2. Applications of the Definite Integral

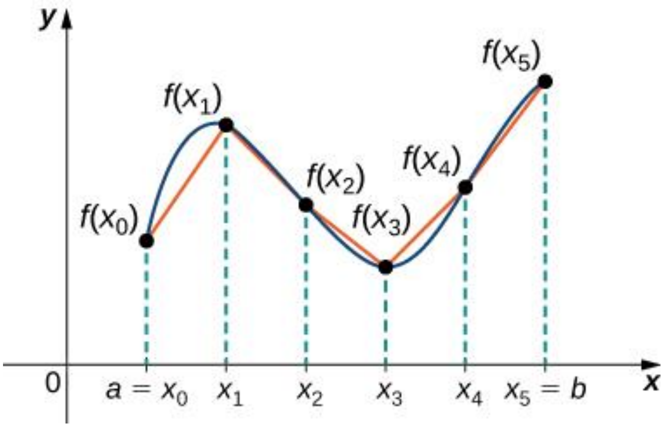
In this section, we use definite integrals to find the arc length of a curve. We can think of arc length as the distance you would travel if you were walking along the path of the curve.

2.3 Arc Length of a Curve

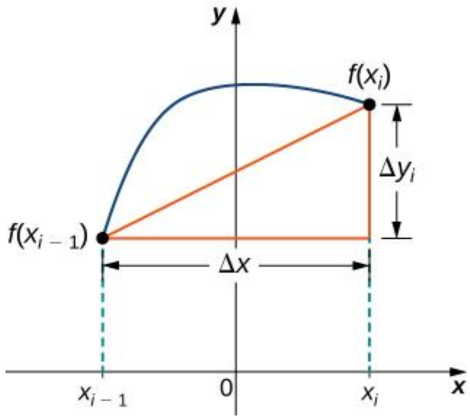
2.3.1 Arc Length of the Curve $y = f(x)$

In previous applications of integration, we required the function $f(x)$ to be integrable, or at most continuous. However, for calculating arc length we have a more stringent requirement for $f(x)$. Here, we require $f(x)$ to be differentiable, and furthermore we require its derivative, $f'(x)$, to be continuous. Functions like this, which have continuous derivatives, are called *smooth*.

Let $f(x)$ be a smooth function defined over $[a, b]$. We want to calculate the length of the curve from the point $(a, f(a))$ to the point $(b, f(b))$. We start by using line segments to approximate the length of the curve. For $i = 0,1,2, \ldots, n$, let $P = x_i$ be a regular partition of $[a, b]$. Then, for $i = 1,2, \ldots, n$, construct a line segment from the point $(x_{i-1}, f(x_{i-1}))$ to the point $(x_i, f(x_i))$. Although it might seem logical to use either horizontal or vertical line segments, we want our line segments to approximate the curve as closely as possible. Figure 1a depicts this construct for $n = 5$.



(a)



(b)

To help us find the length of each line segment, we look at the change in vertical distance as well as the change in horizontal distance over each interval. Because we have used a regular partition, the change in horizontal distance over each interval is given by Δx. The change in vertical distance varies from interval to interval, though, so we use $\Delta y_i = f(x_i) - f(x_{i-1})$ to represent the change in vertical distance over the interval $[x_{i-1}, x_i]$, as shown in Figure 1b . Note that some (or all) Δy_i may be negative.

By the Pythagorean theorem, the length of the line segment is

$$\sqrt{(\Delta x)^2 + (\Delta y_i)^2} = \Delta x\sqrt{1 + ((\Delta y_i)/(\Delta x))^2}.$$

Now, by the Mean Value Theorem, there is a point $x_i^* \in [x_{i-1}, x_i]$ such that $f'(x_i^*) = (\Delta y_i)/(\Delta x)$ Then the length of the line segment is given by

$$\Delta x\sqrt{1 + [f'(x_i^*)]^2}.$$

Adding up the lengths of all the line segments, we get

$$\text{Arc Length} \approx \sum_{i=1}^{n} \sqrt{1 + [f'(x_i^*)]^2}\, \Delta x.$$

This is a Riemann sum. Taking the limit as $n \to \infty$, we have

$$\text{Arc Length} \quad = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + [f'(x_i^*)]^2}\, \Delta x = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx.$$

We summarize these findings in the following rule:

Let $f(x)$ be a smooth function over the interval $[a, b]$. Then the arc length of the portion of the graph of $f(x)$ from the point $(a, f(a))$ to the point $(b, f(b))$ is given by

$$\text{Arc Length} = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx.$$

***Example* 1**. Let $f(x) = 2x^{3/2}$. Calculate the arc length of the graph of $f(x)$ over the interval [0,1].

Solution:

We have $f'(x) = 3x^{1/2}$, so $[f'(x)]^2 = 9x$. Then, the arc length is

$$\text{L} \quad = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx = \int_0^1 \sqrt{1 + 9x}\, dx = \frac{1}{9}\int_0^1 9\sqrt{(1 + 9x)}\, dx =$$

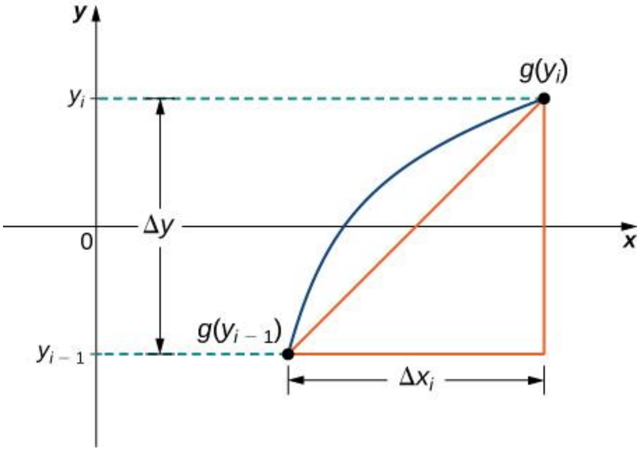
$$= \frac{1}{9}\int_1^{10} \sqrt{u}\, du = \frac{1}{9} \cdot \frac{2}{3} u^{3/2} \mid_1^{10} = \frac{2}{27}[10\sqrt{10} - 1] \approx 2.268 units.$$

2.3.2 *Arc Length of the Curve* $x = g(y)$

We have just seen how to approximate the length of a curve with line segments. If we want to find the arc length of the graph of a function of *y*, we can repeat the same process, except we partition the *y*-axis instead of the *x*-axis. Figure 3 shows a representative line segment.

Then the length of the line segment is $\Delta y\sqrt{1 + \left(\frac{\Delta x_i}{\Delta y}\right)^2}$.

If we now follow the same development we did earlier, we get a formula for arc length of a function $x = g(y)$:



Let $g(y)$ be a smooth function over an interval $[c, d]$. Then, the arc length of the graph of $g(y)$ from the point $(c, g(c))$ to the point $(d, g(d))$ is given by

$$\text{Arc Length} = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy.$$

Example **2.** Let $g(y) = 3y^3$. Calculate the arc length of the graph of $g(y)$ over the interval $[1,2]$.

Solution:

We have $g'(y) = 9y^2$, so $[g'(y)]^2 = 81y^4$. Then the arc length is

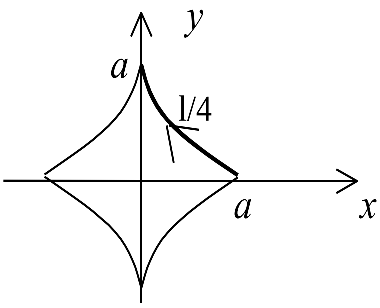
$$\text{Arc Length} \quad = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy = \int_1^2 \sqrt{1 + 81y^4}\, dy$$

2.3.3 Arc Length of the Curve given by Parametric Equations

If the curve is given in parametric form $\begin{cases} x = x(t), \\ y = y(t); \end{cases}$ we obtain

$$L = \int_{\alpha}^{\beta} \sqrt{(x_t^{'})^2 + (y_t^{'})^2}\, dt; \quad (\alpha < \beta).$$

Example **3.** Find the length of the astroid: $x^{2/3} + y^{2/3} = a^{2/3}$.



Solution: *The 1st way.* $l = \int_a^b \sqrt{1 + (y_x')^2}\, dx$.

Differentiating the equation of the astroid we obtain the following:

$$\frac{2}{3} x^{-1/3} + \frac{2}{3} y^{-1/3} y_x' = 0 \;\Rightarrow\; y_x' = \frac{y^{1/3}}{x^{1/3}}.$$

Therefore the length of the quarter of the astroid as follows:

$$\frac{1}{4}L=\int_0^a\sqrt{1+\frac{y^{2/3}}{x^{2/3}}}dx=\int_0^a\frac{a^{1/3}}{x^{1/3}}dx=\left.\frac{3a^{1/3}x^{2/3}}{2}\right|_0^a=\frac{3a}{2}\Rightarrow L=6a\,.$$

The 2nd way (parametric).

$$L=\left|\int_{t_1}^{t_2}\sqrt{(x_t')^2+(y_t')^2}\,dt\right|.$$

$$\begin{cases}x=a\cos^3 t\\ y=a\sin^3 t\end{cases},\quad\begin{cases}x_t'=-3a\cos^2 t\sin t\\ y_t'=3a\sin^2 t\cos t\end{cases}.$$

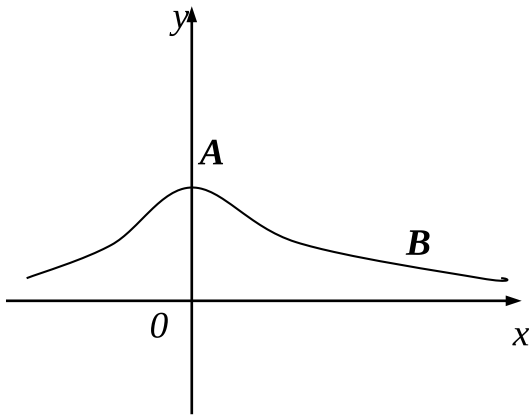
$$(x_t')^2+(y_t')^2=9a^2\cos^2 t\sin^2 t,\quad\sqrt{(x_t')^2+(y_t')^2}=3a\cos t\sin t\,.$$

$$\frac{1}{4}L=3a\int_0^{\frac{\pi}{2}}\cos t\sin t\,dt=\frac{3a}{2}\int_0^{\frac{\pi}{2}}\sin 2t\,dt=-\left.\frac{3a}{4}\cos 2t\right|_0^{\frac{\pi}{2}}=-\frac{3a}{4}(-2)=\frac{3a}{2}\,.$$

$$L=6a\,.$$

Find the length of an arc of the tractrix from point $A(0,a)$ to point $B(x,y)$:

$$\begin{cases}x=a\left(\cos t+\ln tg\frac{t}{2}\right)\\ y=a\sin t\end{cases}.$$



Solution. $l=\int_{t_1}^{t_2}\sqrt{(x_t')^2+(y_t')^2}\,dt.$

$$a\sin t_1=a\Rightarrow t_1=\frac{\pi}{2},\ t_2=\arcsin\frac{y}{a}.$$

$$x_t'=a\left(-\sin t+\frac{1}{\sin t}\right),\ y_t'=a\cos t.$$

$$(x_t')^2+(y_t')^2=a^2\left(\sin^2 t-2+\frac{1}{\sin^2 t}+\cos^2 t\right)=$$

$$=a^2\left(\frac{1}{\sin^2 t}-1\right)=a^2\left(\frac{1-\sin^2}{\sin^2 t}\right)=\frac{a^2\cos^2 t}{\sin^2 t}.$$

As $\arcsin\frac{y}{a}<\frac{\pi}{2}$, then

$$l=\int_{\arcsin y/a}^{\pi/2}\frac{a\cos t}{\sin t}dt=a\ln\sin t\Big|_{\arcsin y/a}^{\pi/2}=-a\ln\frac{y}{a}=a\ln\frac{a}{y}.$$

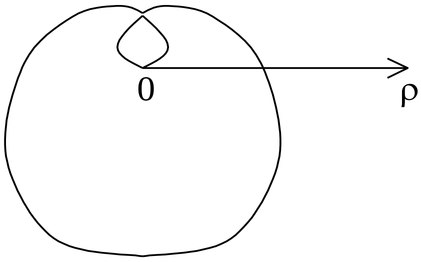
2.3.4 Arc Length of the Curve given in Polar Coordinates

In polar coordinates:

$$L = \int_{\varphi_1}^{\varphi_2} \sqrt{\rho^2(\varphi) + (\rho'(\varphi))^2}\, d\varphi; \ (\varphi_1 \le \varphi_2).$$

Example 5. Find the length of the curve $r = a \sin^3 \frac{\varphi}{3}$, φ is changing from 0 to 3π.

Solution. The derivative is $r' = a\, sin^2 \frac{\varphi}{3} cos \frac{\varphi}{3}$. Then,

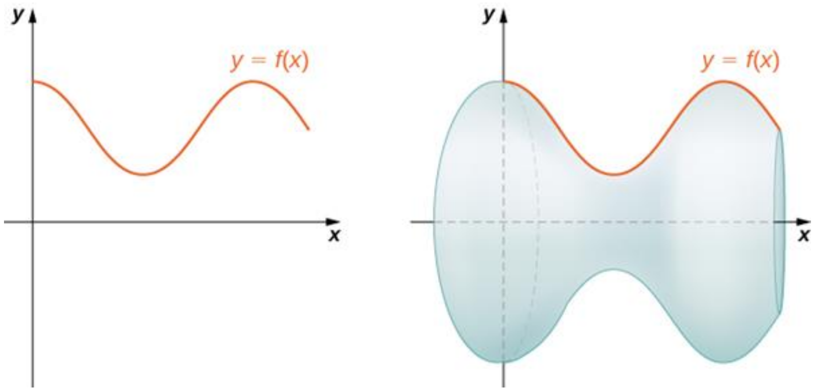


$$l = \int_0^{3\pi} \sqrt{a^2\, sin^6 \frac{\varphi}{3} + a^2\, sin^4 \frac{\varphi}{3} cos^2 \frac{\varphi}{3}}\, d\varphi =$$

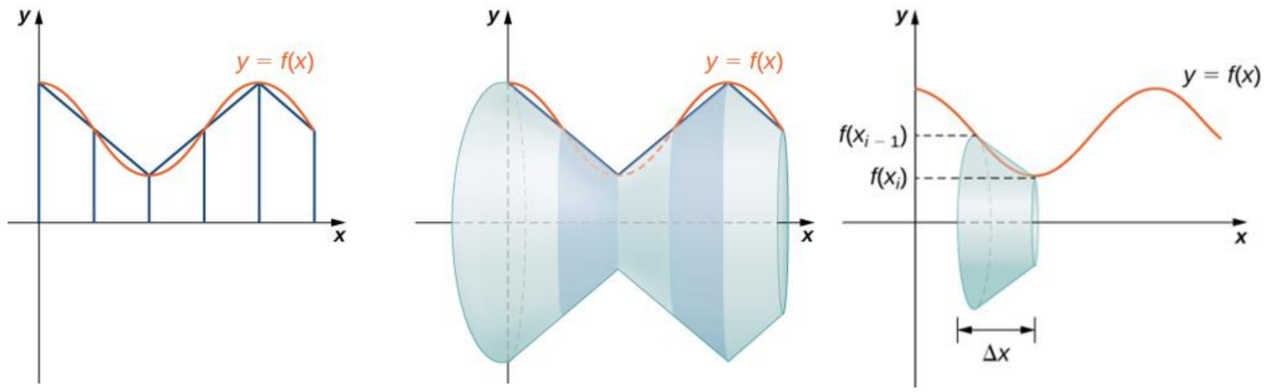
$$= a \int_0^{3\pi} sin^2 \frac{\varphi}{3} d\varphi = = \frac{a}{2} \int_0^{3\pi} \left(1 - cos \frac{2\varphi}{3}\right) d\varphi = \frac{3a\pi}{2}.$$

2.4 Area of a Surface of Revolution

The concepts we used to find the arc length of a curve can be extended to find the surface area of a surface of revolution. Surface area is the total area of the outer layer of an object. For objects such as cubes or bricks, the surface area of the object is the sum of the areas of all of its faces. For curved surfaces, the situation is a little more complex. Let $f(x)$ be a nonnegative smooth function over the interval $[a, b]$. We wish to find the surface area of the surface of revolution created by revolving the graph of $y = f(x)$ around the x-axis as shown in the following Figure 4b.



As we have done many times before, we are going to partition the interval $[a, b]$ and approximate the surface area by calculating the surface area of simpler shapes. We start by using line segments to approximate the curve, as we did earlier in this section. For $i = 0,1,2, \ldots, n$, let $P = x_i$ be a regular partition of $[a, b]$. Then, for $i = 1,2, \ldots, n$, construct a line segment from the point $(x_{i-1}, f(x_{i-1}))$ to the point $(x_i, f(x_i))$. Now, revolve these line segments around the x -axis to generate an approximation of the surface of revolution as shown in the following Figure 5.



(a) (b) (c)

Notice that when each line segment is revolved around the axis, it produces a band. These bands are actually pieces of cones (think of an ice cream cone with the pointy end cut off).

Let's now use this formula to calculate the surface area of each of the bands formed by revolving the line segments around the x–axis . A representative band is shown in the following Figure 5c.

Applying the surface area formula, we have

$$S = \pi(r_1 + r_2)l = \pi(f(x_{i-1}) + f(x_i))\sqrt{\Delta x^2 + (\Delta yi)^2}$$

$$= \pi(f(x_{i-1}) + f(x_i))\Delta x\sqrt{1 + (\frac{\Delta y_i}{\Delta x})^2}$$

Now, as we did in the development of the arc length formula, we apply the Mean Value Theorem to select $x_i^* \in [x_{i-1}, x_i]$ such that $f'(x_i^*) = (\Delta y_i)/\Delta x$. This gives us

$$S = \pi(f(x_{i-1}) + f(x_i))\Delta x\sqrt{1 + (f'(x_i^*))^2}$$

Since $f(x)$ is continuous, by the Intermediate Value Theorem, there is a point $x_i^{**} \in [x_{i-1}, x[i]$ such that $f(x_i^{**}) = \frac{f(x_{i-1})+f(x_i)}{2}$, so we get

$$S = 2\pi f(x_i^{**})\Delta x\sqrt{1 + (f'(x_i^*))^2}.$$

Then the approximate surface area of the whole surface of revolution is given by

$$\text{Surface Area} \approx \sum_{i=1}^{n} 2\pi f(x_i^{**})\Delta x\sqrt{1+(f'(x_i^*))^2}.$$

Taking the limit as $n \to \infty$, we get

$$\begin{aligned}\text{Surface Area} &= \lim_{n\to\infty} \sum_{i=1} n^2\pi f(x_i^{**})\Delta x\sqrt{1+(f'(x_i^*))^2} \\ &= \int_a^b \left(2\pi f(x)\sqrt{1+(f'(x))^2}\right) dx\end{aligned}$$

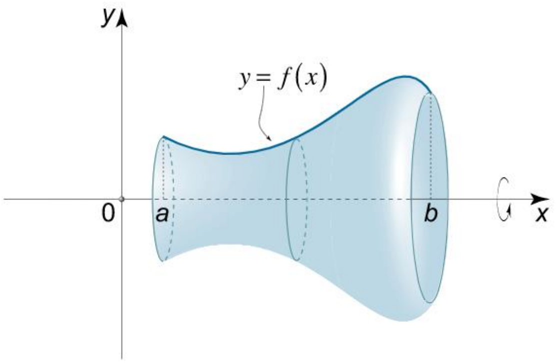
These findings are summarized in the following rule:

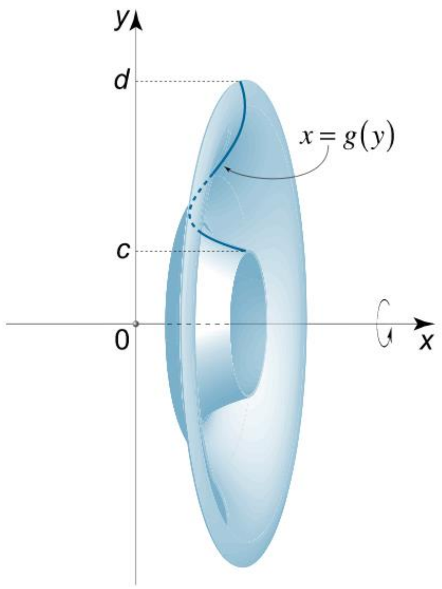
Let $f(x)$ be a nonnegative smooth function over the interval $[a,b]$. Then, the surface area of the surface of revolution formed by revolving the graph of $f(x)$ around the x-axis is given by

$$\text{Surface Area} = \int_a^b \left(2\pi f(x)\sqrt{1+(f'(x))^2}\right) dx$$

Similarly, let $g(y)$ be a nonnegative smooth function over the interval $[c,d]$. Then, the surface area of the surface of revolution formed by revolving the graph of $g(y)$ around the y–axis is given by

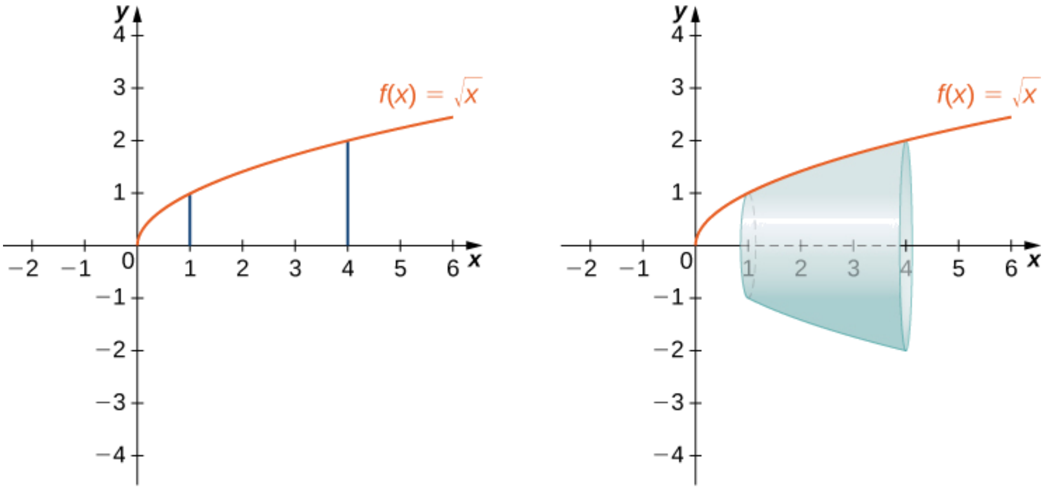
$$\text{Surface Area} = \int_c^d (2\pi g(y)\sqrt{1+(g'(y))^2}\, dy$$





***Example* 6**. Let $f(x) = \sqrt{x}$ over the interval [1,4]. Find the surface area of the surface generated by revolving the graph of $f(x)$ around the x-axis.

Solution:
The graph of $f(x)$ and the surface of rotation are shown in Figure 6.



(a) (b)

We have $f(x) = \sqrt{x}$. Then, $f'(x) = 1/(2\sqrt{x})$ and $(f'(x))^2 = 1/(4x)$. Then,

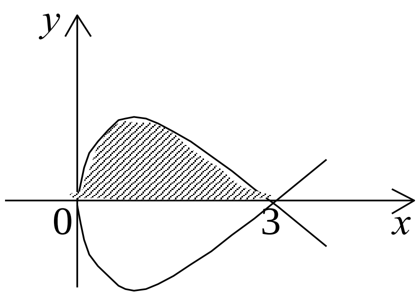
$$S = \int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx$$

$$= \int_1^4 2\pi\sqrt{x}\left(\sqrt{1+\frac{1}{4x}}\right)dx = \int_1^4 2\pi\,\sqrt{x+14}dx.$$

$$= \int_0^1 2\pi\sqrt{\left(x+\frac{1}{4}\right)}\,dx \quad = \int_{5/4}^{17/4} 2\pi\sqrt{u}\,du$$

$$= 2\pi\left[\frac{2}{3}u^{\frac{3}{2}}\right]\Big|_{\frac{5}{4}}^{\frac{17}{4}} = \frac{\pi}{6}[17\sqrt{17} - 5\sqrt{5}] \approx 30.846$$

Example 7. Find area of the surface, formed by gyration of the close loop $9y^2 = x(3-x)^2$ round the axis $0X$.

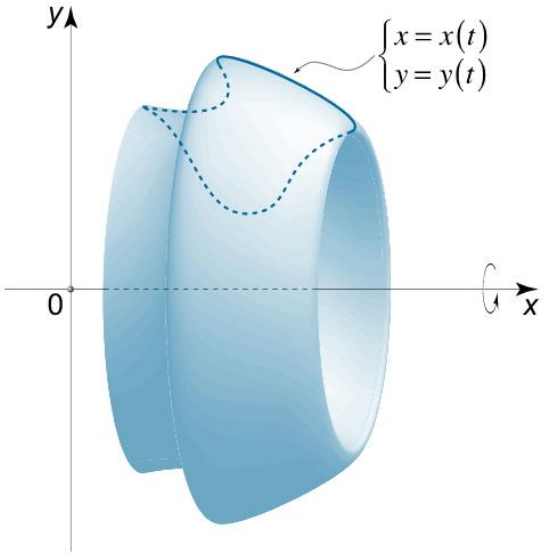


Solution:

For upper part of the curve at $0 \le x \le 3$ we obtain $y = \frac{1}{3}(3-x)\sqrt{x}$. Therefore the differential of the arc makes $dl = \frac{x+1}{2\sqrt{x}}dx$. According to the formula $S = 2\pi\int_a^b y\,dl$, area of the surface

$$S = 2\pi\int_0^3 \frac{1}{3}(3-x)\sqrt{x}\,\frac{x+1}{2\sqrt{x}}\,dx = \frac{\pi}{3}\int_0^3 \left(2x - x^2 + 3\right)dx = \frac{\pi}{3}(9-9+9) = 3\pi$$

If the curve defined by the parametric equations $x = x(t)$, $y = y(t)$, with t ranging over some interval $[\alpha, \beta]$, is rotated about the x–axis, then the surface area is given by the following integral (provided that $y(t)$ is never negative).



Area of the surface, formed by rotation is equal to $S = 2\pi\int_a^b y\,dl$ or if the curve is given in parametric form:

$$S = 2\pi\int_\alpha^\beta y(t)\sqrt{[x'(t)]^2 + [y'(t)]^2}\,dt.$$

***Example* 8.** Calculate area of the surface, formed by gyration of one arch of the cycloid round the axis $0X$:

$$\begin{cases} x = a(t - \sin t) \\ y = a(1 - \cos t) \end{cases} \quad (0 \le t \le \pi).$$

Solution:

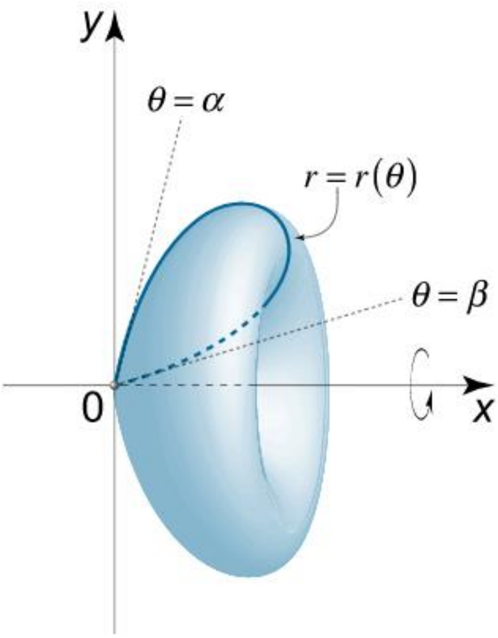
We have in accordance with the formula:

$$S = 2\pi \int_0^{2\pi} a^2 (1 - \cos t)\sqrt{(1 - \cos t)^2 + \sin^2 t}\,dt = 8\pi a^2 \int_0^{2\pi} \sin^3 \frac{t}{2}\,dt =$$

$$= 8\pi a^2 \int_{zz0}^{2\pi} \left(1 - \cos^2 \frac{t}{2}\right) \sin \frac{t}{2}\,dt = 8\pi a^2 \left(-2a \cos \frac{t}{2} + \frac{2}{3} \cos^3 \frac{t}{2}\right)\Bigg|_0^{2\pi} =$$

$$8\pi a^2 \left(2 - \frac{2}{3} + 2 - \frac{2}{3}\right) = \frac{64}{3}\pi a^2.$$

If the curve defined by polar equation $\rho = \rho(\theta)$, with θ ranging over some interval $[\alpha, \beta]$, is rotated about the polar axis, then the area of the resulting surface is given by the following formula (provided that $y = \rho \sin \theta$ is never negative).



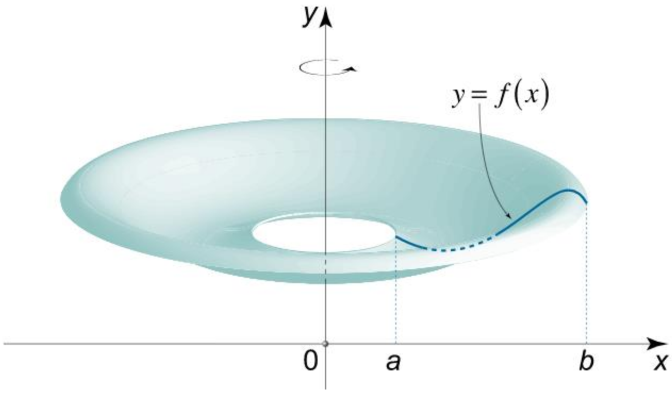
$$S = 2\pi \int_{\alpha}^{\beta} r(\theta) \sin \theta \sqrt{[r(\theta)]^2 + [r'(\theta)]^2}\,d\theta.$$

Revolving about the y−axis

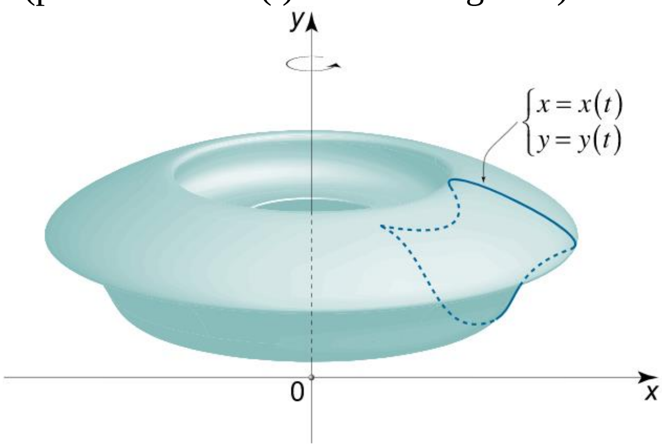
The functions *g(y)*, $x(t)$, and $x(\theta)$ are supposed to be smooth and non-negative on the given interval.

If the curve $y = f(x)$, $a \leq x \leq b$ is rotated about the *y*−axis, then the surface area is given by

$$S = 2\pi \int_a^b x\sqrt{1 + [f'(x)]^2}\,dx.$$

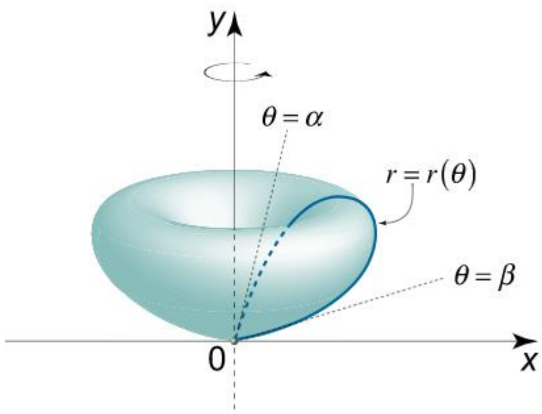


If the curve defined by the parametric equations $x = x(t)$, $y = y(t)$, with t ranging over some interval $[\alpha, \beta]$, is rotated about the y–axis, then the surface area is given by the following integral (provided that $x(t)$ is never negative).



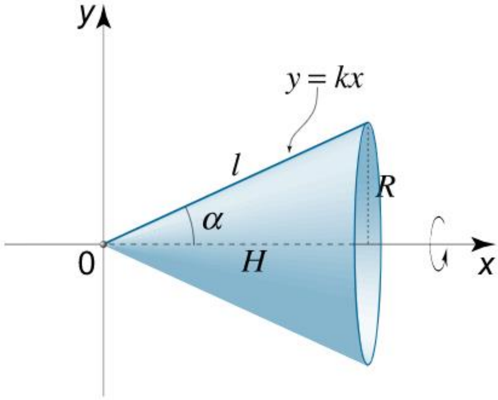
$$S = 2\pi \int_{\alpha}^{\beta} x(t)\sqrt{[x'(t)]^2 + [y'(t)]^2}dt.$$

If the curve defined by polar equation $\rho = \rho(\theta)$, with θ ranging over some interval $[\alpha, \beta]$, is rotated about the polar axis, then the area of the resulting surface is given by the following formula (provided that $x = \rho\cos\theta$ is never negative).



$$S = 2\pi \int_{\alpha}^{\beta} r(\theta)\cos\theta\sqrt{[r(\theta)]^2 + [r'(\theta)]^2}d\theta$$

Example 1. Find the lateral surface area of a right circular cone with slant height ℓ and base radius R.



Let the slant height ℓ be defined by the equation $y = f(x) = kx$. The slope k is given by

$$k = \tan\alpha = \frac{R}{H},$$

where H is the height of the cone. We calculate the lateral surface area of the cone by the formula

$$S = 2\pi \int_a^b f(x)\sqrt{1 + [f'(x)]^2}dx.$$

Substituting, $a = 0$, $b = H$, $f(x) = kx = \frac{R}{H}x$, $f'(x) = k = \frac{R}{H}$, we obtain

$$A = 2\pi \int_0^H kx\sqrt{1 + k^2}dx = 2\pi k\sqrt{1 + k^2} \int_0^H xdx = 2\pi k\sqrt{1 + k^2}\frac{x^2}{2}\Big|_0^H$$

$$= \pi k\sqrt{1 + k^2}H^2 = \pi \frac{R\cancel{H^2}\sqrt{R^2 + H^2}}{\cancel{H^2}} = \pi R\sqrt{R^2 + H^2}.$$

By the Pythagorean theorem, $\sqrt{R^2 + H^2} = \ell$. Hence,

$$S = \pi R\ell.$$

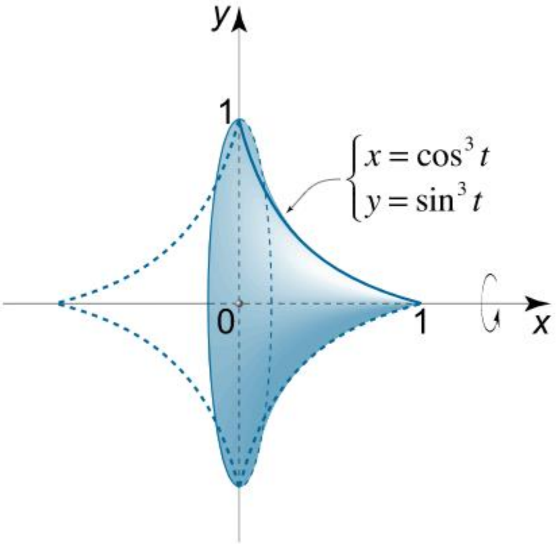
Example 2. Find the area of the surface obtained by revolving the asteroid $x = \cos^3 t$, $y = \sin^3 t$ x −axis.

When calculating the surface area, we consider the part of the astroid lying in the first quadrant and then multiply the result by 2. As the curve is defined in parametric form, we can write

$$S = 2\pi \int_a^b y(t)\sqrt{[x'(t)]^2 + [y'(t)]^2}dt = 4\pi \int_0^{\frac{\pi}{2}} \sin^3 t\sqrt{[(\cos^3 t)']^2 + [(\sin^3 t)']^2}dt.$$

Find the derivatives:

$$(\cos^3 t)' = -3\cos^2 t\sin t, (\sin^3 t)' = 3\sin^2 t\cos t,$$



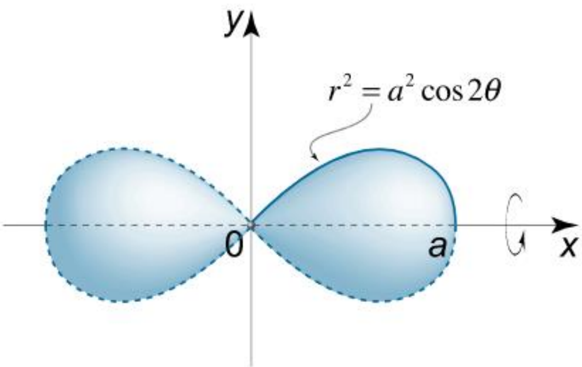
and simplify the radicand:

$$[(\cos^3 t)']^2 + [(\sin^3 t)']^2 = (-3\cos^2 t\sin t)^2 + (3\sin^2 t\cos t)^2$$
$$= 9\cos^4 t\sin^2 t + 9\sin^4 t\cos^2 t = 9\sin^2 t\cos^2 t\underbrace{(\cos^2 t + \sin^2 t)}_{=1}$$
$$= (3\sin t\cos t)^2.$$

Hence, the surface area is

$$S = 4\pi \int_0^{\frac{\pi}{2}} (\sin^3 t \cdot 3\sin t\cos t)dt = 12\pi \int_0^{\frac{\pi}{2}} \sin^4 t\cos t dt = 12\pi \cdot \frac{\sin^5 t}{5}\Big|_0^{\frac{\pi}{2}} = \frac{12\pi}{5}$$

Example 4. The lemniscate of Bernoulli given by the equation $r^2 = a^2\cos 2\theta$ rotates around the polar axis. Find the area of the resulting surface.



Due to symmetry, we can integrate from 0 to $\frac{\pi}{4}$ considering the curve in the first quadrant and then multiply the result by 2. So

$$S = 4\pi \int_0^{\frac{\pi}{4}} r(\theta)\sin\theta\sqrt{[r(\theta)]^2 + [r'(\theta)]^2}d\theta.$$

Take the derivative:

$$r'(\theta) = (a\sqrt{\cos 2\theta})' = \frac{a}{2\sqrt{\cos 2\theta}} \cdot (-\sin 2\theta) \cdot 2 = -\frac{a\sin 2\theta}{\sqrt{\cos 2\theta}}.$$

Let's simplify the expression with the square root:

$$\sqrt{[r(\theta)]^2 + [r'(\theta)]^2} = \sqrt{a^2\cos 2\theta + \frac{a^2(\sin 2\theta)^2}{\cos 2\theta}} = a\sqrt{\frac{(\cos 2\theta)^2 + (\sin 2\theta)^2}{\cos 2\theta}}$$
$$= \frac{a}{\sqrt{\cos 2\theta}}.$$

Then

$$A = 4\pi \int_0^{\frac{\pi}{4}} (a\sqrt{\cos 2\theta}\sin\theta \cdot \frac{a}{\sqrt{\cos 2\theta}})d\theta = 4\pi a^2 \int_0^{\frac{\pi}{4}} \frac{\cancel{\sqrt{\cos 2\theta}}\sin\theta}{\cancel{\sqrt{\cos 2\theta}}} d\theta$$
$$= 4\pi a^2 \int_0^{\frac{\pi}{4}} \sin\theta d\theta = 4\pi a^2(-\cos\theta)|_0^{\frac{\pi}{4}} = 4\pi a^2(-\frac{\sqrt{2}}{2} + 1)$$
$$= 2\pi a^2(2 - \sqrt{2}).$$