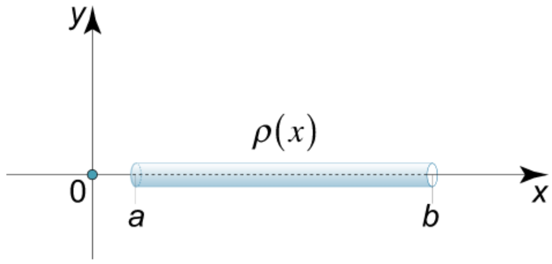
2. Applications of the Definite Integral

2.5 Physical and mechanical applications of a definite integral

Let's consider the applications of a definite integral for solving some physical problems.

Mass of a Body

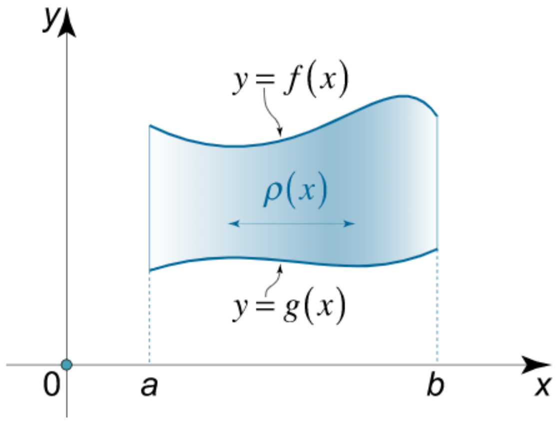
We can use integration for calculating mass based on a density function. Consider a thin wire or rod that is located on an interval $[a, b]$.



The density of the rod at any point x is defined by the density function $\rho(x)$. Assuming that $\rho(x)$is an integrable function, the mass of the rod is given by the integral.

$$m = \int_a^b \rho(x)dx.$$

Suppose a region is enclosed by two curves $y = f(x)$, $y = g(x)$ and by two vertical lines $x = a$ and $x = b$.

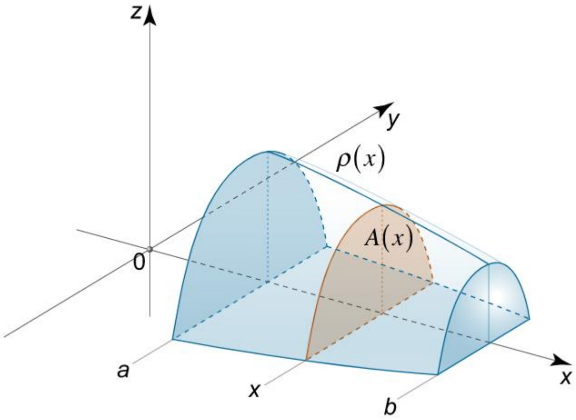


If the density of the lamina which occupies the region only depends on the $x-$coordinate, the total mass of the lamina is given by the integral

$$m = \int_a^b \rho(x)[f(x) - g(x)]dx,$$

where $f(x) \geq g(x)$ on the interval $[a, b]$, and $\rho(x)$ is the density of the material changing along the x–axis.

Consider a solid S that extends in the x–direction from $x = a$ to $x = b$ with cross sectional area $A(x)$.



Suppose that the density function $\rho(x)$ depends on x but is constant inside each cross section $A(x)$. The mass of the solid is

$$m = \int_a^b \rho(x)A(x)dx.$$

Tasks admitting the use of the Defined Integral

Problem 1. Calculate the work, needed to launch the rocket with weight P from the surface of the Earth on the height H. What should be the work equal to if the rocket should be removed on infinite distance (perpetuity)?

Solution.

The value of the force F, determining the force, necessary to launch the rocket from the surface of the Earth, is equal to the value of gravitational force, influencing the rocket, i.e.

$$F(x) = k\frac{MP}{qx^2},$$

where M – mass of the Earth, $\frac{p}{q}$ - mass of the rocket, k – constant coefficient, x – distance between the center of the Earth and the rocket. The force F is directed along the radius of the Earth from its center. Movement of the rocket from $a = R$ (R – radius of the Earth) to $b = R + H$ is carried out in the same direction.

Let's divide the interval $[R, R+H]$ into subintervals $[x_i, x_{i+1}]$ and then calculate an approximate value of work A_i on each subinterval. Due to smallness of each subinterval Δx_i we consider the value of the force F on each subinterval constant on this subinterval and equal to the value of force F at some point x_i of this subinterval. Then

$$A_i \approx F(x_i)\Delta x_i = \frac{kMP}{qx_i^2}\Delta x_i.$$

The work correspondent to the whole interval [*R*, *R*+*H*] is approximately equal to:

$$A = \sum_{i=1}^{n} A_i \approx \sum_{i=1}^{n} \frac{kMP}{qx_i^2} \Delta x_i$$

Passing to a limit in equality (7.1) while $\lambda \to 0$ we obtain an exact value of the work *A* as follows:

$$A = \lim_{\lambda \to 0} \sum \frac{kMP}{qx_i^2} \Delta x_i = \frac{kMP}{q} \int_R^{R+H} \frac{dx}{x^2} = \frac{kMP}{q} \left(\frac{1}{R} - \frac{1}{R+H} \right)$$

Taking into account the fact on the surface of the Earth the gravitational force $F = P$ and $x = R$, we can find the coefficient *k*:

$$P = \frac{kMP}{qR^2},$$

whence

$$k = \frac{qR^2}{M}.$$

And for required work we finally get:

$$A = PR^2 \left(\frac{1}{R} - \frac{1}{R+H} \right)$$

At moving the rocket on indefinite distance (perpetuity) the work makes as follows:

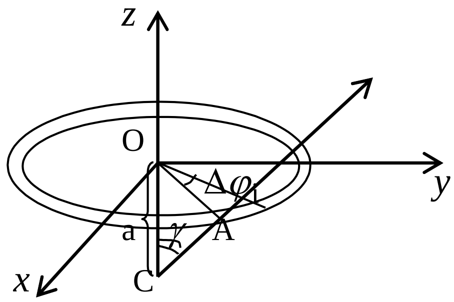
$$\lim_{H \to \infty} A = \lim_{H \to \infty} PR^2 \left(\frac{1}{R} - \frac{1}{R+H} \right) = PR.$$

Problem 2. With what force does wire ring with weight *M*, radius *R* affect the mass point *C* with weight *m*, lying on the straight line passing through the center of the ring perpendicularly to its plane? The distance between the point *C* and the center of the ring makes *a*.

*Solution***.**

Let's divide the ring into elementary segments Δl_i, considering each of them to be a mass point with weight:

$$m_i = \rho \Delta l_i = \frac{M}{2\pi R} \Delta l_i = \frac{M}{2\pi R} R \Delta \varphi_i = \frac{M}{2\pi} \Delta \varphi_i$$



Here ρ – density function of weight, $\Delta \varphi_i$ - angle, correspondent to the part of arch Δl_i (fig). Let's define the force $\vec{F}_i$ of interaction of the material point *C* with the small segment of the ring Δl_i. For this let's decompose $\vec{F}_i$ on basis $\vec{i}, \vec{j}, \vec{k}$.

$$\vec{F}_i = F_{ix}\vec{i} + F_{iy}\vec{j} + F_{iz}\vec{k},$$

where F_{ix}, F_{iy}, F_{iz} - are projections of $\vec{F}_i$ on coordinate axes. Obviously, the required force $\vec{F}$ is a resultant of elementary forces $\vec{F}_i$ and is defined as follows:

$$\vec{F} = \vec{i}\sum_{i=1}^{n} F_{ix} + \vec{j}\sum_{i=1}^{n} F_{iy} + \vec{k}\sum_{i=1}^{n} F_{iz}$$

It should be mentioned that due to a symmetry of the given problem:

$$\sum_{i=1}^{n} F_{ix} = 0, \quad \sum_{i=1}^{n} F_{iy} = 0 .$$

Thus the value of the required value of interaction is determined as a sum of projections of $\vec{F}_i$ on the axis *OZ*. Let's calculate the value F_{iz}

$$F_{iz} = F_i \cos\gamma ,$$

where γ – an angle between the axis *OZ* and the vector $\vec{F}_i$ constant for all $i = 0,1,2,n-1$ and is defined from the rectangle *COA*:

$$cos\,\gamma = \frac{a}{\sqrt{R^2 + a^2}}$$

According to the law of interactions of two material points the value F_i is approximately defined as follows:

$$F_i \approx k\frac{mm_i}{r^2} = \frac{kmM}{\left(R^2 + a^2\right)2\pi}\Delta\varphi_i$$

The value m_i and the distance $r = \sqrt{a^2 + R^2}$ between mass points are taken in account here (fig.).

Then combining the formulas and summarizing on *i* we obtain the following:

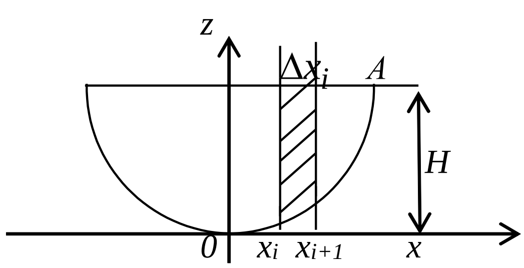
$$F = \sum_{i=1}^{n} F_{iz} \approx \sum_{i=1}^{n} \frac{kmMa}{\left(R^2 + a^2\right)^{3/2} 2\pi}\Delta\varphi_i$$

As an exact value of force of interaction we'll accept the limit the integral sum is aiming to when the length of the longest subinterval Δl_i and therefore $\Delta\varphi_i$ aim to zero.

$$F = \lim_{\lambda \to 0}\sum \frac{kmMa}{\left(R^2 + a^2\right)^{3/2} 2\pi}\Delta\varphi_i = \frac{kmMa}{\left(R^2 + a^2\right)^{3/2} 2\pi}\int_0^{2\pi} d\varphi = \frac{kmMa}{\left(R^2 + a^2\right)^{3/2}} .$$

Problem 3. Find the inertia moment relatively the rotation axis of the paraboloid with radius of the basis equal to *R* and height – *H*.

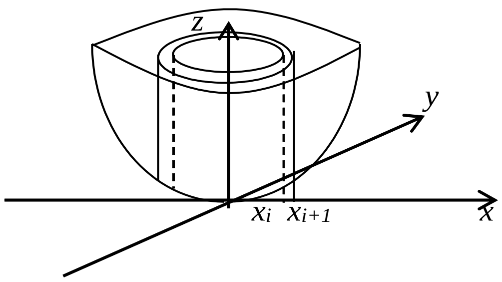
*Solution***.**



The paraboloid of rotation represents the surface, formed at gyration (rotation) of a parabolic segment (*R*, *H*) round the axis *OZ*. Equation of parabola, illustrated on the fig. 7.4, makes $x^2 = 2pz$. Let's put in the equation coordinates of the point A belonging to the mentioned parabola in order to define the parameter p. Then

$$R^2 = 2pH, \quad 2p = \frac{R^2}{H},$$

and equation of parabola can be presented as $x^2 = \frac{R^2}{H} z$. The equation of the surface of rotation (gyration) we'll get by substitution of x^2 on $x^2 + y^2$, i.e. the equation $z = \frac{H}{R^2}\left(x^2 + y^2\right)$ is the equation of the given paraboloid of rotation (gyration).



While solving the problems concerning calculation the inertia moment we should pay attention to the fact division on elementary subintervals should be carried out that all points of each selected segment should be approximately at similar distance from the rotation axis. In the given problem this can be achieved by division the paraboloid of gyration by a set of circumferential cylinders, which axes of rotation correspondent to the axis *OZ*. Then all points of the paraboloid of rotation, located between cylinders with radiuses x_i и x_{i+1} will be at almost similar distance from the axis of rotation due to the smallness of Δx_i.

The weight of the selected segment is approximately calculated as the weight of cylinder ring with width equal to Δx_i and height h_i.

$$h_i = H - Z(x_i,0) = H - \frac{H}{R^2} x_i^2 = \frac{H}{R^2}\left(R^2 - x_i^2\right).$$

By virtue of the created division the selected segment can be considered as a mass point with weight:

$$m_i \approx \pi\left(x_i + \Delta x_i\right)^2 h_i - \pi x_i^2 \approx 2\pi x_i \Delta x_i h_i = 2\pi x_i \frac{H}{R^2}\left(R^2 - x_i^2\right)\Delta x_i$$

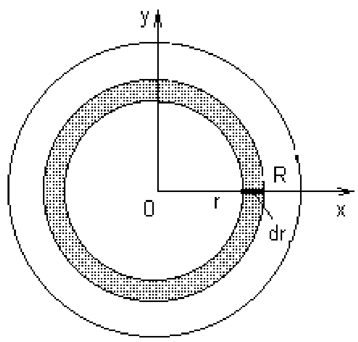
Then the inertia moment of the segment i is approximately equal to the following:

$$J_i \approx 2\pi \frac{H}{R^2} x_i^3\left(R^2 - x_i^2\right)\Delta x_i .$$

An exact meaning of the needed value we can get by summarizing I_i on all $i = 0,1,2,...,n-1$ and then passing to a limit in obtained integral sum while $\max \Delta x_i \to 0$.

$$J = \int_0^R 2\pi \frac{H}{R^2}\left(x^3 R^2 - x^5\right)dx = 2\pi \frac{H}{R^2}\left(\frac{x^4 R^2}{4} - \frac{x^6}{6}\right)\Bigg|_0^R = \frac{\pi R^4 H}{6}.$$

Problem 4. Calculate kinetic energy of the disk of mass M and radius R, rotating with angular speed ω round the axis, passing through its center perpendicularly to its plane.



The kinetic energy of an element of the disk makes:

$$dK = \frac{mV^2}{2} = \frac{\rho r^2\omega^2}{2} ds,$$

where r – distance between an element of the disk (circumferential ring) and the rotation axis

Area density of mass makes:

$$\rho = \frac{M}{\pi R^2},$$

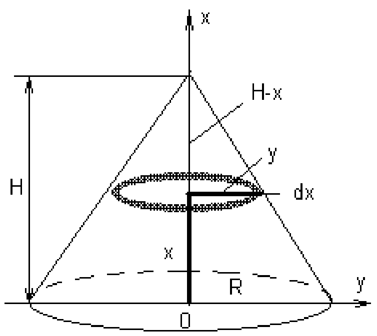
then

$$dK = \frac{M\omega^2}{2\pi R^2} r^2 ds; \quad ds = 2\pi r dr;$$

Kinetic energy of the disk is equal to

$$K = \frac{M\omega^2}{R^2}\int_0^R r^3 dr = \frac{M\omega^2}{R^2}\left.\frac{r^4}{4}\right|_0^R = \frac{M\omega^2 R^2}{4}.$$

Problem 5. Find the static moment of the homogeneous cone with radius R of its basis and height H relatively its basis.



From similarity of triangles we have:

$$\frac{y}{R} = \frac{H-x}{H},$$

whence $y = R\left(1 - \dfrac{x}{H}\right)$.

The division is carried out the way the element of the volume (shaded) is located on the similar distance from the basis.

We have $dV = \pi y^2 dx = \pi R^2\left(1 - \dfrac{x}{H}\right)^2 dx.$

An elementary static moment makes:

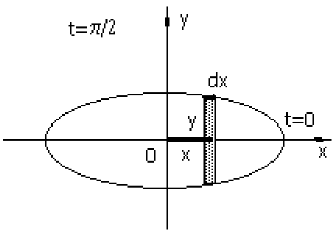
$$dM = \pi R^2\left(1 - \frac{x}{H}\right)^2 x dx,$$

whence

$$M=\pi R^2\int_0^H x\left(1-\frac{x}{H}\right)^2dx=\pi R^2\int_0^H x\left(1-\frac{2x}{H}+\frac{x^2}{H^2}\right)dx=$$

$$=\pi R^2\left(\frac{x^2}{2}-\frac{2x^3}{3H}+\frac{x^4}{4H^2}\right)\Bigg|_0^H=\pi R^2\left(\frac{H^2}{2}-\frac{2}{3}H^2+\frac{H^2}{4}\right)=\frac{\pi R^2H^2}{12}.$$

Problem 6. Find the inertia moment of area of the ellipse $x=a\cos t,\ y=b\sin t$ relatively the axis *0Y*. Let's consider $\rho=1$.



As it is clear from the figure

$$I_y=2\int_0^a x^2dS,$$

where

$$dS=2ydx=-2b\sin t\,a\sin t dt=-2ab\sin^2 t dt$$

an element of the area. By virtue of equality of area density to 1 we have $dS=dm$.

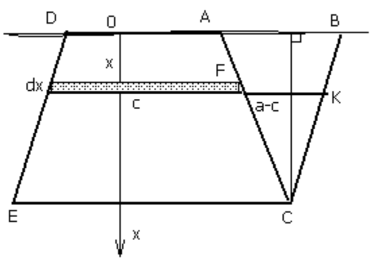
Therefore

$$I_y=-4ab\int_{\pi/2}^{0}a^2\cos^2 t\sin^2 t dt=a^3b\int_0^{\pi/2}\sin^2 2t dt=$$

$$=a^3b\int_0^{\pi/2}\frac{1-\cos 4t}{2}dt=a^3b\left(\frac{t}{2}-\frac{\sin 4t}{8}\right)\Bigg|_0^{\pi/2}=\frac{\pi a^3b}{4}.$$

Problem 7. Find the force of water pressure on a vertical wall, having the form of a trapezoid, which lower basis is *a*, upper – *b*, height *H*, basis *b* is on the surface of the water $(a>b)$.

According to the Pascal law the liquid pressure on a horizontal plate, immersed in it, is equal to the weight of a liquid pile, resting on this plate, i.e. product of the plate area on the distance between the plate and free surface of the liquid and on the specific weight of the liquid. Let's divide the plate into elementary layers, placed on the diving depth equal to *x*. ($AD=b,\ EC=a$).



$$\Delta ABC\sim\Delta FKC,$$

then

$$\frac{a-c}{a-b}=\frac{H-x}{H}=1-\frac{x}{H};\ c=b+(a-b)\frac{x}{H}.$$

The pressure of the liquid on elementary layer dx makes

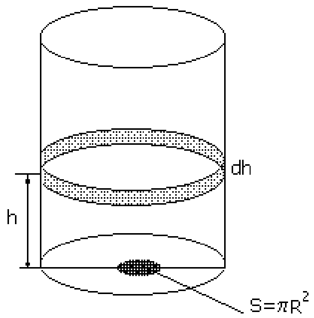
$$dP = \left((b + a - b) \frac{x}{H} xdx \right) \text{ (specific weight of the water 1)},$$

then for calculating the pressure influencing the whole plate we should integrate the pressure element dP in the limits of its changing from 0 to H. We obtain the following:

$$P = \int_0^H \left(b + (a - b) \frac{x}{H} \right) xdx = \int_0^H \left(bx + \frac{ax^2}{H} - \frac{b}{H} x^2 \right) dx =$$

$$= \left(b \frac{x^2}{2} + \frac{(a - b)}{H} \frac{x^3}{3} \right) \Bigg|_0^H = \frac{b}{2} H^2 + \frac{(a - b) H^3}{3H} = \frac{b + 2a}{6} H^2.$$

Problem 8. In what time will be bleeded (will become empty) the vertical cylindrical barrel filled up to the top with radius R, height (altitude) H through a round hole with radius r in the bottom of the barrel?



According to the law of Torricelli the speed of a liquid outflow is equal to $v = k\sqrt{2gh}$, where h – height of the liquid level above the hole. The amount of flowing out liquid at the time dt is equal to the volume $\pi R^2 dh$, where πR^2 is an area of the cylinder basis. Then the speed of a liquid outflow through the hole with area $S = \pi r^2$ will be equal

to $\frac{\pi r^2 dh}{dt}$, and from another side $kS\sqrt{2gh}.$ Here we obtain the equality

$$\frac{\pi R^2 dh}{dt} = k\pi r^2 \sqrt{2gh},$$

whence

$$dt = \frac{\pi R^2 dh}{k\pi r^2 \sqrt{2gh}} = \frac{R^2}{kr^2 \sqrt{2g}} \frac{dh}{\sqrt{h}}.$$

The container will be bleeded (become empty) while changing h from 0 to H. Integrating the obtained equality we find the needed time T.

$$T = \frac{R^2}{kr^2 \sqrt{2g}} \int_0^H \frac{dh}{\sqrt{h}} = \frac{R^2}{kr^2 \sqrt{2g}} 2\sqrt{h}\Big|_0^H = \sqrt{\frac{2H}{g}} \frac{R^2}{kr^2}.$$