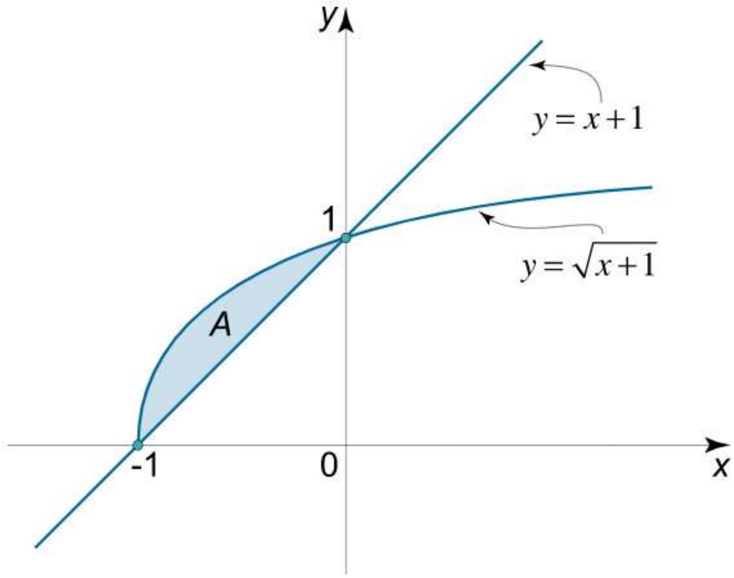
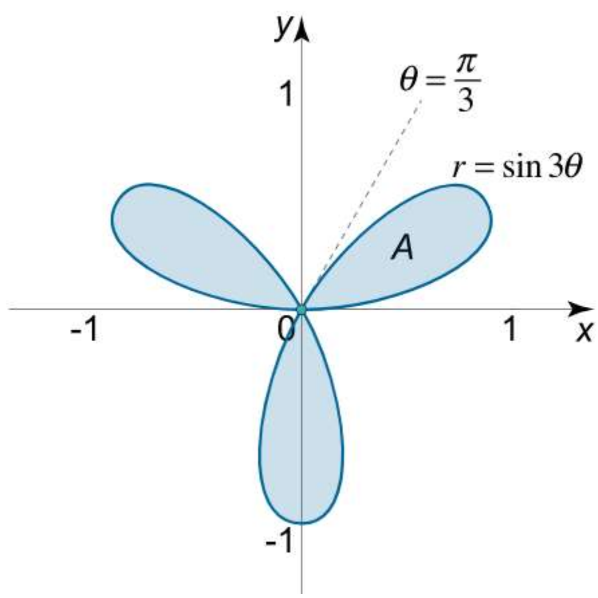
Example 1. Find the area of the region enclosed by the curve $y = \sqrt{x+1}$ and the line $y = x + 1$.




Then, $A = \int_{-1}^{0}[\sqrt{x+1} - (x+1)]dx = \frac{2(x+1)^{\frac{3}{2}}}{3} - \frac{x^2}{2} - x|_{-1}^{0} = (\frac{2}{3} - 0 - 0) - (0 - \frac{1}{2} + 1) = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$.

Example 2. Find the area enclosed by the three petaled rose $\rho = sin3\theta$



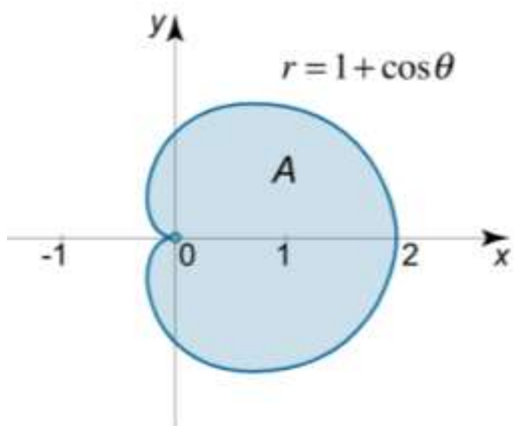

Since each petal has the same area, we calculate the area of one petal and multiply the result by three. So we have

$$A = \frac{1}{2}\int_0^{\frac{\pi}{3}} r^2(\theta)d\theta = \frac{1}{2}\int_0^{\frac{\pi}{3}} \sin^2(3\theta)d\theta = \frac{1}{4}\int_0^{\frac{\pi}{3}} [1 - \cos(6\theta)]d\theta$$

$$= \frac{1}{4}[\theta - \frac{\sin(6\theta)}{6}]|_0^{\frac{\pi}{3}} = \frac{1}{4}\cdot\frac{\pi}{3} = \frac{\pi}{12}$$

Hence, the area of the all region is $\frac{\pi}{4}$ (units$^2$).

Example 3. Find the area enclosed by the cardioid $\rho = 1 + cos\theta$




We can easily the area of the cardioid by integrating the polar equation in the interval $[0,2\pi]$. This yields:

$$A = \frac{1}{2}\int_0^{2\pi} r^2(\theta)d\theta = \frac{1}{2}\int_0^{2\pi} (1 + \cos\theta)^2 d\theta = \frac{1}{2}\int_0^{2\pi} (1 + 2\cos\theta + \cos^2\theta)d\theta$$

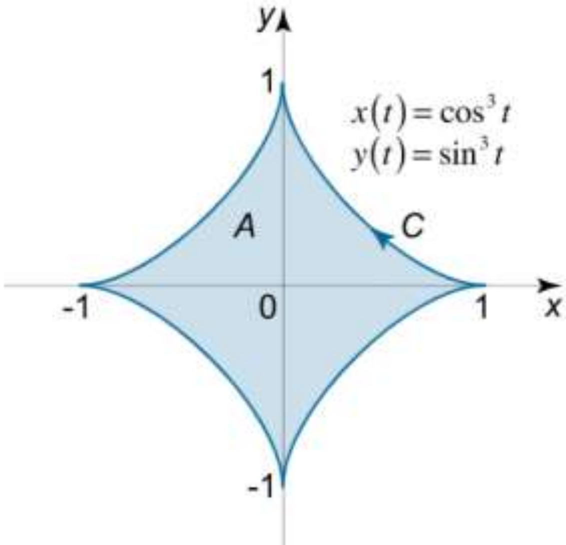
$$= \frac{1}{2}\int_0^{2\pi} (1 + 2\cos\theta + \frac{1 + \cos 2\theta}{2})d\theta$$

$$= \frac{1}{4}\int_0^{2\pi} (3 + 4\cos\theta + \cos 2\theta)d\theta = \frac{1}{4}[3\theta + 4\sin\theta + \frac{\sin 2\theta}{2}]|_0^{2\pi}$$

$$= \frac{1}{4}\cdot 6\pi = \frac{3\pi}{2}$$

Example 4. Find the area of the region bounded by the asteroid. We represent the equation of the astroid in parametric form:

$$x(t) = \cos^3 t, y(t) = \sin^3 t.$$




We apply the following integration formula:

$$A = \frac{1}{2}\int_0^T [x(t)y'(t) - x'(t)y(t)]dt.$$

As

$$x'(t) = -3\cos^2 t\sin t, y'(t) = 3\sin^2 t\cos t,$$

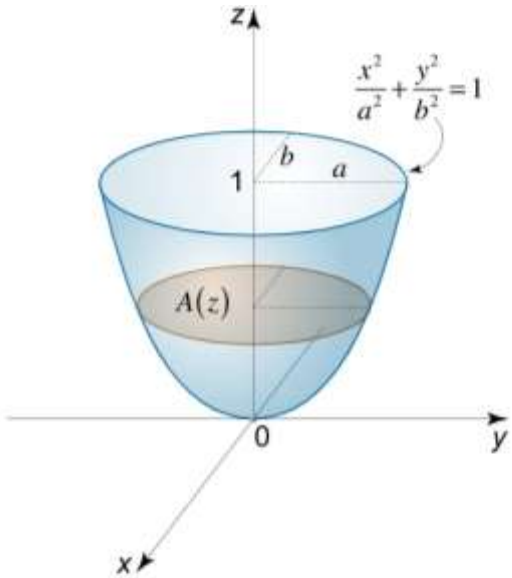
we have

$$A = \frac{1}{2}\int_0^{2\pi} [x(t)y'(t) - x'(t)y(t)]dt = \frac{1}{2}\int_0^{2\pi} [3\cos^4 t\sin^2 t + 3\cos^2 t\sin^4 t]dt$$

$$= \frac{3}{2}\int_0^{2\pi} [\cos^2 t\sin^2 t\underbrace{(\cos^2 t + \sin^2 t)}_{1}]dt = \frac{3}{8}\int_0^{2\pi} \sin^2(2t)dt$$

$$= \frac{3}{16}\int_0^{2\pi} [1 - \cos(4t)]dt = \frac{3}{16}[t - \frac{\sin(4t)}{4}]|_0^{2\pi} = \frac{3}{16}\cdot 2\pi = \frac{3\pi}{8}$$

Example 5.Find the volume of a solid bounded by the elliptic paraboloid $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ and the plane $z = 1$.




Consider an arbitrary planar section perpendicular to the $z$–axis at a point $z$, where $0 < z \leq 1$. The cross section is an ellipse defined by the equation

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}, \Rightarrow \frac{x^2}{(a\sqrt{z})^2} + \frac{y^2}{(b\sqrt{z})^2} = 1.$$

The area of the cross section is

$$A(z) = \pi \cdot (a\sqrt{z}) \cdot (b\sqrt{z}) = \pi abz.$$

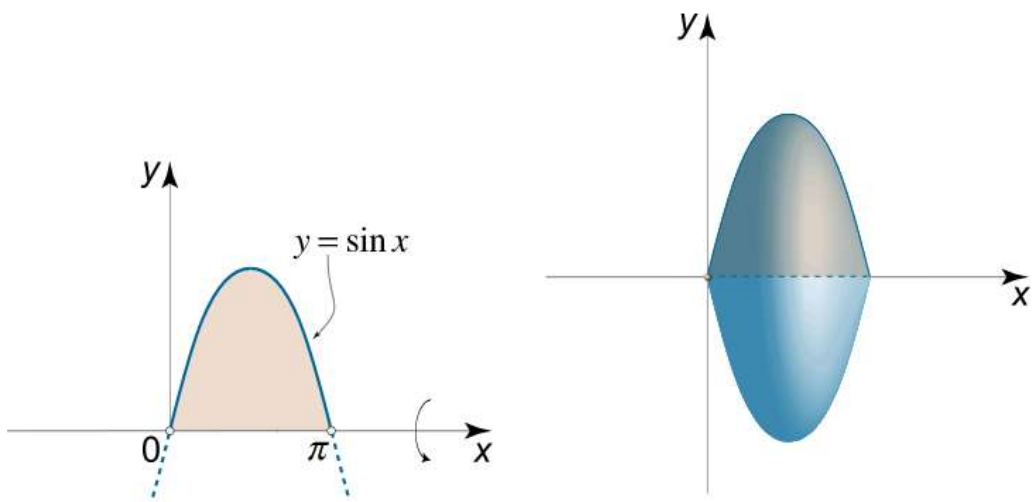
Then, by the slice method where the cross-section formula is known,

$$V = \int_0^1 A(z)dz = \int_0^1 \pi abzdz = \pi ab \int_0^1 zdz = \pi ab \cdot \frac{z^2}{2}\Big|_0^1 = \frac{\pi ab}{2}$$

Example 6. Find the volume of the solid obtained by rotating the sine function between $x = 0$ and $x = \pi$ about the $x$–axis.

By the disk method,

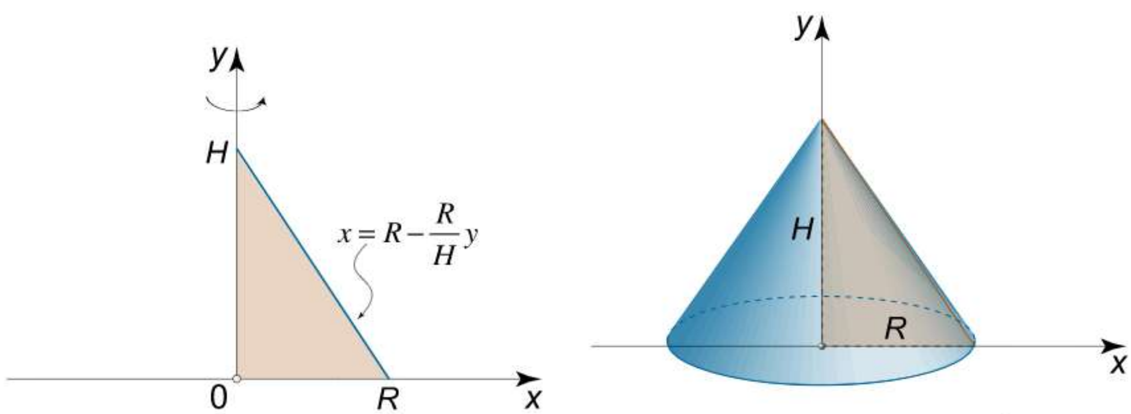
$$V = \pi \int_0^\pi [\sin x]^2 dx = \frac{\pi}{2} \int_0^\pi (1 - \cos 2x)dx = \frac{\pi}{2}(x - \frac{\sin 2x}{2})\Big|_0^\pi$$




$$= \frac{\pi}{2}[(\pi - 0) - (0 - 0)] = \frac{\pi^2}{2}.$$

Example 6. Calculate the volume of the right circular cone of height $H$ and base radius $R$.

The slant height of the cone is defined by the equation: $x = R - \frac{R}{H}y$.

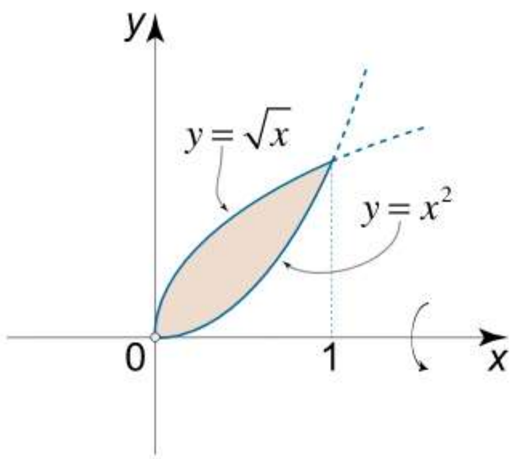


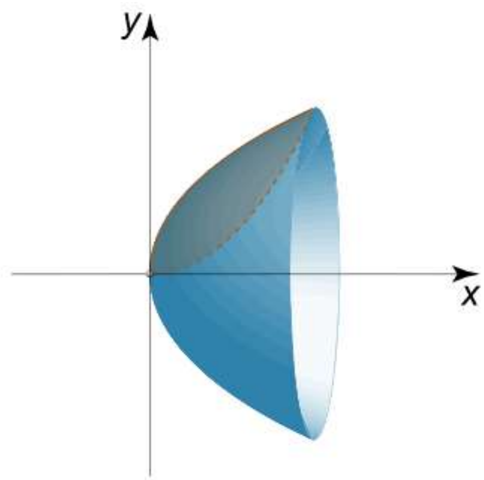

Hence, the volume of the cone is given by

$$V = \pi\int_0^H [x(y)]^2 dy = \pi\int_0^H [R - \frac{R}{H}y]^2 dy = \pi R^2\int_0^H (1 - \frac{2y}{H} + \frac{y^2}{H^2})dy$$

$$= \pi R^2(H - \frac{y^2}{H} + \frac{y^3}{3H^2})\Big|_0^H = \pi R^2(\cancel{H} - \cancel{H} + \frac{H}{3}) = \frac{\pi R^2 H}{3}.$$

Example 7. Calculate the volume of the solid obtained by rotating the region bounded by the parabola $y = x^2$ and the square root function $y = \sqrt{x}$ around the $x$–axis.




Both curves intersect at the points $x = 0$ and $x = 1$. Using the washer method, we have

$$V = \pi \int_0^1 ([\sqrt{x}]^2 - [x^2]^2)dx = \pi \int_0^1 (x - x^4)dx = \pi(\frac{x^2}{2} - \frac{x^5}{5})|_0^1 = \pi\left(\frac{1}{2} - \frac{1}{5}\right)$$
$$= \frac{3\pi}{10}.$$

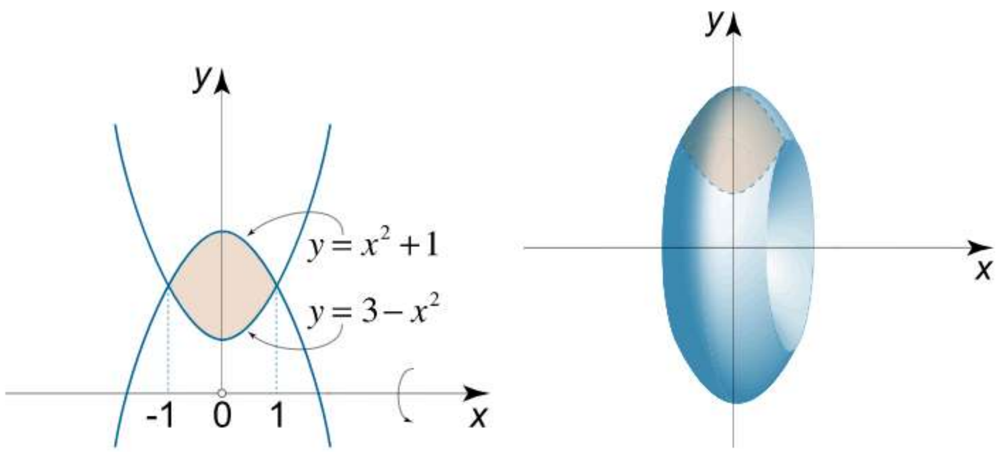
Example 8. Find the volume of the solid obtained by rotating the region bounded by two parabolas $y = x^2 + 1$ and $y = 3 - x^2$ about the $x$–axis.

First we determine the boundaries $a$ and $b$:

$$x^2 + 1 = 3 - x^2, \Rightarrow 2x^2 = 2, \Rightarrow x^2 = 1, \Rightarrow x_{1,2} = \pm 1.$$

Hence the limits of integration are $a = -1$, $b = 1$. We sketch the bounding region and the solid of revolution. Using the washer method, we find the volume of the solid:

$$V = \pi \int_a^b ([f(x)]^2 - [g(x)]^2)dx = \pi \int_{-1}^1 ((3 - x^2)^2 - (x^2 + 1)^2)dx$$




$$= \pi \int_{-1}^{1} ([3-x^2]^2 - [x^2+1]^2)dx = \pi \int_{-1}^{1} (8-8x^2)dx = 8\pi \int_{-1}^{1} (1-x^2)dx$$

$$= 8\pi(x - \frac{x^3}{3})|_{-1}^{1} = 8\pi[(1-\frac{1}{3}) - (-1+\frac{1}{3})] = 8\pi \cdot \frac{4}{3} = \frac{32\pi}{3}$$

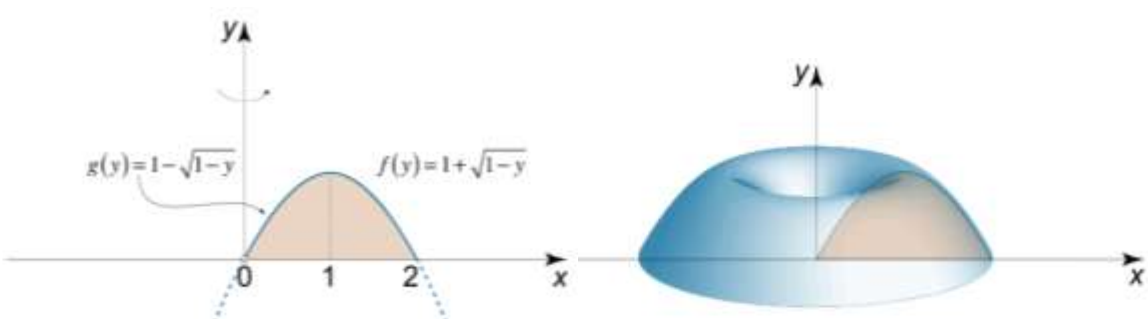
Example 9. Calculate the volume of the solid obtained by rotating the region bounded by the curve $y = 2x - x^2$ and the $x$–axis about the $y$–axis.

Find the points of intersection of the parabola with the $x$–axis:

$$2x - x^2 = 0, \Rightarrow x(2-x) = 0, \Rightarrow x_1 = 0, x_2 = 2.$$

As the region is revolved about the $y$–axis, we express the equation of the bounding curve in terms of $y$:

$$y = 2x - x^2, \Rightarrow x^2 - 2x + 1 = 1 - y, \Rightarrow (x-1)^2 = 1 - y, \Rightarrow x - 1 = \pm\sqrt{1-y},$$
$$\Rightarrow x = 1 \pm \sqrt{1-y}.$$




The signs “plus” and “minus” correspond to the two branches of the curve:

$$x = g(y) = 1 - \sqrt{1-y}, x = f(y) = 1 + \sqrt{1-y}.$$

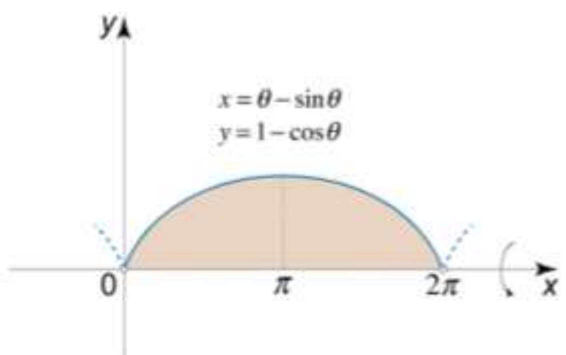
Given that the variable $y$ varies from 0 to 1 and using the washer method, we find the volume of the solid:

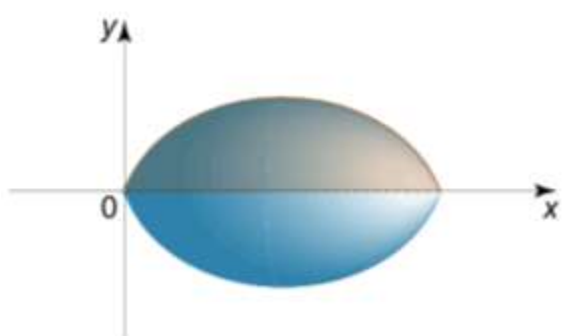
$$V = \pi \int_0^1 ([f(y)]^2 - [g(y)]^2)dy = \pi \int_0^1 ([1+\sqrt{1-y}]^2 - [1-\sqrt{1-y}]^2)dy$$

$$= \pi \int_0^1 (4\sqrt{1-y})dy = 4\pi \int_0^1 \sqrt{1-y}dy = [4\pi \cdot \frac{2(1-y)^{\frac{3}{2}}}{3} \cdot (-1)]|_0^1$$

$$= [-\frac{8\pi\sqrt{(1-y)^3}}{3}]|_0^1 = \frac{8\pi}{3}.$$

Example 10. One arch of the cycloid $x = \theta - \sin\theta, y = 1 - \cos\theta$ revolves around its base. Calculate the volume of the body bounded by the given surface.




The cycloid is given in parametric form. Therefore we express the integral $V = \pi \int_0^{2\pi} y^2 dx$ in terms of the parameter:

$$y^2 = (1-\cos\theta)^2, dx = d(\theta - \sin\theta) = (1-\cos\theta)d\theta.$$

Note that the variable $x$ and the parameter $\theta$ change in the same range from 0 to 2π. Hence, the volume of the solid is given by the integral

$$V = \pi \int_0^{2\pi} y^2 dx = \pi \int_0^{2\pi} (1-\cos\theta)^3 d\theta.$$

To calculate the integral we use the following algebraic and trigonometric identities:

$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3, \quad \cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta, \quad \cos^3\theta = \frac{3}{4}\cos\theta + \frac{1}{4}\cos 3\theta.$

Hence, the volume of the solid is

$$V = \pi \int_0^{2\pi} (1-\cos\theta)^3 d\theta = \pi \int_0^{2\pi} (1 - 3\cos\theta + 3\cos^2\theta - \cos^3\theta) d\theta$$

$$= \pi \int_0^{2\pi} (1 - 3\cos\theta + \frac{3}{2} + \frac{3}{2}\cos 2\theta - \frac{3}{4}\cos\theta - \frac{1}{4}\cos 3\theta) d\theta$$

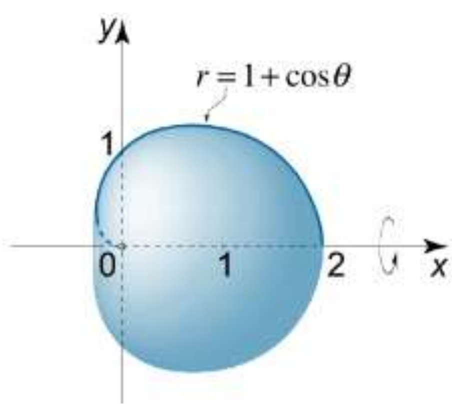
$$= \pi \int_0^{2\pi} (\frac{5}{2} - \frac{15}{4}\cos\theta + \frac{3}{2}\cos 2\theta - \frac{1}{4}\cos 3\theta) d\theta$$

$$= \pi[\frac{5\theta}{2} - \frac{15}{4}\sin\theta + \frac{3}{4}\sin 2\theta - \frac{1}{12}\sin 3\theta]|_0^{2\pi} = 5\pi^2.$$

Example 11. The cardioid $r = 1 + \cos\theta$ rotates around the polar axis. Find the area of the resulting surface.

As the curve is defined in polar coordinates and rotated about the $x$–axis, we calculate the surface area by the formula

$$S = 2\pi \int_\alpha^\beta r(\theta)\sin\theta\sqrt{[r(\theta)]^2 + [r'(\theta)]^2}d\theta$$




Here

$$r(\theta) = 1 + \cos\theta, r'(\theta) = (1 + \cos\theta)' = -\sin\theta.$$

Simplify the expression under the square root sign:

$$[r(\theta)]^2 + [r'(\theta)]^2 = (1+\cos\theta)^2 + (-\sin\theta)^2 = 1 + 2\cos\theta + \cos^2\theta + \sin^2\theta$$

$$= 2(1+\cos\theta).$$

Let's recall now the double angle identities:

$$1+\cos\theta = 2\cos^2\frac{\theta}{2}, \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}.$$

Substituting these formulas we can write the integral in the form

$$S = 2\pi\int_{\alpha}^{\beta} r(\theta)\sin\theta\sqrt{[r(\theta)]^2 + [r'(\theta)]^2}d\theta$$

$$= 2\pi\int_0^{\pi}(1+\cos\theta)\sin\theta\sqrt{2(1+\cos\theta)}d\theta$$

$$= 2\pi\int_0^{\pi}(2\cos^2\frac{\theta}{2}\cdot 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cdot 2\cos\frac{\theta}{2})d\theta = 16\pi\int_0^{\pi}\cos^4\frac{\theta}{2}\sin\frac{\theta}{2}d\theta.$$

It's convenient to change variable:

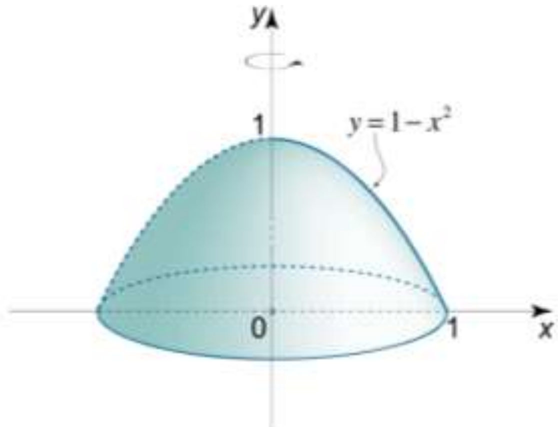
$$\cos\frac{\theta}{2} = z, \Rightarrow -\frac{1}{2}\sin\frac{\theta}{2}d\theta = dz, \Rightarrow \sin\frac{\theta}{2}d\theta = -2dz.$$

When $\theta = 0$, $z = 1$, and when $\theta = \pi$, $z = 0$. Hence, the surface area is equal to

$$S = 16\pi\int_0^{\pi}\cos^4\frac{\theta}{2}\sin\frac{\theta}{2}d\theta = 16\pi\int_1^0 z^4(-2dz) = 32\pi\int_0^1 z^4dz = 32\pi\cdot\frac{z^5}{5}\Big|_0^1$$

$$= \frac{32\pi}{5}$$

Example 12. Find the area of the surface formed by rotating the parabola $y = 1 - x^2$ on the interval $[0,1]$ around the $y-$axis.




Here $a = 0, b = 1, f'(x) = (1 - x^2)' = -2x$. Hence

$$S = 2\pi \int_0^1 x\sqrt{1 + (-2x)^2}dx = 2\pi \int_0^1 x\sqrt{1 + 4x^2}dx.$$

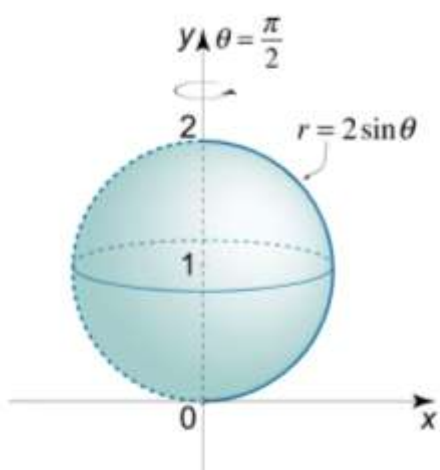
We make the substitution:

$$1 + 4x^2 = t^2, \Rightarrow 8xdx = 2tdt, \Rightarrow xdx = \frac{1}{4}tdt.$$

When $x = 0$, $t = 1$, and when $x = 1$, $t = \sqrt{5}$. This yields

$$S = 2\pi \int_1^{\sqrt{5}} (t \cdot \frac{1}{4}t)dt = \frac{\pi}{2}\int_1^{\sqrt{5}} t^2 dt = \frac{\pi}{2} \cdot \frac{t^3}{3}|_1^{\sqrt{5}} = \frac{\pi}{6}(5\sqrt{5} - 1).$$

Example 13. Find the area of the surface obtained by rotating the circle $r = 2\sin\theta$ around the y–axis.



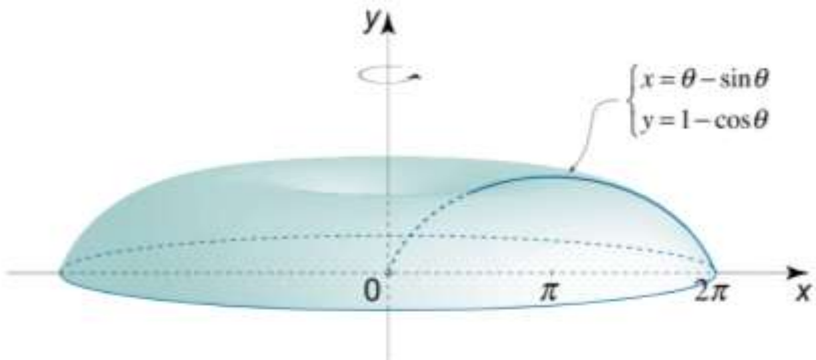
Integrating from 0 to $\frac{\pi}{2}$ and substituting $r(\theta) = 2\sin\theta, r'(\theta) = 2\cos\theta$, we have

$$S = 2\pi \int_0^{\frac{\pi}{2}} 2\sin\theta\cos\theta\sqrt{[2\sin\theta]^2 + [2\cos\theta]^2}d\theta$$

$$= 4\pi \int_0^{\frac{\pi}{2}} \sin 2\theta \underbrace{\sqrt{\sin^2\theta + \cos^2\theta}}_{=1}d\theta = 4\pi \int_0^{\frac{\pi}{2}} \sin 2\theta\, d\theta$$

$$= 4\pi(-\frac{\cos 2\theta}{2})|_0^{\frac{\pi}{2}} = 2\pi(-\cos\pi + \cos 0) = 4\pi.$$

Example 14. One arch of the cycloid $x = \theta - \sin\theta, y = 1 - \cos\theta$ is rotated around the $y$–axis. Calculate the area of the resulting surface.




The curve is given in parametric form. Therefore, we use the following integration formula

$$S = 2\pi \int_\alpha^\beta x(\theta)\sqrt{[x'(\theta)]^2 + [y'(\theta)]^2}d\theta,$$

where the parameter θ varies from 0 to 2π.

Take the derivatives:

$$x'(\theta) = (\theta - \sin\theta)' = 1 - \cos\theta,$$

$$y'(\theta) = (1 - \cos\theta)' = \sin\theta,$$

and simplify the expression under the root square sign:

$$[x'(\theta)]^2 + [y'(\theta)]^2 = (1-\cos\theta)^2 + \sin^2\theta = 1 - 2\cos\theta + \underbrace{\cos^2\theta + \sin^2\theta}_{=1}$$

$$= 2 - 2\cos\theta = 4\sin^2\frac{\theta}{2}.$$

Then the surface area is given by

$$S = 2\pi\int_0^{2\pi}[(\theta-\sin\theta)\cdot 2\sin\frac{\theta}{2}]d\theta = 4\pi[\int_0^{2\pi}\theta\sin\frac{\theta}{2}d\theta - \int_0^{2\pi}\sin\theta\sin\frac{\theta}{2}d\theta]$$

$$= 4\pi[I_1 - I_2].$$

We calculate the first integral using integration by parts:

$$I_1 = \int_0^{2\pi}\theta\sin\frac{\theta}{2}d\theta = \left[\begin{matrix} u = \theta \\ dv = \sin\frac{\theta}{2}d\theta \\ u' = 1 \\ v = -2\cos\frac{\theta}{2}\end{matrix}\right] = -2\theta\cos\frac{\theta}{2}|_0^{2\pi} - \int_0^{2\pi}(-2\cos\frac{\theta}{2})d\theta$$

$$= -2\theta\cos\frac{\theta}{2}|_0^{2\pi} + 2\int_0^{2\pi}\cos\frac{\theta}{2}d\theta = -2\theta\cos\frac{\theta}{2}|_0^{2\pi} + 4\sin\frac{\theta}{2}|_0^{2\pi}$$

$$= [4\sin\frac{\theta}{2} - 2\theta\cos\frac{\theta}{2}]|_0^{2\pi} = 4\pi.$$

Consider now the second integral. Notice that

$$\int\sin\theta\sin\frac{\theta}{2}d\theta = 2\int\sin^2\frac{\theta}{2}\cos\frac{\theta}{2}d\theta = 4\int\sin^2\frac{\theta}{2}d(\sin\frac{\theta}{2}) = \frac{4}{3}\sin^3\frac{\theta}{2} + C.$$

Hence,

$$I_2 = \int_0^{2\pi}\sin\theta\sin\frac{\theta}{2}d\theta = \frac{4}{3}\sin^3\frac{\theta}{2}|_0^{2\pi} = 0.$$

So the area of the surface is

$$A = 4\pi[I_1 - I_2] = 16\pi^2.$$