected.

When the solid of revolution has a cavity in the middle, the slices used to approximate the volume are not disks, but *washers* (disks with holes in the center). For example, consider the region bounded above by the graph of the function $f(x) = \sqrt{x}$ and below by the graph of the function $g(x) = 1$ over the interval $[1,4]$. When this region is revolved around the x -axis, the result is a solid with a cavity in the middle, and the slices are washers. The graph of the function and a representative washer are shown in Figure 6 (a) and (b). The region of revolution and the resulting solid are shown in Figure 6 (c) and (d).



(*a*) (*b*)



(c) (d)

The cross-sectional area, then, is the area of the outer circle less the area of the inner circle. In this case,

$$A(x) = \pi(\sqrt{x})^2 - \pi(1)^2 = \pi(x-1).$$

Then the volume of the solid is

$$V = \int_a^b A(x)\,dx = \int_1^4 \pi\,(x-1)dx = \pi[\frac{x^2}{2} - x]|_1^4 = \frac{9}{2}\pi \text{ units}^3.$$

Generalizing this process gives the *Washer Method*:

Suppose $f(x)$ and $g(x)$ are continuous, nonnegative functions such that $f(x) \geq g(x)$ over $[a, b]$. Let *R* denote the region bounded above by the graph of $f(x)$, below by the graph of $g(x)$, on the left by the line $x = a$, and on the right by the line $x = b$. Then, the volume of the solid of revolution formed by revolving *R* around the *x*-axis is given by

$$V_{Ox} = \int_a^b \pi\,[(f(x))^2 - (g(x))^2]dx.$$

As with the disk method, we can also apply the washer method to solids of revolution that result from revolving a region around the *y* -axis. In this case, the following rule applies.

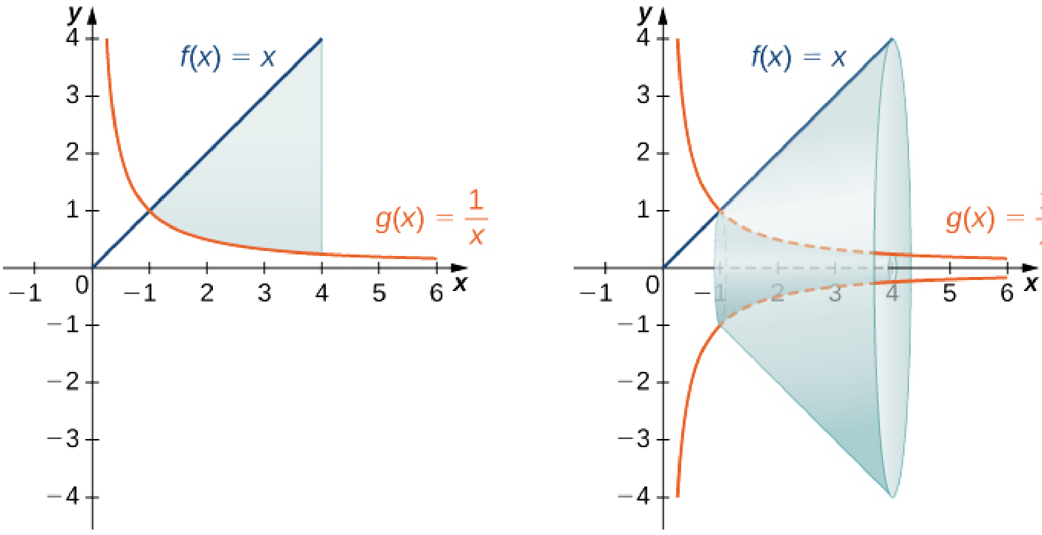
Suppose $u(y)$ and $v(y)$ are continuous, nonnegative functions such that $v(y) \leq u(y)$ for $y \in [c, d]$. Let *Q* denote the region bounded on the right by the graph of $u(y)$, on the left by the graph of $v(y)$, below by the line $y = c$, and above by the line $y = d$. Then, the volume of the solid of revolution formed by revolving *Q* around the *y* -axis is given by

$$V_{Oy} = \int_c^d \pi\,[(u(y))^2 - (v(y))^2]dy.$$

***Example* 4**. Find the volume of a solid of revolution formed by revolving the region bounded above by the graph of $f(x) = x$ and below by the graph of $g(x) = \frac{1}{x}$ over the interval $[1,4]$ around the x-axis.

Solution:

The graphs of the functions and the solid of revolution are shown in the following figure.



(a) (b)

Then, we have

$$V = \int_a^b \pi\,[(f(x))^2 - (g(x))^2]dx = \pi \int_1^4 [x^2 - (\frac{1}{x})^2]\,dx$$

$$= \pi[\frac{x^3}{3} + \frac{1}{x}]|_1^4 = \frac{81\pi}{4}\text{units}^3.$$

***Example* 5**. Find the volume of a solid of revolution formed by revolving the region bounded above by $f(x) = 4 - x$ and below by the x-axis over the interval $[0,4]$ around the line $y = -2$.

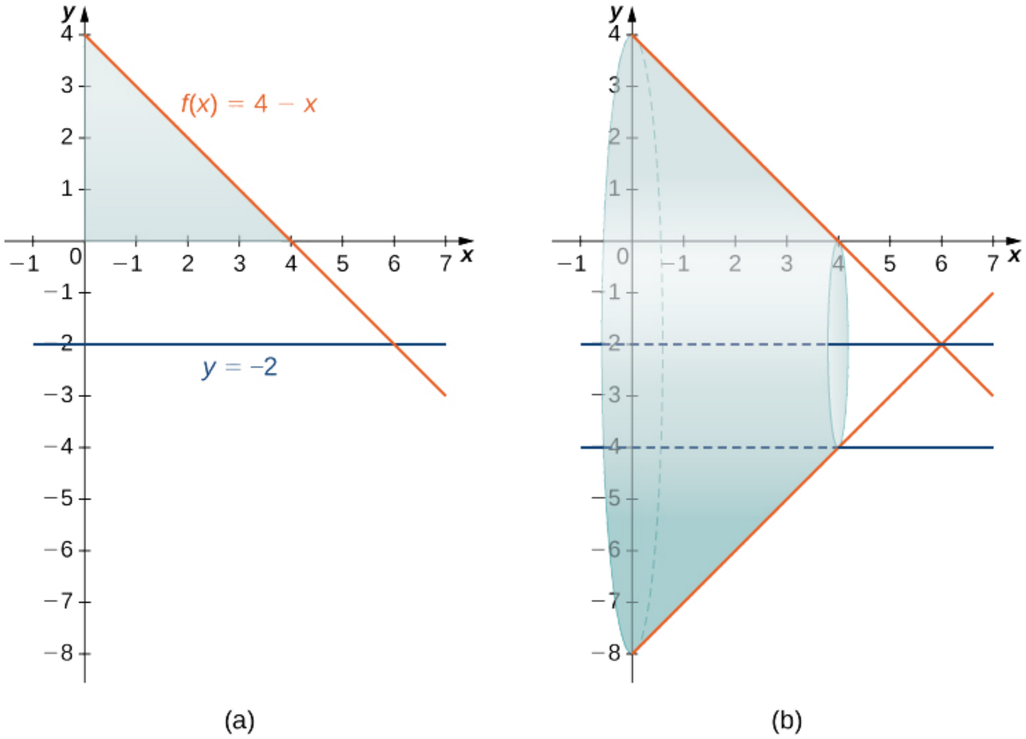
Solution:

The graph of the region and the solid of revolution are shown in the following figure.

We can’t apply the volume formula to this problem directly because the axis of revolution is not one of the coordinate axes. However, we still know that the area of the cross-section is the area of the outer circle less the area of the inner circle. Looking at the graph of the function, we see the radius of the outer circle is given by $f(x) + 2$, which simplifies to

$$f(x) + 2 = (4 - x) + 2 = 6 - x.$$

The radius of the inner circle is g(x)=2. Therefore, we have



$$V = \int_0^4 \pi\,[(6-x)^2 - (2)^2]dx$$

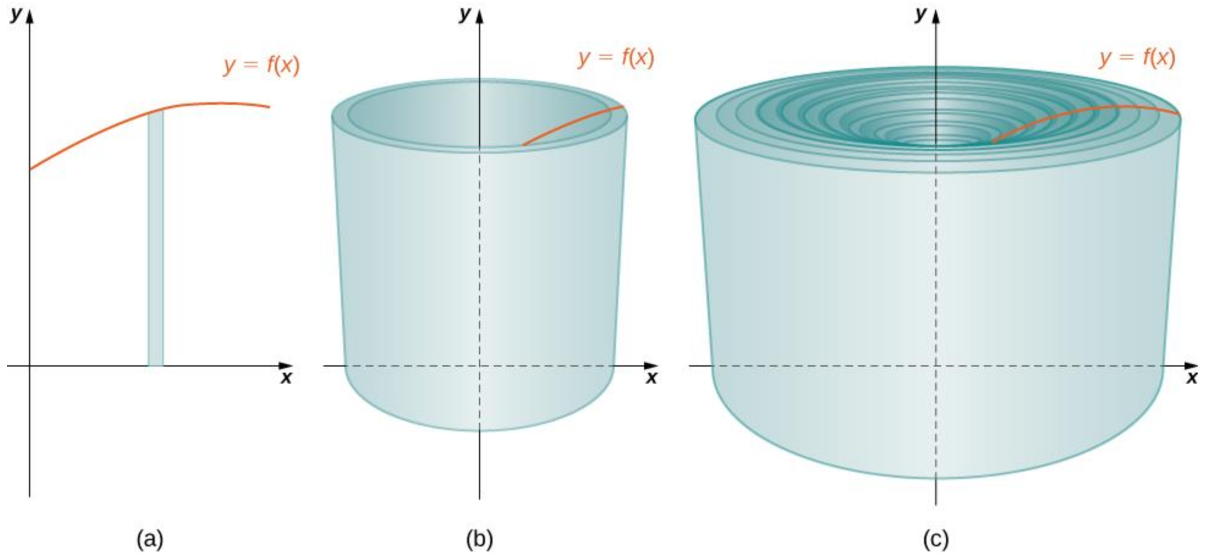
$$= \pi\int_0^4 (x^2 - 12x + 32)\,dx = \pi[\frac{x^3}{3} - 6x^2 + 32x]|_0^4 = \frac{160\pi}{3}\text{units}^3.$$

2.3.3 The Method of Cylindrical Shells

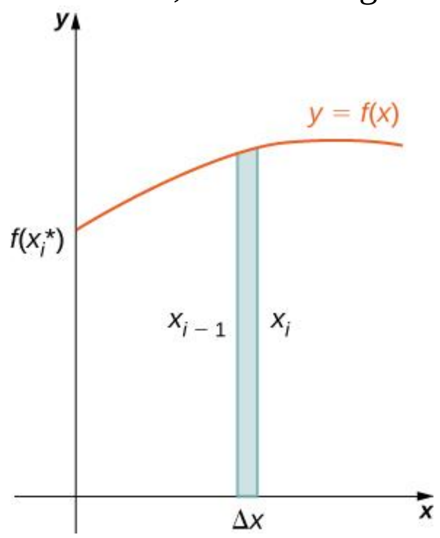
In this section, we examine the method of cylindrical shells, the final method for finding the volume of a solid of revolution. We can use this method on the same kinds of solids as the disk method or the washer method; however, with the disk and washer methods, we integrate along the coordinate *axis parallel to the axis of revolution*. With the method of cylindrical shells, we integrate along the coordinate *axis perpendicular to the axis of revolution*.

As before, we define a region R , bounded above by the graph of a function $y = f(x)$, below by the *x*-axis, and on the left and right by the lines $x = a$ and $x = b$, respectively, as shown in Figure 7a. We then revolve this region around the *y* -axis, as shown in Figure 7c. Note that this is different from what we have done before. Previously, regions defined in terms of functions of *x* were revolved around the *x*-axis or a line parallel to it.

As we have done many times before, partition the interval $[a, b]$ using a regular partition, $P = x_0, x_1, \dots, x_n$ and, for $i = 1,2, \dots, n$, choose a point $x_i^* \in [x_{i-1}, x_i]$. Then, construct a rectangle over the interval $[x_{i-1}, x_i]$ of height $f(x_i^*)$ and width Δx. A representative rectangle is shown in Figure 7c. When that rectangle is revolved around the *y*-axis, instead of a disk or a washer, we get a cylindrical shell, as shown in Fig. 7b.



To calculate the volume of this shell, consider Figure 7d.



Then the volume of the shell is

$$V_{shell} = f(x_i^*)(\pi x_i^2 - \pi x_{i-1}^2) = \pi f(x_i^*)(x_i^2 - x_{i-1}^2)$$
$$= \pi f(x_i^*)(x_i + x_{i-1})(x_i - x_{i-1}) = 2\pi f(x_i^*)(\frac{x_i + x_{i-1}}{2})(x_i - x_{i-1}).$$

Here, $\frac{x_i+x_{i-1}}{2}$ is both the midpoint of the interval $[x_{i-1}, x_i]$ and the average radius of the shell, and we can approximate this by x_i^* . We then have

$$V_{shell} \approx 2\pi f(x_i^*) x_i^* \Delta x.$$

To calculate the volume of the entire solid, we then add the volumes of all the shells and obtain

$$V \approx \sum_{i=1}^{n} (2\pi x_i^* f(x_i^*)\, \Delta x).$$

Here we have another Riemann sum, this time for the function $2\pi x f(x)$. Taking the limit as $n \to \infty$ gives us

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} (2\pi x_i^* f(x_i^*)\,\Delta x) = \int_a^b (2\pi x f(x))\,dx.$$

This leads to the following rule for the method of cylindrical shells.

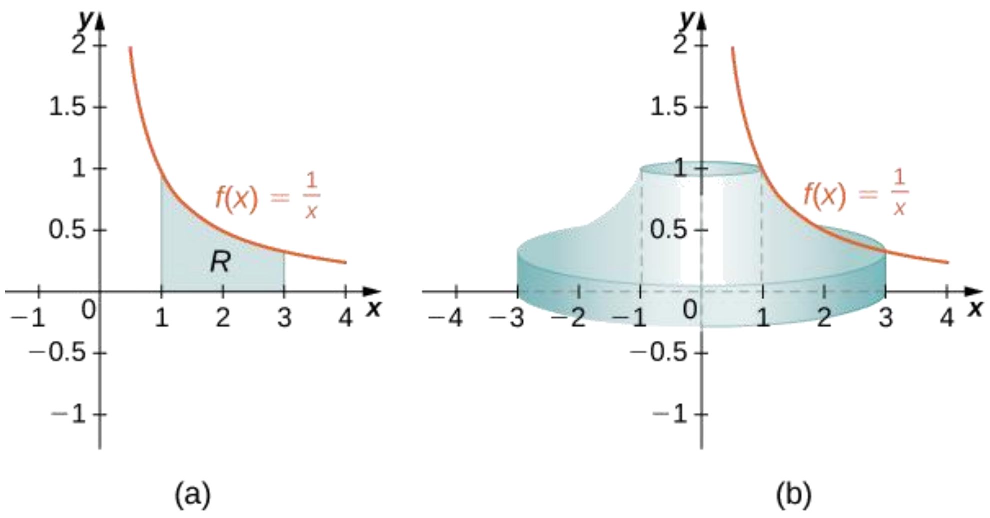
Let $f(x)$ be continuous and nonnegative. Define R as the region bounded above by the graph of *f(x)*, below by the x -axis, on the left by the line $x=a$, and on the right by the line $x=b$. Then the volume of the solid of revolution formed by revolving R around the y -axis is given by

$$V_{Oy} = \int_a^b (2\pi x f(x))\,dx.$$

***Example* 6**. Define R as the region bounded above by the graph of $f(x) = \frac{1}{x}$ and below by the x -axis over the interval $[1,3]$. Find the volume of the solid of revolution formed by revolving R around the y -axis.

Solution

First we must graph the region R and the associated solid of revolution, as shown in Figure 8.



Then the volume of the solid is given by

$$V = \int_a^b (2\pi x f(x))\,dx = \int_1^3 \left(2\pi x \left(\frac{1}{x}\right)\right) dx = \int_1^3 2\pi\,dx = 2\pi x|_1^3 = 4\pi \text{ units}^3.$$

As with the disk method and the washer method, we can use the method of cylindrical shells with solids of revolution, revolved around the x -axis, when we want to integrate with respect to y. The analogous rule for this type of solid is given here.

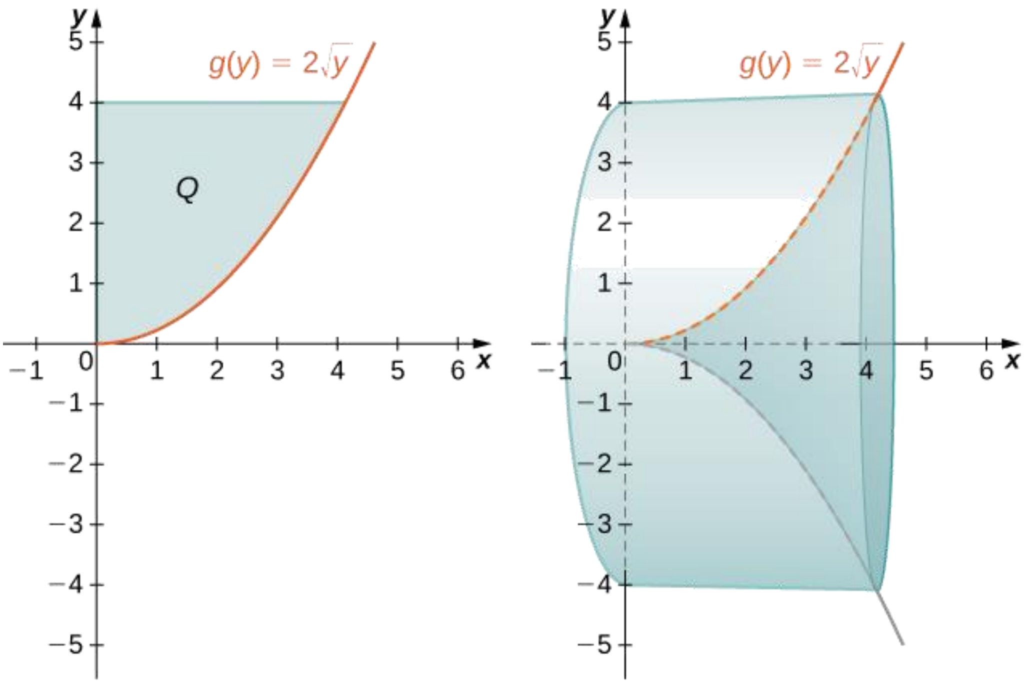
Let $g(y)$ be continuous and nonnegative. Define Q as the region bounded on the right by the graph of $g(y)$, on the left by the y-axis, below by the line $y = c$, and above by the line $y = d$. Then, the volume of the solid of revolution formed by revolving Q around the x-axis is given by

$$V_{Ox} = \int_c^d (2\pi y g(y))\, dy.$$

***Example* 7**. Define Q as the region bounded on the right by the graph of $g(y) = 2\sqrt{y}$ and on the left by the y-axis for $y \in [0,4]$. Find the volume of the solid of revolution formed by revolving Q around the x-axis.

Solution

First, we need to graph the region Q and the associated solid of revolution, as shown in Figure 8.



(a)

(b)

Then the volume of the solid is given by

$$V = \int_c^d (2\pi y g(y))\, dy = \int_0^4 (2\pi y(2\sqrt{y}))\, dy = 4\pi \int_0^4 y^{3/2}\, dy$$
$$= 4\pi[\frac{2y^{5/2}}{5}]\,|_0^4 = \frac{256\pi}{5}\,\text{units}^3$$

Note that the axis of revolution is the y-axis, so the radius of a shell is given simply by x. We don't need to make any adjustments to the x-term of our integrand.

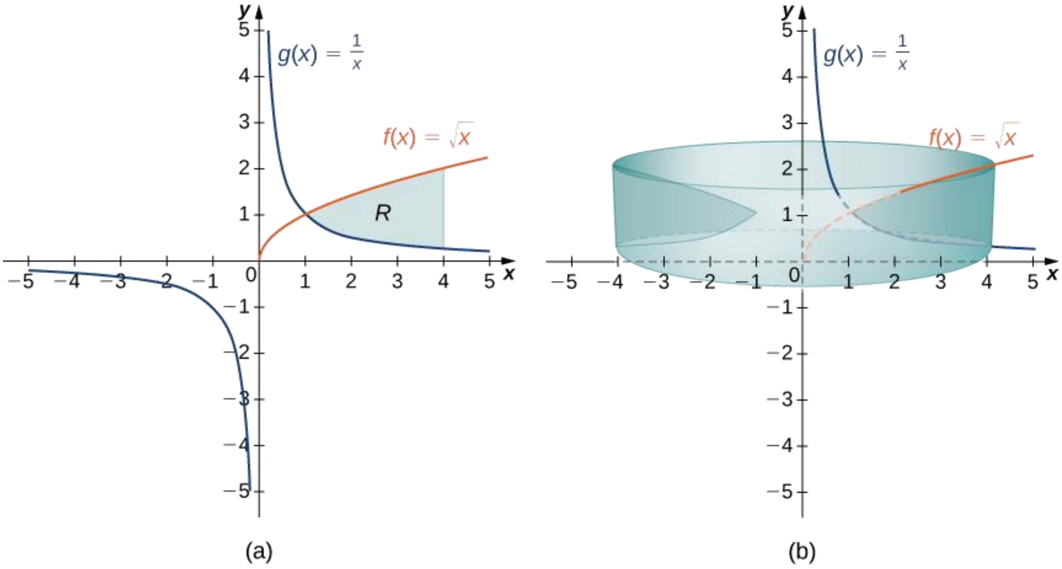
The height of a shell, though, is given by $f(x) - g(x)$, so in this case we need to adjust the $f(x)$ term of the integrand. Then the volume of the solid is given by

$$V = \int_1^4 (2\pi x(f(x) - g(x)))\,dx$$

***Example* 8**. Define R as the region bounded above by the graph of the function $f(x) = \sqrt{x}$ and below by the graph of the function $g(x) = \frac{1}{x}$ over the interval [1,4]. Find the volume of the solid of revolution generated by revolving R around the y-axis.

Solution:

First, graph the region R and the associated solid of revolution, as shown in Figure 9.



Then the volume of the solid is given by

$$V = \int_1^4 \left(2\pi x\left(\sqrt{x} - \frac{1}{x}\right)\right) dx = 2\pi \int_1^4 (x^{3/2} - 1)\,dx$$

$$= 2\pi\left[\frac{2x^{5/2}}{5} - x\right]\Big|_1^4 = \frac{94\pi}{5}\,\text{units}^3.$$