LECTURE 5
FUNDAMENTALS OF THE DIFFERENTIAL CALCULUS FOR ONE VARIABLE FUNCTIONS

4.1. Derivative of a Function and its Geometric Sense

Let us consider a function $y = f(x)$ defined on a given interval. We denote as $x_0 \in D_f$ the fixed value of the argument *x*. Let the argument *x* receive the certain increment Δ*x*, such that $[x_0, x_0 + \Delta x] \in D_f$ (note, that it is immaterial whether increment is positive or negative). Then the function *y* will receive certain increment Δy. Let us find the increment in the function Δy:

$$\Delta y = f(x_0 + \Delta x) - f(x_0).$$

Forming the ratio of the increment in the function to the increment in the argument, we get $\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$. Let us find the limit of this ratio as Δ*x*→0. If this limit $\left(\lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \right)$ exists, *it is called the derivative of the given function* $y = f(x)$ with respect to *x* at the point x_0 and is denoted by $f'(x_0)$. The function $f(x)$ in this case is called *differentiable* at the point x_0. So, by definition

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = y' . \qquad (4.1)$$

Definition. The *derivative* of the given function $y = f(x)$ with respect to the argument *x* is limit of the ratio of the increment in the function Δy to the increment in the argument Δ*x*, when the latter approaches zero in arbitrary fashion.

Example 1. Let us find the derivative of exponential function $y = a^x$.

$$f'(x_0)=\lim_{\Delta x\to 0}\frac{a^{x_0+\Delta x}-a^{x_0}}{\Delta x}=a^{x_0}\lim_{\Delta x\to 0}\frac{a^{\Delta x}-1}{\Delta x}=a^{x_0}\ln a. \quad (4.2)$$

Example 3. Let function $y=\ln x$ be given. Then

$$y'=\lim_{\Delta x\to 0}\frac{\ln(x+\Delta x)-\ln(x)}{\Delta x}=\lim_{\Delta x\to 0}\frac{\ln\dfrac{(x+\Delta x)}{x}}{\Delta x}=\lim_{\Delta x\to 0}\frac{\ln\left(1+\dfrac{\Delta x}{x}\right)}{\Delta x},$$

since $\ln\left(1+\dfrac{\Delta x}{x}\right)\sim\dfrac{\Delta x}{x}$, as $\Delta x\to 0$, then we obtain

$$y'=\lim_{\Delta x\to 0}\frac{\dfrac{\Delta x}{x}}{\Delta x},$$

or

$$\boxed{(\ln x)'=\frac{1}{x}}. \quad (4.5)$$

Note. If $y=\log_a x$, then $y'=\dfrac{1}{x\ln a}$ or

$$\boxed{(\log_a x)'=\frac{1}{x\ln a}}. \quad (4.6)$$

Example 5. Derivatives of the Power Function

Let us consider the power function $y=x^a$, where a is any real number. Then

$$y'=\lim_{\Delta x\to 0}\frac{(x+\Delta x)^a-x^a}{\Delta x}=\lim_{\Delta x\to 0}\frac{\left(1+\dfrac{\Delta x}{x}\right)^a-1}{\Delta x}x^a=\lim_{\Delta x\to 0}a\frac{\Delta x}{x}\frac{x^a}{\Delta x}=ax^{a-1}.$$

Using consequence of the second remarkable limit we get finally

$$\boxed{y'=ax^{a-1}}.$$

Note 1. If $y=x$, then $y'=\left(x^1\right)'=1\cdot x^0=1$. So $(x)'=1$

or $y' = 1$.

Note 2. Let function $y = \sqrt{x}$ be given. Then

$$y' = \left(x^{1/2}\right)' = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}.$$

Note 3. If $y = \frac{1}{x}$, then $y' = \left(x^{-1}\right)' = -x^{-2} = -\frac{1}{x^2}$.

It should be noted that designation $f'(x)$ is not only for designation of the derivative. Alternative symbols are

$$y',\ y'_x,\ \frac{dy}{dx}.$$

The operation of finding the derivative of a function is called differentiation of the function.

4.3. The Basic Rules for Finding Derivatives

Theorem 2. The derivative of a constant is equal to zero.

■ Let $y = C$, where $C = const$. Then for any increament in argument by Δx corresponding increament in the function will be equal to zero, that is $\Delta y = 0$, so

$$y' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 0,\ (C)' = 0. \square \tag{4.11}$$

Theorem 3. The derivative of the sum of a finite number of differentiable functions is equal to the corresponding sum of the derivatives of these functions.

■ Let us prove this theorem for two functions. Consider the sum $y = u(x) + v(x)$. Then

$$y' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta u + \Delta v}{\Delta x} = \lim_{\Delta x \to 0}\left(\frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x}\right) = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} + \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x}.$$

It means

$$y' = u' + v'$$

or

$$(u+v)' = u' + v'. \tag{4.12}$$

Which is what had to be proved.□

Example. If $y = x^4 + \frac{1}{\sqrt[3]{x}} - 2$.

Then

$$y' = 4x^3 - \frac{1}{3}x^{-4/3} + 0 = 4x^3 - \frac{1}{3x\sqrt[3]{x}}.$$

Theorem 4. The derivative of a product of two differentiable functions is equal to the product of the derivative of the first function by the second one plus the product of the first function by the derivative of the second function, that is, if

$$y = u(x)v(x),$$

then

$$y' = u'v + v'u.$$

or

$$(uv)' = u'v + v'u. \square \tag{4.13}$$

Corollary. A constant factor may be taken outside the derivative sign, i.e. if $y = Cf(x)$, then $y' = Cf'(x)$ or

$$(Cf(x))' = Cf'(x). \tag{4.14}$$

Example. Consider the function $y = (3x^2 + 4x)\left(\frac{5}{x} + \sqrt[3]{x}\right)$. Then

$$y' = (6x+4)\left(\frac{5}{x} + \sqrt[3]{x}\right) + (3x^2 + 4x)\left(-\frac{5}{x^2} + \frac{1}{3\sqrt[3]{x^2}}\right).$$

Theorem 5. The derivative of a fraction is equal to a fraction whose denominator is the square of the denominator of the given fraction and the numerator is the difference between the product of the denominator by the derivative of the numerator, and the product of the numerator by the derivative of the denominator, i.e., if

$$y = \frac{u(x)}{v(x)},$$

then

$$y' = \frac{u'v - uv'}{v^2}.$$

or

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}. \square \qquad (4.15)$$

Example. Let the function $y = \frac{x^2}{x^3 + 4}$ be given. Then

$$y' = \frac{2x(x^3+4) - x^2 3x^2}{(x^3+4)^2} = \frac{8x - x^4}{(x^3+4)^2}.$$

Example. Let us consider the function $y = \tan x$ and $y = \cot x$. If $y = \tan x$

$$y' = \left(\frac{\sin x}{\cos x}\right)' = \frac{(\sin x)' \cos x - \sin x(\cos x)'}{\cos^2 x} = \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} =$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x},$$

that is

$$\boxed{(\tan x)' = \frac{1}{\cos^2 x}}. \qquad (4.16)$$

Analogously, if $y = \cot x$, then

$$y'=\left(\frac{\cos x}{\sin x}\right)'=\frac{-\sin x\sin x-\cos x\cos x}{\sin^2 x}=-\frac{\cos^2 x+\sin^2 x}{\sin^2 x}=-\frac{1}{\sin^2 x},$$

that is

$$\boxed{(\cot x)'=-\frac{1}{\sin^2 x}}. \qquad (4.17)$$

4.4. Derivative of the Inverse Function

Theorem 6. If a function $y=f(x)$ is continuous and strictly monotonic on some interval E and there exists the derivative $y_0'=f'(x_0)\neq 0$ at some point $x_0\in E$, then the inverse function $x=\varphi(y)$ has the derivative $x_0'=\varphi'(y_0)$ at the corresponding point $y_0=f(x_0)$, which is defined as

$$\varphi'(y_0)=\frac{1}{f'(x_0)}. \qquad (4.18)$$

or

$$x_y'=\frac{1}{y_x'}. \qquad (4.19)$$

4.5. Derivatives of the Inverse Trigonometric Functions

Let us find the derivative of the function $y=\arcsin x$. Determine the inverse function to the given one $x=\sin y$. Then

$$y_x'=\frac{1}{x_y'}=\frac{1}{\cos y}=\frac{1}{\sqrt{1-\sin^2 y}}=\frac{1}{\sqrt{1-x^2}}.$$

In the given case $\cos y=+\sqrt{1-\sin^2 y}$, because the function

$y = \arcsin x$ has the range $\left(-\frac{\pi}{2}; \frac{\pi}{2}\right)$ (Fig.4.2). So

$$\boxed{(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}}. \tag{4.20}$$

Analogously we can find the derivative of the function $y = \arccos x$. However it is simply if take into account the following relation

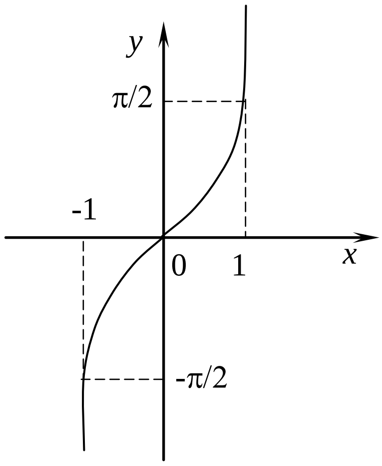

Fig. 4.2

$$\arcsin x + \arccos x = \frac{\pi}{2},$$

hence

$$\arccos x = \frac{\pi}{2} - \arcsin x,$$

then

$$(\arccos x)' = \left(\frac{\pi}{2}\right)' - (\arcsin x)' = -\frac{1}{\sqrt{1-x^2}}.$$

So

$$\boxed{(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}}. \tag{4.21}$$

Let us find the derivative of the function $y = \arctan x$. From this it follows that $x = \tan y$, consequently

$$y'_x = \frac{1}{x'_y} = \frac{1}{(\tan y)'} = \frac{1}{\frac{1}{\cos^2 y}} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}.$$

So

$$\boxed{(\arctan x)' = \frac{1}{1+x^2}}. \tag{4.22}$$

By virtue of identity $\arctan x + \operatorname{arccot} x = \frac{\pi}{2}$, we get

$$\boxed{(\operatorname{arc}\cot x)' = -\frac{1}{1+x^2}}. \tag{4.23}$$

4.7. The Table of the Basic Formulas and Rules of Differentiation

Below we present the table of differentiation rules and table with derivatives of the basic elementary functions.

Table 4.1.

Rules of Differentiation

$(Const)' \equiv 0$	$(u \cdot v)' = u'v + uv'$
$(u+v)' = u' + v'$	$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$

Table 4.2.

Basic Formulas of Differentiation

$(x^a)' = ax^{a-1}$	$(\sin x)' = \cos x$	$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$	$(\operatorname{sh} x)' = \operatorname{ch} x$
$(a^x)' = a^x \ln a$ $(e^x)' = e^x$	$(\cos x)' = -\sin x$	$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$	$(\operatorname{ch} x)' = \operatorname{sh} x$
$(\log_a x)' = \frac{1}{x \cdot \ln a}$	$(\tan x)' = \frac{1}{\cos^2 x}$	$(\arctan x)' = \frac{1}{1+x^2}$	$(\operatorname{th} x)' = \frac{1}{\operatorname{ch}^2 x}$
$(\ln x)' = \frac{1}{x}$	$(\cot x)' = -\frac{1}{\sin^2 x}$	$(\operatorname{arccot} x)' = -\frac{1}{1+x^2}$	$(\operatorname{cth} x)' = -\frac{1}{\operatorname{sh}^2 x}$

4.8. Derivative of the Composite Function

Theorem 4.7. If a function $u=\varphi(x)$ is differentiable at the point x_0, and a function $y=f(u)$ is differentiable at the point u_0, where $u_0=\varphi(x_0)$, then the composite function $F(x)=f[\varphi(x)]$ will be differentiable at the point x_0 and its derivative is calculated by the following formula

$$F'(x_0)=f'(u_0)\varphi'(x_0).$$

$$y'_x = y'_u u'_x.$$

So the derivative of the composite function is equal to the product of the derivative of this function with respect to intermediate argument by derivative of the intermediate argument with respect to independent variable.

Example 1. Calculate the derivative of the function $y=\sin 5x$. It means that $y=\sin u$, where $u=5x$ and due to formula (4.25) we obtain

$$y'=(\sin u)'_u(5x)'_x=\cos u\cdot 5=5\cos 5x.$$

Example 2. Let $y=3^{\tan x}$ the function be given. It means that $y=3^u$, where $u=\tan x$, thus

$$y'=3^u\ln 3\frac{1}{\cos^2 x}=3^{\tan x}\cdot\ln 3\cdot\frac{1}{\cos^2 x}.$$

Example 3. Consider the function $y=\left(x^3+2\right)^{100}$. It means that $y=u^{100}$, where $u=x^3+2$. So we obtain $y'=100\cdot u^{99}\cdot 3x^2=300\cdot\left(x^3+2\right)^{99}x^2$.

Example 4. Let the function $y=e^{\sqrt{\arcsin x}}$ be given.

Denote. $y=e^u$, where $u=\sqrt{v}$, $v=\arcsin x$.

$$y'=e^u\frac{1}{2\sqrt{v}}\cdot\frac{1}{\sqrt{1-x^2}}=\frac{e^{\sqrt{\arcsin x}}}{2\sqrt{\arcsin x}\sqrt{1-x^2}}.$$

Example 5. Consider the function $y=\mathrm{ch}\left[\ln^2(1-x)\right]$. Then its derivative is defined as

$$y'=\mathrm{sh}\left[\ln^2(1-x)\right]\cdot 2\cdot\ln(1-x)\cdot\frac{1}{1-x}(-1)=$$
$$=-\frac{2}{1-x}\cdot\ln(1-x)\cdot\mathrm{sh}\left[\ln^2(1-x)\right].$$

Table 4.3.

Differentiation of the Composite Function $y=f(u)$, where $u=u(x)$

1. $\left(u^\alpha\right)'=\alpha\cdot u^{\alpha-1}\cdot u'_x$	10. $(\cot u)'=-\frac{1}{\sin^2 u}\cdot u'_x$
2. $\left(a^u\right)'=a^u\cdot\ln a\cdot u'_x$	11. $(\arcsin u)'=\frac{1}{\sqrt{1-u^2}}\cdot u'_x$
3. $\left(e^u\right)'=e^u\cdot u'_x$	12. $(\arccos u)'=-\frac{1}{\sqrt{1-u^2}}\cdot u'_x$
4. $(\log_a u)'=\frac{1}{u\cdot\ln a}\cdot u'$	13. $(\arctan u)'=\frac{1}{1+u^2}\cdot u'_x$
5. $(\ln u)'=\frac{1}{u}\cdot u'$	14. $(\mathrm{arccot}\, u)'=-\frac{1}{1+u^2}\cdot u'_x$
6. $\left(u(x)^{v(x)}\right)'=u^v\cdot\ln u\cdot v'+v\cdot u^{v-1}\cdot u'$	15. $(\mathrm{ch} u)'=-\mathrm{sh} u\cdot u'_x$

7. $(\sin u)' = \cos u \cdot u'_x$	16. $(\mathrm{sh} u)' = \mathrm{ch} u \cdot u'_x$
8. $(\cos u)' = -\sin u \cdot u'_x$	17. $(\mathrm{th} u)' = \dfrac{1}{\mathrm{ch}^2 u} \cdot u'_x$
9. $(\tan u)' = \dfrac{1}{\cos^2 u} \cdot u'_x$	18. $(\mathrm{cth} u)' = -\dfrac{1}{\mathrm{sh}^2 u} \cdot u'_x$

4.10. Logarithmic Differentiation

Let us find the derivative of the function

$$y = \frac{\sqrt[3]{1+x^2} \cdot 2^{\sin x}(\tan x - 1)^5}{x^3 \ln^2 x}.$$

Direct differentiation of this function is possible, but it is connected with difficulties, because here is a large number of the multipliers. There is more comfortable first to take the logarithm of the given function and then carry out differentiation. Indeed

$$\ln y = \frac{1}{3}\ln\left(1+x^2\right) + \sin x \cdot \ln 2 + 5\ln(\tan x - 1) - 3\ln x - 2\ln(\ln x).$$

Then

$$\frac{1}{y}y' = \frac{1}{3}\frac{2x}{1+x^2} + \cos x \cdot \ln 2 + \frac{5}{\tan x - 1} \cdot \frac{1}{\cos^2 x} - \frac{3}{x} - \frac{2}{\ln x} \cdot \frac{1}{x}.$$

whence

$$y' = \left[\frac{2x}{3\left(1+x^2\right)} + \cos x \cdot \ln 2 + \frac{5}{(\tan x - 1)\cos^2 x} - \frac{3}{x} - \frac{2}{x \ln x}\right] y,$$

i. e.

$$y' = \left[\frac{2x}{3(1+x^2)} + \cos x \cdot \ln 2 + \frac{5}{(\tan x - 1)\cos^2 x} - \frac{3}{x} - \frac{2}{x \ln x}\right] *$$

$$* \frac{\sqrt[3]{1+x^2} \cdot 2^{\sin x}(\tan x - 1)^5}{x^3 \ln^2 x}.$$

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This approach is called *logarithmic differentiation*. It is also applied to differentiation of the power-exponential function.

Let us consider the composite function $y = (u(x))^{v(x)}$, so called *power-exponential function*. To find the derivative of this function let us differentiate it

$$\ln y = v(x) \cdot \ln(u(x)).$$

Calculate derivative from the both parts

$$\frac{1}{y} \cdot y' = v' \ln u + v \cdot \frac{1}{u} \cdot u'.$$

From this it follows that

$$y' = u^v \left(v' \cdot \ln u + v \cdot \frac{1}{u} \cdot u'\right).$$

The obtained relation may be presented as

$$y' = u^v \cdot \ln u \cdot v' + v \cdot u^{v-1} \cdot u'. \qquad (4.25)$$

The last formula may be remembered easily if take into account, that the first term is derivative of the given function if it is considered as power composite function, the second term is derivative of the given function if it is considered as a exponential one.

Example 1. Consider the function $y = (\sin x)^{\cos x}$. Then

$$\ln y = \cos x \cdot \ln \sin x,$$

consequently

$$\frac{1}{y} \cdot y' = -\sin x \cdot \ln \sin x + \cos x \cdot \frac{1}{\sin x} \cdot \cos x,$$

whence

$$y' = \left(-\sin x \cdot \ln\sin x + \frac{\cos^2 x}{\sin x}\right) y = (\text{ctg}x \cdot \cos x - \sin x \cdot \ln\sin x) \cdot (\sin x)^{\cos x}$$

Example 2. Find derivative of the function $y = x^x$. Applying formula (4.25) we can obtain at once

$$y' = x^x \cdot \ln x + x \cdot x^{x-1} = x^x(\ln x + 1).$$

We can also apply the method of logarithmic differentiation. Then,

$$\ln y = x\ln x \qquad \Rightarrow \qquad \frac{1}{y}y' = \ln x + 1,$$

$$y' = x^x \cdot (\ln x + 1).$$

4.15. The Differential

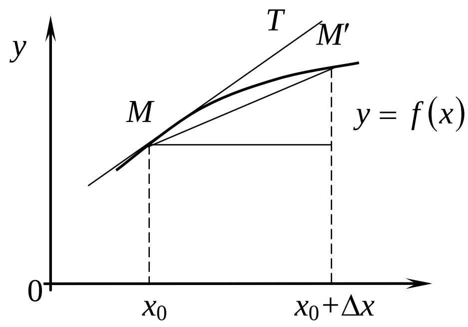

Fig. 4.1

Let a function $y = f(x)$ be differentiable on an interval (a,b). The derivative of this function at some point x of (a,b) is determined by the equation

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(x).$$

As $\Delta x \to 0$ the ratio $\frac{\Delta y}{\Delta x}$ approaches a definite number $f'(x)$ and, consequently, differs from the derivative $f'(x)$ by an infinitesimal:

$$\frac{\Delta y}{\Delta x} = f'(x) + \alpha,$$

where $\alpha \to 0$ as $\Delta x \to 0$.

Multiplying all terms by Δx we get

$$\Delta y = f'(x)\Delta x + \alpha \Delta x. \tag{4.36}$$

Since in the general case $f'(x) \neq 0$ for a constant x and variable $\Delta x \to 0$, the product $f'(x)\Delta x$ is an infinitesimal of the first order relative to Δx. But the product $\alpha \Delta x$ is always an infinitesimal of higher order than Δx because

$$\lim_{\Delta x \to 0} \frac{\alpha \Delta x}{\Delta x} = \lim_{\Delta x \to 0} \alpha = 0.$$

Thus, the increment Δy of the function consists of two terms. The first one is (when $f'(x) \neq 0$) the so-called *principal part* of the increment, and it is linear in Δx. The product $f'(x)\Delta x$ is called the *differential* of the function and is denoted by dy or $df(x)$.

And so if a function $y = f(x)$ has a derivative $f'(x)$ at the point x, the product of the derivative $f'(x)$ by the increment Δx in the argument is called the *differential of the function* and is denoted by the symbol dy:

$$\boxed{dy = f'(x)\Delta x}. \tag{4.37}$$

Find the differential of the function $y = x$; here,

$$y' = (x)' = 1,$$

and, consequently, $dy = dx = \Delta x$ or $dx = \Delta x$. Thus, *the differential dx of the independent variable x coincides with its increment Δx*. The equation $dx = \Delta x$ might be regarded likewise as a definition of the differential of the independent variable, and then the foregoing example would indicate that this does not contradict the definition of the differential of the function. In any case we can write formula (4.37) as

$$dy = f'(x)dx.$$

But from this relationship it follows that

$$f'(x) = \frac{dy}{dx}.$$

Hence, the derivative $f'(x)$ may be regarded as the ratio of the differential of the function to the differential of the independent variable.

Let us return to expression (4.36), which may be rewritten thus taking (4.37) into account

$$\Delta y = dy + \alpha \Delta x. \tag{4.38}$$

Thus, the increment of a function differs from the differential of a function by an infinitesimal of higher order than Δx. If $f'(x) \neq 0$, then $\alpha \Delta x$ is an infinitesimal of higher order that dy and

$$\lim_{\Delta x \to 0} \frac{\Delta y}{dy} = 1 + \lim_{\Delta x \to 0} \frac{\alpha \Delta x}{f'(x)\Delta x} = 1 + \lim_{\Delta x \to 0} \frac{\alpha}{f'(x)} = 1.$$

For this reason in approximate calculations one sometimes uses the approximate equation

$$\Delta y \approx dy \tag{4.39}$$

or in expanded form

$$f(x + \Delta x) - f(x) \approx f'(x)\Delta x. \tag{4.40}$$

Or

$$f(x + \Delta x) \approx f(x) + f'(x)\Delta x. \tag{4.41}$$

Example 1. Let $f(x) = \sin x$, then $f'(x) = \cos x$. In this case the approximate equation (4.41) takes the form

$$\sin(x + \Delta x) \approx \sin x + \cos x \Delta x.$$

Let us calculate the approximate value of $\sin 46^{\circ}$. Put $x = 45^{\circ} = \frac{\pi}{4}$, $\Delta x = 1^{\circ} = \frac{\pi}{180}$, $x + \Delta x = \frac{\pi}{4} + \frac{\pi}{180}$. Substituting into (4.41) we get

$$\sin 46^{\circ} \approx \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\frac{\pi}{180} \approx 0.7071 + 0.7071 \cdot 0.0175 = 0.7191.$$

Example 2. If in (4.41) we put $x = 0$, $\Delta x = \alpha$, we will get the following approximate equation:

$$\sin\alpha \approx \alpha.$$

Example 3. If $f(x) = \tan x$, then by (4.41) we get the following approximate equation:

$$\tan(x + \Delta x) \approx \tan x + \frac{1}{\cos^2 x}\Delta x,$$

for $x = 0$, $\Delta x = \alpha$ we get

$$\tan\alpha \approx \alpha.$$

Example 4. If $f(x) = \sqrt{x}$, then (4.41) yields

$$\sqrt{x + \Delta x} \approx \sqrt{x} + \frac{1}{2\sqrt{x}}\Delta x.$$

Putting $x = 1$, $\Delta x = \alpha$ we get the approximate equation

$$\sqrt{1 + \alpha} \approx 1 + \frac{1}{2}\alpha.$$

The problem of finding the differential of a function is equivalent to finding the derivative, since, multiplying the latter by the differential of the argument we get the differential of the function. Consequently, most of theorems and formulas pertaining to derivatives are also valid for differentials. Let us illustrate this.

The differential of the sum of two differentiable functions u *and* v *is equal to the sum of the differentials of these functions:*

$$d(u + v) = du + dv.$$

The differential of the product of two differentiable functions u *and* v *is determined by the formula:*

$$d(uv) = udv + vdu.$$

Let us prove the latter formula by way of illustration. If $y = uv$, then

$$y = f(x),$$

but

$$v'dx = dv,\ u'dx = du,$$

therefore

$$dy = udv + vdu.$$

Other formulas (for instance, the formula defining the differential of a quotient) are proved in similar fashion:

$$\text{if } y = \frac{u}{v}, \text{ then } dy = \frac{vdu - udv}{v^2}.$$

Let us solve some examples in calculating the differential of a function.

Example 5. $y = \tan^2 x,\ dy = 2\tan x \frac{1}{\cos^2 x} dx.$

Example 6. $y = \sqrt{1 + \ln x},\ dy = \frac{1}{2\sqrt{1 + \ln x}} \cdot \frac{1}{x} dx.$

Find the expression for the differential of a composite function. Let

$$y = f(u),\ u = \varphi(x) \text{ or } y = f(\varphi(x)).$$

Then by the rule for differentiating of the composite function

$$\frac{dy}{dx} = f_u'(u)\varphi'(x).$$

Hence,

$$dy = f_u'(u)\varphi'(x)dx,$$

but $\varphi'(x)dx = du$, therefore

$$dy = f'(u)du.$$

Thus, *the differential of a composite function has the same form as it would have if the intermediate argument is the independent* *variable.* In other words, *the form of the differential does not depend on whether the argument of the function is an independent variable or the function of another argument.* This important property of a differential, called *the preservation of the form of the differential*, will be widely used later on.

Example 7. Given a function $y = \sin\sqrt{x}$. Find dy.

Solution. Representing the given function as a composite one:

$$y = \sin u,\ u = \sqrt{x},$$

we find

$$dy = \cos u \frac{1}{2\sqrt{x}} dx,$$

but

$$\frac{1}{2\sqrt{x}} dx = du,$$

so we can write

$$dy = \cos u du$$

or

$$dy = \cos\left(\sqrt{x}\right) d\left(\sqrt{x}\right).$$

4.2. Connection Between Continuity and Differentiability of a Function

Let a function $y = f(x)$ be differentiable at a point x_0. It means that there exists the following limit

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

Whence it follows that the value $\dfrac{\Delta y}{\Delta x}$ differs from its limit by infinitesimal value, that is,

$$\frac{\Delta y}{\Delta x} = f'(x_0) + \alpha, \tag{4.9}$$

where $\lim\limits_{\Delta x \to 0} \alpha = 0$. So,

$$\Delta y = f'(x_0)\Delta x + \alpha \Delta x. \tag{4.10}$$

Obviously formula (4.10) is true as Δx approaches zero in any manner.

Theorem 1. If a function is differentiable at same point, then it is continuous at this point.

■ Indeed by virtue of theorem condition the value $f'(x_0)$ is finite number. But on base of equality (4.10), we get

$$\lim_{\Delta x \to 0} \Delta y = \lim_{\Delta x \to 0} \left[f'(x_0) + \alpha \right] \Delta x = 0,$$

which is what had to be proved.□

The contrary statement is not always true. For example, the function $\sqrt[3]{x}$ is continuous at the point $x = 0$, but it is not differentiable at this point (see example 4).

4.16. The Geometric Meaning of the Derivative and Differential

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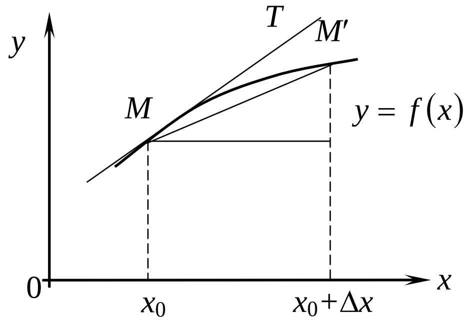

Fig. 4.1

Let us give geometric interpretation of the derivative. To do this we should first define a *tangent line* to a curve at a given point. For this purpose let us take a curve with fixed point M on it. Taking the point M' on the curve we draw the secant MM' (Fig. 4.1). If the point M' approaches the point M without limit, the secant MM' will occupy various positions. If in the unbounded approach of the point M' (along the curve) to the point M, the secant tends to occupy the position of a definite straight line MT, then this *line is called the tangent to the* curve at the point M.

Let us denote the point coordinates M as $(x_0, f(x_0))$. Increasing the argument x by Δx we obtain the value of the function $y_0 + \Delta y = f(x_0 + \Delta x)$ corresponding to the new value of the argument $x_0 + \Delta x$. The corresponding point on the curve we denote as $M'(x_0 + \Delta x, y_0 + \Delta y)$. Draw the secant MM' and denote by φ the angle formed by the secant and the positive x-axis. Form the ratio $\frac{\Delta y}{\Delta x}$. From Fig.4.1 it follows that $\frac{\Delta y}{\Delta x} = \tan\varphi$. It means that the ratio $\frac{\Delta y}{\Delta x}$ is a slope of the secant MM'. If $\Delta x \to 0$, then the point M' will move along the curve approaching point M. The secant will turn about point M and angle φ will be changed with changing Δx and approach the certain value of angle α, formed tangent line with the positive *x*-axis. So we have

$$\tan\alpha = \lim_{\Delta x\to 0}\tan\varphi = \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = f'(x_0).$$

Hence

$$f'(x) = \tan\alpha,$$

what means that *the value of the derivative $f'(x)$ for the given value of the argument x is equal to the tangent of the angle formed with the positive x-axis by the tangent line to the graph of the function $f(x)$ at the corresponding point $M(x,y)$*. Using the geometric sense of the derivative function we can write the equation of the tangent line to the graph of the function at any point with coordinates $(x_0, f(x_0))$. It is obvious that such equation has the following form

$$\boxed{y - y_0 = f'(x_0)(x - x_0)\,.} \tag{4.7}$$

Then the equation of the normal line to the given curve at the same point may be written as

$$\boxed{y - y_0 = -\frac{1}{f'(x_0)}(x - x_0)\,.} \tag{4.8}$$

Let us consider the function $y = f(x)$ and a curve of its graph in Fig. 4.10.

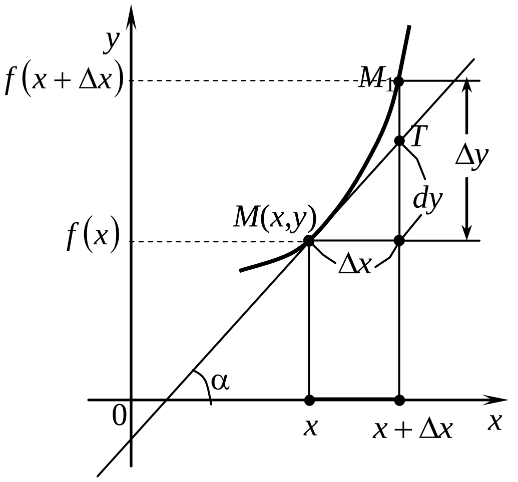

Fig. 4.10

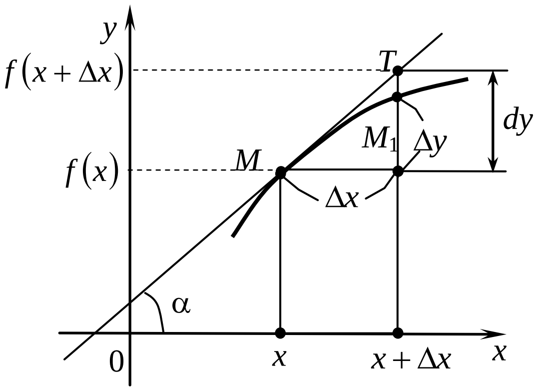

Fig. 4.11

Take an arbitrary point $M(x, y)$ on the curve $y = f(x)$, draw a tangent line to the curve at this point and denote by α the angle which the tangent line forms with the positive x-axis. So the function $f(x)$ has a finite derivative at the point x, we assume $\alpha \neq \frac{\pi}{2}$. Increase the independent variable by Δx; then the function will be changed by $\Delta y = NM_1$. To the values $x + \Delta x$, $y + \Delta y$ on the curve $y = f(x)$ there will correspond the point $M_1(x + \Delta x, y + \Delta y)$.

From the triangle MNT we find

$$NT = MN \tan\alpha .$$

Since

$$\tan\alpha = f'(x),\ MN = \Delta x,$$

we get

$$NT = f'(x)\Delta x .$$

But by the definition of a differential the relation $f'(x)\Delta x = dy$ is valid. Thus,

$$NT = dy .$$

The equation signifies that *the differential of a function $f(x)$, is equal to the increment in the ordinate of the tangent to the curve $y = f(x)$ at the given point x.*

From (Fig.4.10) it follows directly that

$$M_1T = \Delta y - dy.$$

By what has already been proved $\frac{M_1T}{NT} \to 0$ as $\Delta x \to 0$.

It should be nosed that the increment Δy is not always greater than dy. For instance, in Fig. 4.11

$$\Delta y = M_1N,\ dy = NT \text{ and } \Delta y < dy.$$

4.11. Geometric and Physical Applications of the Derivatives

It was shown that the equation of the tangent line to the given curve at the point x_0 has the form (4.7)

$$y - y_0 = f'(x_0)(x - x_0),$$

where $y_0 = f(x_0)$, $f'(x_0)$ is a slope of the tangent line (Fig. 4.3).

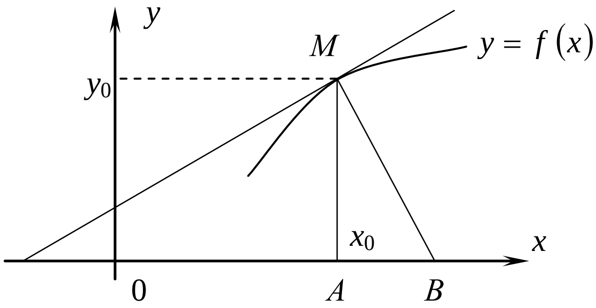

Fig. 4.3

The straight line passing through tangency point perpendicular to tangent line is called *normal to the given* curve at the given point. Obviously its slope is $-\frac{1}{f'(x_0)}$, therefore an equation of the normal may be presented in the following form $y - y_0 = -\frac{1}{f'(x_0)}(x - x_0)$.

Example 1. Form an equation of the tangent line and the normal to the curve $y = \frac{1}{x}$ at the point M(2, 1/2).

Solution. Let us determine the derivative of the given function

$$y' = -\frac{1}{x^2}$$

and calculate its value at the given point $y'(2) = -\frac{1}{4}$. Then the equation of the
tangent line is:

$$y - \frac{1}{2} = -4(x-2).$$

Example 2. Find the angle of the intersection of the curves $y = x^2$ and $y = \sqrt{x}$ at the point $M(1,1)$, (Fig. 4.4). From the equations we find the derivatives

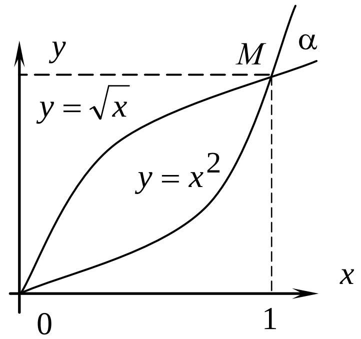

Fig. 4.4

$$y' = 2x,\ y' = \frac{1}{2\sqrt{x}},$$

then the slopes of the tangent lines at the point M (1,1) are

$$k_1 = 2, \quad k_2 = \frac{1}{2}.$$

To find a sought angle α we can use the following formula $\tan\alpha = \frac{k_2 - k_1}{1 + k_1 k_2}$ (tangent of an angle between two straight lines with slopes k_1 and k_2). Then

$$\tan\alpha = \frac{2 - 1/2}{1 + 2 \cdot 1/2} = \frac{3}{4}.$$

Example 3. Find the equation of the tangent line to the curve $y = 2x^2 + 4x - 1$ if it is known that the sought tangent line is parallel to straight line $x + 2y - 3 = 0$.

Solution. In the given case a point (x_0, y_0) of the tangency is not known. But it is known the slope of tangent $k = -\frac{1}{2}$. Therefore we can determine the derivative and put it equal to $-\frac{1}{2}$. So, $y' = 4x + 4$, then

$$4x_0 + 4 = -\frac{1}{2},$$

whence

$$x_0 + 1 = -\frac{1}{8},$$

from this it follows

$$x_0 = -\frac{9}{8},\ y_0 = \frac{81}{32} - \frac{9}{2} - 1 = -\frac{95}{32}.$$

So, the equation of sought tangent is:

$$y + \frac{95}{32} = -\frac{1}{2}\left(x + \frac{9}{8}\right).$$

Example 4. To form the equation of the tangent line to the curve $y = x^2 - 4x + 1$, if it is known that tangent line passes through point of the origin (Fig. 4.5).

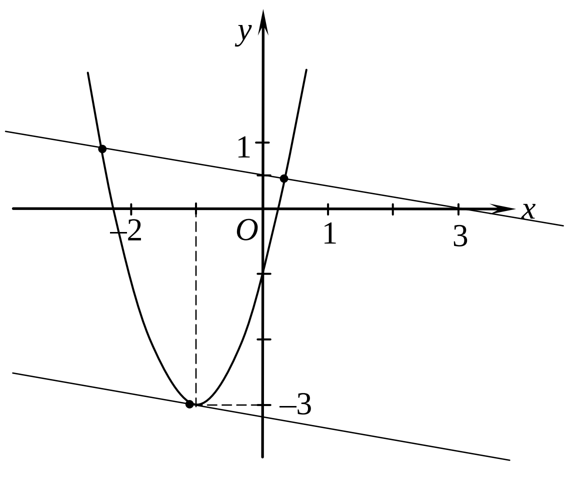

Fig. 4.5

Solution. Let us denote unknown coordinates of the tangency point as (x_0, y_0).

The sought equation of the tangent line is defined as $y - y_0 = (2x_0 - 4)(x - x_0)$.

Let us require passing of the tangent line through origin O. We get

$$-y_0 = (2x_0 - 4)(-x_0),$$

that is

$$y_0 = 2x_0^2 - 4x_0,$$

or

$$x_0^2 - 4x_0 + 1 = 2x_0^2 - 4x_0,$$

whence

$$x_0^2 = 1,$$

hence

$$x_{0_1} = 1, \;\; x_{0_2} = -1.$$

We has got two points of tangency $M_1(1,-2), M_2(-1,6)$. The equations of the tangent lines are:

$$y + 2 = -2(x-1), \;\; y - 6 = -6(x+1).$$