3. Improper Integrals

We extend the concept of the definite integral $\int_a^b f(x)dx$ to improper integrals.

There are two types of improper integrals:

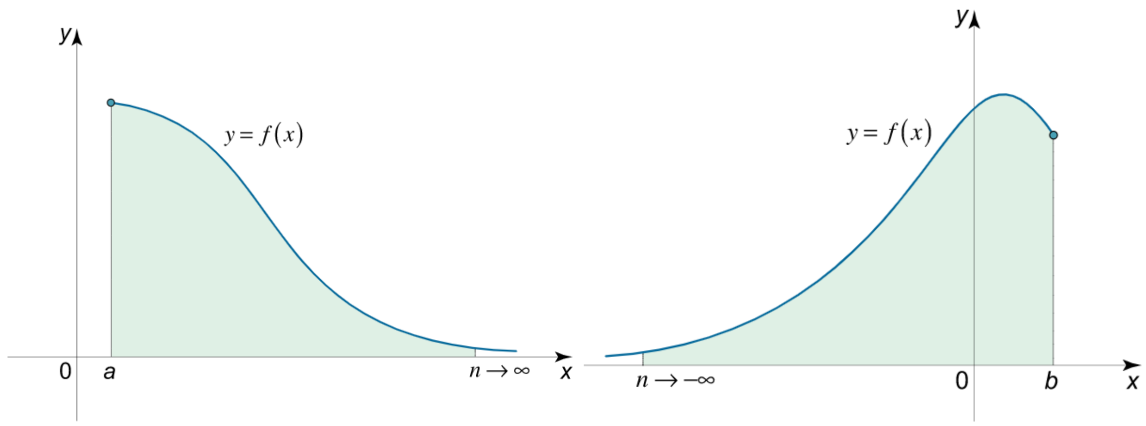
1. The limit a or b (or both the limits) are *infinite*;
2. The function $f(x)$ has one or more points of *discontinuity* in the interval $[a, b]$.

3.1 Integration over an Infinite Domain

Let $f(x)$ be a continuous function on the interval $[a, \infty)$. We define the improper integral as $\int_a^\infty f(x)dx$.

In order to integrate over the infinite domain [a,∞), we consider the limit of the form

$$\int_a^\infty f(x)dx = \lim_{n\to\infty} \int_a^n f(x)dx.$$



(a) (b)

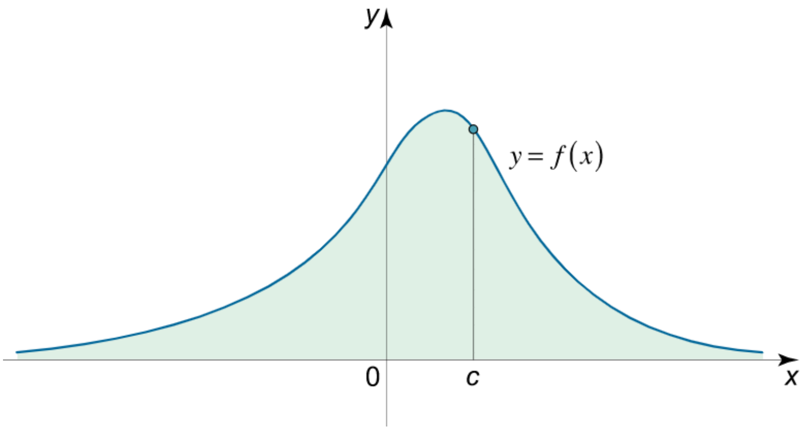
Similarly, if a continuous function $f(x)$ is given on the interval $(-\infty, b]$, the improper integral of $f(x)$ is defined as

$$\int_{-\infty}^b f(x)dx = \lim_{n\to-\infty} \int_n^b f(x)dx.$$

If these limits exist and are finite then we say that the improper integrals are convergent. Otherwise the integrals are divergent.

An improper integral might have two infinite limits. In this case, we can pick an arbitrary point c and break the integral up there. As a result, we obtain two improper integrals, each with one infinite limit:

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_{c}^{\infty} f(x)dx.$$



If, for some real number c, both of the integrals in the right-hand side are convergent, then we say that the integral $\int_{-\infty}^{\infty} f(x)dx$ is also convergent; otherwise it is divergent.

Comparison Tests

Let $f(x)$ and $g(x)$ be continuous functions on the interval $[a,\infty)$. Suppose that $0 \le g(x) \le f(x)$ or all x in the interval $[a,\infty)$.

1. If $\int_a^\infty f(x)dx$ is convergent, then $\int_a^\infty g(x)dx$ is also convergent;
2. If $\int_a^\infty g(x)dx$ is divergent, then $\int_a^\infty f(x)dx$ is also divergent;
3. If $\int_a^\infty |f(x)|dx$ is convergent, then $\int_a^\infty f(x)dx$ is also convergent. In this case, we say that the integral $\int_a^\infty f(x)dx$ is absolutely convergent.

It is often convenient to make comparisons with improper integrals of the form:

$$\int_1^\infty \frac{dx}{x^p},$$

where $p > 0$ is a real number.

The integral $\int_1^{\infty} \frac{dx}{x^p}$ converges if $p > 1$, and diverges if $p < 1$. If $p = 1$, then the integral also diverges:

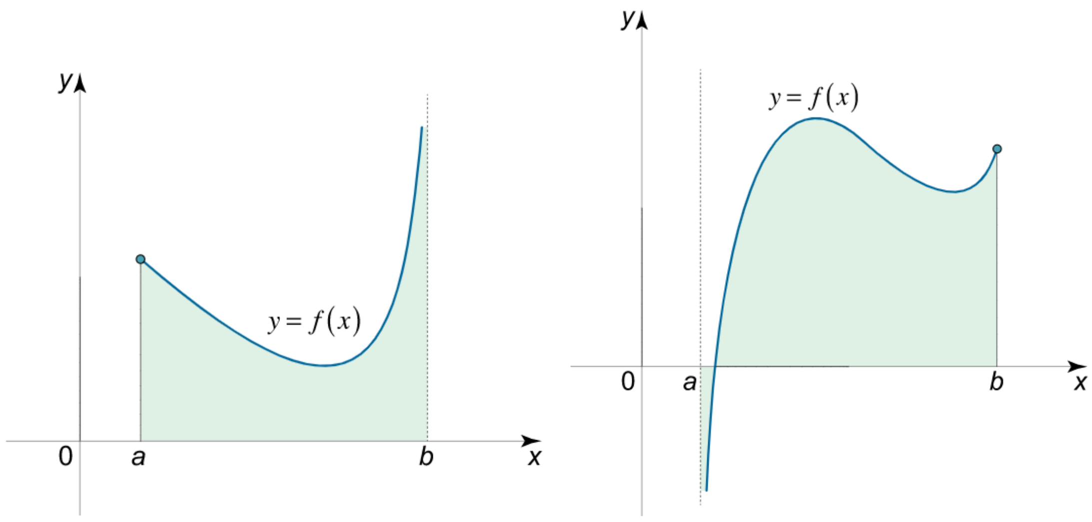
$$\lim_{n\to\infty} \int_1^n \frac{dx}{x^p} = \lim_{n\to\infty} \int_1^n \frac{dx}{x} = \lim_{n\to\infty} \ln x \,|_1^n = \infty.$$

3.2 Improper Integrals with Infinite Discontinuities

This type of improper integrals refers to integrands that are undefined at one or more points of the domain of integration $[a, b]$.

Let $f(x)$ be a function which is continuous on the interval $[a, b)$, but is discontinuous at $x = b$. We define the improper integral as

$$\int_a^b f(x)dx = \lim_{\varepsilon\to 0+} \int_a^{b-\varepsilon} f(x)dx.$$



(a) (b)

Similarly we can consider the case when the function $f(x)$ is continuous on the interval $(a, b]$, but is discontinuous at $x = a$. Then

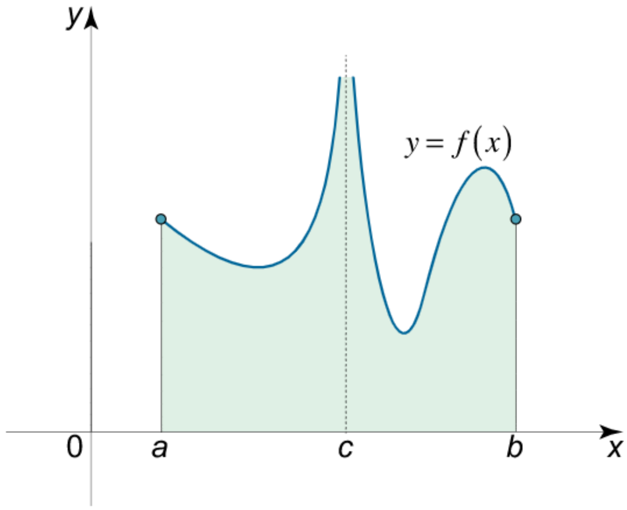
$$\int_a^b f(x)dx = \lim_{\varepsilon\to 0+} \int_{a+\varepsilon}^{b} f(x)dx.$$

If these limits exist and are finite then we say that the integrals are convergent;

otherwise the integrals are divergent.

Finally, if the function $f(x)$ is continuous on $[a,c) \cup (c,b]$ with an infinite discontinuity at $x = c$, then we define the improper integral as

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx,$$



We say that the integral $\int_a^b f(x)dx$ is convergent if both of the integrals in the right side are also convergent. Otherwise the improper integral is divergent.

Solved Problems

Example 1. Calculate the integral $\int_1^\infty \frac{dx}{x^2+1}$.

$$\int_1^\infty \frac{dx}{x^2+1} = \lim_{n\to\infty} \int_1^n \frac{dx}{x^2+1} = \lim_{n\to\infty} [\arctan x]_1^n = \lim_{n\to\infty} [\arctan n - \arctan 1] = \frac{\pi}{2} - \frac{\pi}{4}$$
$$= \frac{\pi}{4}.$$

Hence, the integral converges.

Example 2. Calculate the integral $\int_0^\infty \frac{dx}{x^2+16}$.

$$\int_0^{\infty} \frac{dx}{x^2+16} = \lim_{n\to\infty} \int_0^{n} \frac{dx}{x^2+16} = \lim_{n\to\infty} \int_0^{n} \frac{dx}{x^2+4^2} = \lim_{n\to\infty} (\frac{1}{4}\arctan\frac{x}{4})|_0^n$$

$$= \frac{1}{4}\lim_{n\to\infty}(\arctan\frac{n}{4} - \arctan\frac{0}{4}) = \frac{1}{4}\lim_{n\to\infty}(\arctan\frac{n}{4} - 0) = \frac{1}{4}\cdot\frac{\pi}{2} = \frac{\pi}{8}.$$

The given integral converges.

Example 3. Calculate the integral $\int_{-\infty}^{\infty} \frac{dx}{x^2+4}$.

The original integral has two infinite limits. Therefore we split it into two integrals and evaluate each as a one-sided improper integral:

$$I = \int_{-\infty}^{\infty} \frac{dx}{x^2+4} = \int_{-\infty}^{0} \frac{dx}{x^2+2^2} + \int_0^{\infty} \frac{dx}{x^2+2^2} = I_1 + I_2.$$

Calculate each integral:

$$I_1 = \int_{-\infty}^{0} \frac{dx}{x^2+2^2} = \lim_{n\to-\infty} \int_n^{0} \frac{dx}{x^2+2^2} = \lim_{n\to-\infty} [\frac{1}{2}\arctan\frac{x}{2}]_n^0$$

$$= \frac{1}{2}\lim_{n\to-\infty}[\arctan 0 - \arctan\frac{n}{2}] = \frac{1}{2}(0-(-\frac{\pi}{2})) = \frac{\pi}{4};$$

$$I_2 = \int_0^{\infty} \frac{dx}{x^2+2^2} = \lim_{n\to\infty} \int_0^{n} \frac{dx}{x^2+2^2} = \lim_{n\to\infty} [\frac{1}{2}\arctan\frac{x}{2}]_0^n$$

$$= \frac{1}{2}\lim_{n\to-\infty}[\arctan\frac{n}{2} - \arctan 0] = \frac{1}{2}(\frac{\pi}{2} - 0) = \frac{\pi}{4}.$$

Hence,

$$I = I_1 + I_2 = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}.$$

We see that the integral converges.

Example 4. Determine whether the integral $\int_1^{\infty} \frac{dx}{x^2 e^x}$ converges or diverges?

Note, that $\frac{1}{x^2 e^x} \le \frac{1}{x^2}$ for all values $x \ge 1$. Since the improper integral $\int_1^{\infty} \frac{dx}{x^2}$ is convergent then the given integral is also convergent by Comparison Test 1.

Example 5. Determine whether the integral $\int_1^\infty \frac{\sin x}{\sqrt{x^3}}\,dx$ converges or diverges?

We can write the obvious inequality for the absolute values:

$$|\frac{\sin x}{\sqrt{x^3}}| \le |\frac{1}{\sqrt{x^3}}| = |\frac{1}{x^{\frac{3}{2}}}|.$$

It's easy to show that the integral $\int_1^\infty |\frac{1}{\sqrt{x^3}}|\,dx$ converges. Indeed

$$\int_1^\infty |\frac{1}{\sqrt{x^3}}|dx = \int_1^\infty \frac{dx}{\sqrt{x^3}} = \int_1^\infty \frac{dx}{x^{\frac{3}{2}}} = \int_1^\infty x^{-\frac{3}{2}}dx = \lim_{n\to\infty}\int_1^n x^{-\frac{3}{2}}dx = \lim_{n\to\infty}(\frac{x^{-\frac{3}{2}+1}}{-\frac{3}{2}+1})|_1^n$$

$$= -2\lim_{n\to\infty}(\frac{1}{\sqrt{x}})|_1^n = -2\lim_{n\to\infty}(\frac{1}{\sqrt{n}} - 1) = -2(0-1) = 2.$$

Then we conclude that the integral $\int_1^\infty |\frac{\sin x}{\sqrt{x^3}}|\,dx$ also converges by Comparison Test 1.

Example 6. Calculate the integral $\int_{-2}^{2} \frac{dx}{x^3}$.

There is a discontinuity at $x = 0$, so that we must consider two improper integrals:

$$\int_{-2}^{2} \frac{dx}{x^3} = \int_{-2}^{0} \frac{dx}{x^3} + \int_{0}^{2} \frac{dx}{x^3}.$$

Using the definition of improper integral, we obtain

$$\int_{-2}^{2} \frac{dx}{x^3} = \int_{-2}^{0} \frac{dx}{x^3} + \int_{0}^{2} \frac{dx}{x^3} = \lim_{\varepsilon\to 0+}\int_{-2}^{-\varepsilon} \frac{dx}{x^3} + \lim_{\varepsilon\to 0+}\int_{\varepsilon}^{2} \frac{dx}{x^3}.$$

For the first integral,

$$\lim_{\varepsilon\to 0+}\int_{-2}^{-\varepsilon}\frac{dx}{x^3}=\lim_{\varepsilon\to 0+}\left(\frac{x^{-2}}{-2}\right)\Big|_{-2}^{-\varepsilon}=-\frac{1}{2}\lim_{\varepsilon\to 0+}\left(\frac{1}{x^2}\right)\Big|_{-2}^{-\varepsilon}=-\frac{1}{2}\lim_{\varepsilon\to 0+}\left[\frac{1}{(-\varepsilon)^2}-\frac{1}{(-2)^2}\right]$$

$$=-\frac{1}{2}\lim_{\varepsilon\to 0+}\left(\frac{1}{\varepsilon^2}+\frac{1}{8}\right)=\infty.$$

Since it is divergent, the initial integral also diverges.

Example 7. Determine whether the improper integral $\int_{-\infty}^{\infty}\frac{dx}{x^2+2x+8}$ converges or diverges?

We can write this integral as

$$I=\int_{-\infty}^{\infty}\frac{dx}{x^2+2x+8}=\int_{-\infty}^{0}\frac{dx}{x^2+2x+8}+\int_{0}^{\infty}\frac{dx}{x^2+2x+8}.$$

By the definition of an improper integral, we have

$$I=\int_{-\infty}^{\infty}\frac{dx}{x^2+2x+8}=\int_{-\infty}^{0}\frac{dx}{x^2+2x+8}+\int_{0}^{\infty}\frac{dx}{x^2+2x+8}$$

$$=\lim_{M\to-\infty}\int_{M}^{0}\frac{dx}{x^2+2x+8}+\lim_{N\to\infty}\int_{0}^{N}\frac{dx}{x^2+2x+8}$$

$$=\lim_{M\to-\infty}\int_{M}^{0}\frac{dx}{(x+1)^2+7}+\lim_{N\to\infty}\int_{0}^{N}\frac{dx}{(x+1)^2+7}$$

$$=\lim_{M\to-\infty}\left(\frac{1}{\sqrt{7}}\arctan\frac{x+1}{\sqrt{7}}\right)\Big|_{M}^{0}+\lim_{N\to\infty}\left(\frac{1}{\sqrt{7}}\arctan\frac{x+1}{\sqrt{7}}\right)\Big|_{0}^{N}$$

$$=\frac{1}{\sqrt{7}}\left(\arctan\frac{1}{\sqrt{7}}-\lim_{M\to-\infty}\arctan\frac{M+1}{\sqrt{7}}\right)+\frac{1}{\sqrt{7}}\left(\lim_{N\to\infty}\arctan\frac{N+1}{\sqrt{7}}\right.$$

$$\left.-\arctan\frac{1}{\sqrt{7}}\right)=\cancel{\frac{1}{\sqrt{7}}\arctan\frac{1}{\sqrt{7}}}-\frac{1}{\sqrt{7}}\cdot\left(-\frac{\pi}{2}\right)+\frac{1}{\sqrt{7}}\cdot\frac{\pi}{2}-\cancel{\frac{1}{\sqrt{7}}\arctan\frac{1}{\sqrt{7}}}$$

$$=\frac{1}{\sqrt{7}}\cdot\frac{\pi}{2}+\frac{1}{\sqrt{7}}\cdot\frac{\pi}{2}=\frac{\pi}{\sqrt{7}}.$$

As both the limits exist and are finite, the given integral converges.

Example 8. Determine whether the integral $\int_0^4 \frac{dx}{(x-2)^3}$ converges or diverges?

There is a discontinuity in the integrand at $x = 2$, so that we must consider two improper integrals:

$$\int_0^4 \frac{dx}{(x-2)^3} = \int_0^2 \frac{dx}{(x-2)^3} + \int_2^4 \frac{dx}{(x-2)^3}.$$

Using the definition of an improper integral, we obtain

$$\int_0^2 \frac{dx}{(x-2)^3} + \int_2^4 \frac{dx}{(x-2)^3} = \lim_{\varepsilon\to 0+} \int_0^{2-\varepsilon} \frac{dx}{(x-2)^3} + \lim_{\varepsilon\to 0+} \int_{2+\varepsilon}^4 \frac{dx}{(x-2)^3}.$$

For the first integral,

$$\lim_{\varepsilon\to 0+} \int_0^{2-\varepsilon} \frac{dx}{(x-2)^3} = \lim_{\varepsilon\to 0+} \int_0^{2-\varepsilon} (x-2)^{-3} d(x-2) = \lim_{\varepsilon\to 0+} [\frac{(x-2)^{-3+1}}{-3+1}]_0^{2-\varepsilon}$$

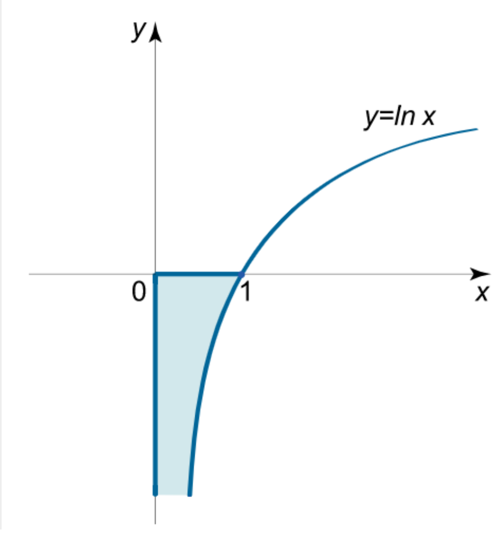
$$= -\frac{1}{2} \lim_{\varepsilon\to 0+} [\frac{1}{(x-2)^2}]|_0^{2-\varepsilon} = -\frac{1}{2} \lim_{\varepsilon\to 0+} [\frac{1}{(2-\varepsilon-2)^2} - \frac{1}{(0-2)^2}]$$

$$= -\frac{1}{2} \lim_{\varepsilon\to 0+} (\frac{1}{\varepsilon^2} - \frac{1}{4}) = -\infty.$$

As it is divergent, the given integral $\int_0^4 \frac{dx}{(x-2)^3}$ is also divergent.

Example 9. Find the area above the curve $y = lnx$ in the lower half-plane between $x = 0$ and $x = 1$.

The given region is sketched in Figure



Since it is infinite, we calculate the improper integral to find the area:

$$\int_0^1 \ln x dx = \lim_{\varepsilon \to 0+} \int_\varepsilon^1 \ln x dx.$$

Use integration by parts. Let $u = \ln x$, $dv = dx$. Then $du = \frac{dx}{x}$, $v = x$.

Thus

$$\lim_{\varepsilon \to 0+} \int_\varepsilon^1 \ln x dx = \lim_{\varepsilon \to 0+} [(x \ln x)|_\varepsilon^1 - \int_\varepsilon^1 x \frac{dx}{x}] = \lim_{\varepsilon \to 0+} [x \ln x - x]|_\varepsilon^1$$

$$= \lim_{\varepsilon \to 0+} [(\ln 1 - 1) - (\varepsilon \ln \varepsilon - \varepsilon)] = (0 - 1) - \lim_{\varepsilon \to 0+} [\varepsilon(\ln \varepsilon - 1)]$$

$$= -1 - \lim_{\varepsilon \to 0+} \frac{\ln \varepsilon - 1}{\frac{1}{\varepsilon}}.$$

We can apply L'Hopital's rule to find the limit:

$$\lim_{\varepsilon \to 0+} \frac{\ln \varepsilon - 1}{\frac{1}{\varepsilon}} = \lim_{\varepsilon \to 0+} \frac{\frac{1}{\varepsilon}}{-\frac{1}{\varepsilon^2}} = -\lim_{\varepsilon \to 0+} \frac{\varepsilon^2}{\varepsilon} = -\lim_{\varepsilon \to 0+} \varepsilon = 0.$$

Hence, the improper integral is

$$\lim_{\varepsilon \to 0+} \int_\varepsilon^1 \ln x dx = -1 - \lim_{\varepsilon \to 0+} \frac{\ln \varepsilon - 1}{\frac{1}{\varepsilon}} = -1 - 0 = -1.$$

As you can see from the figure above, the required area is

$$A = |\lim_{\varepsilon \to 0+} \int_\varepsilon^1 \ln x dx|$$

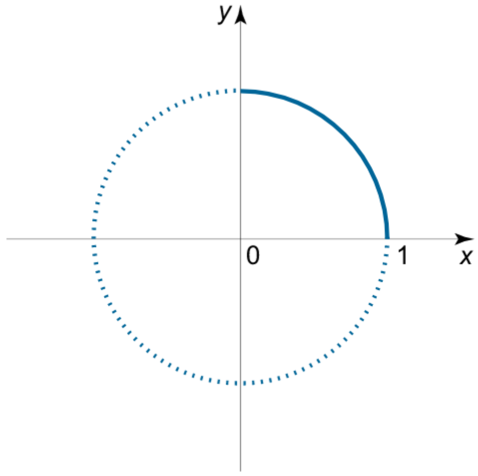
Example 10. Find the circumference of the unit circle.

We calculate the length of the arc of the circle in the first quadrant between $x = 0$ and $x = 1$ and then multiply the result by 4. The equation of the circle centered at the origin is

$$x^2 + y^2 = 1.$$

Then the arc of the circle in the first quadrant (Figure) is described by the function

$$y = \sqrt{1 - x^2}, 0 \leq x \leq 1.$$



Find the derivative of the function:

$$y' = \frac{d}{dx}\sqrt{1-x^2} = \frac{-\cancel{2}x}{\cancel{2}\sqrt{1-x^2}} = \frac{-x}{\sqrt{1-x^2}}.$$

Since the length of an arc is given by $\int_{x=\alpha}^{x=\beta}\sqrt{1+(y')^2}dx$, we obtain

$$\int_0^1 \sqrt{1+\left(\frac{-x}{\sqrt{1-x^2}}\right)^2}dx = \int_0^1 \sqrt{1+\frac{x^2}{1-x^2}}dx = \int_0^1 \sqrt{\frac{1-\cancel{x^2}+\cancel{x^2}}{1-x^2}}dx$$

$$= \int_0^1 \frac{dx}{\sqrt{1-x^2}}.$$

Now we calculate the improper integral $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$:

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{\varepsilon\to 0+}\int_0^{1-\varepsilon}\frac{dx}{\sqrt{1-x^2}} = \lim_{\varepsilon\to 0+}(\arcsin x)|_0^{1-\varepsilon}$$

$$= \lim_{\varepsilon\to 0+}[\arcsin(1-\varepsilon) - \arcsin 0] = \arcsin 1 - 0 = \frac{\pi}{2}.$$

Thus, the circumference of the unit circle is $\frac{\pi}{2}\cdot 4$