Lecture 10.

Methods of integration

1. Direct way (continue)

Example.

1.

$$\int \sin^2 x dx = \int \frac{1-\cos 2x}{2} dx = \frac{1}{2}\int dx - \frac{1}{2}\int \cos 2x dx = \frac{1}{2}x - \frac{1}{4}\int \cos 2x d(2x) =$$

$$\frac{1}{2}x - \frac{1}{4}\sin 2x + C$$

2.

$$\int \frac{\cos^3 x}{\sin^4 x} dx = \int \frac{\cos^2 x \cdot d(\sin x)}{\sin^4 x} = \int \frac{(1-\sin^2 x)}{\sin^4 x} d(\sin x) = -\frac{1}{3\sin^3 x} + \frac{1}{\sin x} + C$$

3.

$$\int \frac{dx}{\sin x} = \int \frac{dx}{2\sin\frac{x}{2}\cos\frac{x}{2}} = \frac{1}{2}\int \frac{dx}{\cos^2\frac{x}{2} \cdot \dfrac{\sin\frac{x}{2}}{\cos\frac{x}{2}}} = \int \frac{d\left(\tan\frac{x}{2}\right)}{\tan\frac{x}{2}} = \ln\left|\tan\frac{x}{2}\right| + C.$$

4. $$\int \frac{dx}{2\cos^2 x + \sin^2 x} = \int \frac{dx}{\cos^2 x(2+\tan^2 x)} = \int \frac{d(\tan x)}{2+\tan^2 x} =$$

$$= \frac{1}{\sqrt{2}}\arctan\frac{\tan x}{\sqrt{2}} + C.$$

5.

$$\int \frac{dx}{x\sqrt{4-\ln^2 x}} = \int \frac{d(\ln x)}{\sqrt{4-\ln^2 x}} = \|u = \ln x\| = \int \frac{du}{\sqrt{4-u^2}} = \arcsin\left(\frac{\ln x}{2}\right) + C$$

.

6. $$\int \frac{x^2 dx}{\sqrt{8-x^3}} = -\frac{1}{3}\int \frac{d(8-x^3)}{\sqrt{8-x^3}} = -\frac{2}{3}\sqrt{8-x^3} + C.$$

6.4.1. Method of Integration by Parts

Let u and v be two differentiable functions of x. Then the differential of the product $u \cdot v$ is found as

$$d(uv) = udv + vdu.$$

Whence, by integration of the left and right sides, we obtain

$$uv = \int udv + \int vdu$$

or transposing one of the integrals on the left side, we get

$$\int udv = uv - \int vdu. \tag{6.8}$$

This formula is called *the formula of integration by parts*.

Note. The formula (6.8) does not produce the final result but only transforms the problem from calculating $\int udv$ to calculating the integral $\int vdu$, which is simpler at the successful choice u and v.

It should be mentioned there are no common methods for the right choice of u and dv. Bellow some successful recommendations for certain cases are proposed.

As a rule this method is used in the case when the integrand contains a product of rational and transcendental functions and other methods are unsuitable. The typical integrals, which are calculated by method of integration by parts are following: $\int P_n(x)\cos\alpha x dx$, $\int P_n(x)\sin ax dx$, $\int P_n(x)e^{\alpha x}dx$, $\int x^k \arctan x dx$, $\int x^k \ln x dx$.

1. If the integrand looks like $P_n(x)\cos\alpha x$, $P_n(x)\sin ax$, $P_n(x)e^{\alpha x}$, then the polynomiall P_n is selected as u.
2. If the integrand looks like $x^k \arctan x$, $x^k \ln x$, that is as product of logarithmic or inverse trigonometric functions by a polynomial then the $\arctan x$ (inverse trigonometric function) or $\ln x$ are selected as ”u”.

$$\int udv = uv - \int vdu$$

Example 1.

$$\int x\sin xdx = \left\| \begin{matrix} x = u \Rightarrow du = dx \\ \sin xdx = dv \Rightarrow v = -\cos x \end{matrix} \right\| = -x\cdot\cos x + \int \cos xdu =$$

$= -x\cos x + \sin x + C$.

Example 2. $\int udv = uv - \int vdu$

$$\int \arctan xdx = \left\| \begin{matrix} \arctan x = u \Rightarrow du = \dfrac{1}{l + x^2}dx \\ dx = dv \Rightarrow v = x \end{matrix} \right\|.$$

Solution. If we try to match $\arctan xdx$ with udv, we may take $u = \arctan x$ and $dv = dx$. To use (1.8) it is required to find $du = \dfrac{1}{1+x^2}dx$, $v = x$, then

$$I = x\arctan x - \int \frac{x}{1+x^2}dx = x\arctan x - \frac{1}{2}\int \frac{d\left(x^2+1\right)}{x^2+1} =$$

$$= x\arctan x - \frac{1}{2}\ln\left(x^2+1\right) + C.$$

Sometimes an integration by parts must be repeated to obtain an answer, as in the following example.

Example 3. $\int udv = uv - \int vdu$

$$\int x^2e^xdx = \left\| \begin{matrix} let\ x^2 = u \Rightarrow 2xdx = du \\ e^xdx = dv \Rightarrow v = e^x \end{matrix} \right\| = x^2e^x - 2\int xe^xdx =$$

The integral on the right is similar to the original integral, except that we have reduced the power of x from 2 to 1.

If we could now reduce it from 1 to 0 we could see success ahead. In integral $\int xe^xdx$ we therefore put

$$\begin{matrix} u = x \\ du = dx \end{matrix} \quad \text{and} \quad \begin{matrix} dv = e^xdx \\ v = e^x \end{matrix}, \text{ so that.}$$

Then

$$I = x^2e^x - 2\left(xe^x - e^x\right) + C = e^x\left(x^2 - 2x + 2\right) + C.$$

Example: a) $\int (x+5)\sin 3x dx$;

b) $\int x \arcsin x dx$;

c) $\int x \ln x dx$;

d) $\int \sin \ln x dx,\ \int e^x \cos x dx$.

Solution.

a) $$\int (x+5)\sin 3x dx = \left\| \begin{matrix} x+5=u, & du=dx \\ \sin 3x dx = dv, & v=\int \sin 3x dx = -\frac{1}{3}\cos 3x \end{matrix} \right\| =$$

$$= -\frac{1}{3}(x+5)\cdot\cos 3x + \frac{1}{3}\int \cos 3x \cdot dx = -\frac{1}{3}(x+5)\cdot \cos 3x + \frac{1}{9}\sin 3x + C.$$

$\int u dv = uv - \int v du$

b) $$\int x \arcsin x dx = \left\| \begin{matrix} \arcsin x = u, & du = \frac{1}{\sqrt{1-x^2}}dx \\ x dx = dv, & v = \int x dx = \frac{1}{2}x^2 \end{matrix} \right\| =$$

$$= \frac{1}{2}x^2 \cdot \arcsin x - \frac{1}{2}\int \frac{x^2 dx}{\sqrt{1-x^2}} = \frac{1}{2}\left(x^2 \arcsin x - I\right).$$

Where I denotes integral $I = \int \frac{x^2 dx}{\sqrt{1-x^2}}$.

To evaluate this integral we can use trigonometric substitution $x = \cos t$, then $dx = -\sin t dt$ and

$$I = -\int \frac{\cos^2 t \cdot \sin t}{\sin t} dt = -\int \frac{1+\cos 2t}{2} dt = -\left(\frac{1}{2}t + \frac{1}{4}\sin 2t\right) + C =$$

$$= \left\| \begin{matrix} t = \arccos x \\ \sin 2t = 2\sin t \cdot \cos t = 2\sqrt{1-\cos^2 t}\cdot \cos t = 2x\cdot\sqrt{1-x^2} \end{matrix} \right\| =$$

$$= -\frac{1}{2}\left(\arccos x + x\sqrt{1-x^2}\right) + C.$$

Finally,

$$\int x \arcsin x = \frac{1}{2}\left(x^2 \arcsin x + \frac{1}{2}\arccos x + \frac{1}{2}x\sqrt{1-x^2}\right) + C.$$

a) $\int x \ln x dx = \left\| \begin{array}{ll} \ln x = u, & du = \frac{1}{x} dx \\ xdx = dv, & v = \frac{x^2}{2} \end{array} \right\| = \frac{1}{2} x^2 \ln x - \frac{1}{2} \int \frac{x^2}{x} dx =$

$$= \frac{1}{2} x^2 \ln x - \frac{1}{4} x^2 + C.$$

b)

$$\int \sin \ln x dx = \left\| \begin{array}{ll} \sin \ln x = u, & du = \frac{\cos \ln x}{x} dx \\ dx = dv, & v = x \end{array} \right\| = x \sin \ln x - \int x \frac{\cos \ln x}{x} dx =$$

$$= x \sin \ln x - \int \cos \ln x dx = \left\| \begin{array}{ll} \cos \ln x = u, & du = \frac{\sin \ln x}{x} dx \\ dx = dv, & v = x \end{array} \right\| =$$

$$= x \sin \ln x - \left(x \cos \ln x + \int x \frac{\sin \ln x}{x} dx \right) =$$

$$= x(\sin \ln x - \cos \ln x) - \int \sin \ln x dx.$$

On the right side we got the same integral, which is on the left side. In other words we received the equation concerning integral. Transporting the unknown integral from right side to left one we get

$$\int \sin \ln x dx = \frac{1}{2} x(\sin \ln x - \cos \ln x) + C.$$

6.4.2. Integration by Substitution (Change of Variable)

Let it be required to find the integral

$$\int f(x) dx.$$

Suppose that we cannot directly find antiderivative of $f(x)$ but we know that it exists. Let us change the variable in the expression under the integral sign, putting

$$x = \varphi(t), \tag{6.6}$$

where $\varphi(t)$ is a continuous monotone function with continuous derivative (it is known, that such function has inverse function). Then $dx = \varphi'(t)dt$. Let us prove that in this case the following equality

$$\int f(x)dx = \int f(\varphi(t))\varphi'(t)dt \tag{6.7}$$

is valid.

Here it is assumed that after integration we will substitute on the right side the expression of *t* in terms of *x* on the basis (6.6).

To establish the equality (6.7) treated in the sense indicated early, it is necessary to prove that derivatives with respect to *x* on the both side are equal each to other.

Find the derivative of the left side

$$\left(\int f(x)dx\right)'_x = f(x).$$

Let us differentiate the right side of (6.7) with respect to *x* as a composite function, where *t* is the intermediate argument. The dependence of *t* from *x* is expressed by (6.6). We thus have

$$\left(\int f(\varphi(t))\varphi'(t)dt\right)'_x = \left(\int f(\varphi(t))\varphi'(t)dt\right)'_t \frac{dt}{dx} =$$

$$= f(\varphi(t))\varphi'(t)\frac{1}{\varphi'(t)} = f(\varphi(t)) = f(x).$$

So the derivatives with respect to *x* of the right and left sides of (6.7) are equal each to other as it is required.

The function $x = \varphi(t)$ should be chosen so that one can evaluate the indefinite integral on the right side of (6.7).

For example.

a) $\int \sqrt{1-x^2}dx = \left\| \begin{matrix} x = \sin t \\ dx = \cos t dt \end{matrix} \right\| = \int \sqrt{1-\sin^2 t}\cos t\, dt =$

$$=\int\cos^2 t\,dt=\int\frac{1+\cos 2t}{2}dt=\frac{1}{2}\left(t+\frac{1}{2}\sin 2t\right)=\left(\arcsin x+x\sqrt{1-x^2}\right)+C$$

b) $$\int\frac{dx}{x\cdot\sqrt{1+x^2}}=\left\|\begin{matrix}x=\tan t\\ dx=\dfrac{1}{\cos^2 t}dt\end{matrix}\right\|=\int\frac{\cos t dt}{\cos^2 t\sin t\sqrt{1+\tan^2 t}}=\int\frac{dt}{\sin t}=$$

$$=\ln\left|\tan\frac{t}{2}\right|+C=\|t=\arctan x\|=\ln\left|\tan\left(\arctan\frac{x}{2}\right)+C\right|.$$

Note. This integral may be calculated more simple.

$$\int\frac{dx}{x\cdot\sqrt{1+x^2}}=\int\frac{dx}{x^2\sqrt{1+\left(\frac{1}{x}\right)^2}}=-\int\frac{d\left(\frac{1}{x}\right)}{\sqrt{1+\left(\frac{1}{x}\right)^2}}=-\frac{1}{x}+\ln\left|1/x+\sqrt{1+\left(\frac{1}{x}\right)^2}\right|+C$$

.

c) $$\int\frac{\sqrt{x^2-4}}{x}dx=\left\|\begin{matrix}x=\dfrac{2}{\cos t}\\ dx=\dfrac{2\sin t}{\cos^2 t}dt\end{matrix}\right\|=\int\frac{2\sqrt{1-\cos^2 t}\cos t\,2\sin t}{\cos t\,2\cos^2 t}dt=$$

$$=2\int\tan^2 dt=2\int\frac{1-\cos^2 t}{\cos^2 t}dt=2(\tan t-t)+C,$$

where $t=\arccos\frac{2}{x}$.

Note. It is sometimes better to choose a change of the variable at the form of $t=\psi(x)$, but not at $x=\varphi(t)$. For example let it be required to calculate an integral of the form

$$\int\frac{\psi'(x)}{\psi(x)}dx.$$

Here it is convenient to put $\psi(x)=t$ then $\psi'(x)dx=dt$, and

$$\int\frac{\psi'(x)}{\psi(x)}dx=\int\frac{dt}{t}=\ln|t|+C=\ln|\psi(x)|+C.$$

Example.

1) $\int \tan x dx = \int \frac{\sin x}{\cos x} dx = -\int \frac{d(\cos x)}{\cos x} = -\ln|\cos x| + C$.

2) $\int \frac{2x+5}{x^2+5x+25} dx = \ln\left(x^2+5x+25\right) + C$.

3) $\int \frac{2x+3}{x^2+3x+10} dx = \ln\left|x^2+3x+10\right| + C$;

4) $\int \frac{e^x}{e^x+5} dx = \ln\left(e^x+5\right) + C$;

5) $\int \frac{dx}{\left(1+x^2\right)\mathrm{arctg} x} = \ln|\mathrm{arctg} x| + C$; $\quad \int \frac{d(\arctan x)}{\arctan x}$

6) $\int \frac{\left(3x^2+5x\right)dx}{x^3+\frac{5}{2}x^2+10} = \ln\left|x^3+\frac{5}{2}x^2+10\right| + C$.

You have to remember that success of integration depends largely on how appropriate substitution simplifies the given integral.

Example.

1) $\int \frac{\arctan x\sqrt{1+\arctan^2 x}}{1+x^2} dx = \left\| \begin{array}{l} 1+\arctan^2 x = t^2 \\ \frac{2\arctan x}{1+x^2} dx = 2tdt \end{array} \right\| = \int t^2 dt = \frac{t^3}{3} + C =$

$= \frac{\left(\sqrt{1+\arctan^2 x}\right)^3}{3} + C$.

2)

$$\int \frac{\cos x}{\sqrt{e^{\sin x}-1}} dx = \left\| \begin{array}{l} e^{\sin x} - 1 = t^2 \\ e^{\sin x}\cos x dx = 2tdt \\ \cos x dx = \frac{2tdt}{t^2+1} \end{array} \right\| = \int \frac{2tdt}{\left(t^2+1\right)t} = 2\arctan\sqrt{e^{\sin x}-1} + C$$

.

As you can see the method of substitution is one of the basic method for calculating indefinite integrals. It should be noted that even when we integrate by some other method we often resort to substitution in the intermediate stages of calculation.

Essentially, the study of methods of integration is reduced to finding out what kind of substitution has to be performed for a given element of integration.

Some useful substitution.

Bellow some useful recommendations for integration some irrational functions are given.

Let integrand be expression like $R\left(x, \sqrt{x^2 \pm a^2}\right)$, where R – is sign of rational function of its arguments. What means "rational function"?

It means that operations of addition, multiplication and involution in integer power can be only carried out under arguments of this function.

While integrating the expressions like $R\left(x, \sqrt{x^2 \pm a^2}\right)$ we use such substitutions that allow to get rid of irrationality

a) If $R\left(x, \sqrt{a^2 + x^2}\right)$,

then you can use substitutions $x = a \tan t$ or $x = a\,\mathrm{sh}\,t$;

b) if $R\left(x, \sqrt{a^2 - x^2}\right)$,

then the substitutions $x = a \sin t$ or $x = a \cos t$ are convenient;

c) if $R\left(x, \sqrt{x^2 - a^2}\right)$ there are convenient the following substitutions

$$x = \frac{a}{\cos t}, \quad x = \frac{a}{\sin t} \quad \text{or} \quad x = a\,\mathrm{ch}\,t.$$

It should be noted that the last integrals could be calculated by another substitution.

6.4.3. Integrals of Function Containing a Quadratic Trinomial

Let the integral be given

$$I = \int \frac{dx}{ax^2 + bx + c}$$

Let us first transform the trinomial in the denominator by representing it in the form of the sum or the difference of squares. It may be done as follows

$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = a\left[x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2} + \frac{c}{a}\right] =$$

$$= a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2}\right].$$

Then let us change the variable of integration:

$$x + \frac{b}{2a} = u;\ dx = du,\ I = \frac{1}{a}\int \frac{du}{u^2 \pm k^2}.$$

Where $k^2 = \pm\left(\frac{c}{a} - \frac{b^2}{2a}\right)$. These are tabular integrals.

Example 1. Calculate the integral.

$$\int \frac{dx}{2x^2 + 8x + 20} = \frac{1}{2}\int \frac{dx}{x^2 + 4x + 10} = \left\| \begin{matrix} x^2 + 4x + 10 = (x+2)^2 + 6 \\ x + 2 = u; \qquad dx = du \end{matrix} \right\| =$$

$$= \frac{1}{2}\int \frac{du}{u^2 + 6} = \frac{1}{2}\cdot\frac{1}{\sqrt{6}}\arctan\frac{x+2}{\sqrt{6}} + C.$$

Example 2.

$$\int \frac{x+3}{x^2 - 2x + 5}dx = \int \frac{x+3}{(x-1)^2 + 4}dx = \left\| \begin{matrix} x - 1 = u \\ dx = du \\ x = u + 1 \end{matrix} \right\| = \int \frac{u+4}{u^2 + 4}du =$$

$$=\int\frac{udu}{u^2+4}+4\int\frac{du}{u^2+4}=\frac{1}{2}\int\frac{d\left(u^2+4\right)}{u^2+4}+\frac{4}{2}\arctan\frac{u}{2}=$$

$$=\frac{1}{2}\ln\left(u^2+4\right)+2\arctan\frac{u}{2}+C=\frac{1}{2}\ln\left(x^2-2x+5\right)+2\arctan\frac{x-1}{2}+C..$$

Example 3. $\int\frac{5x+3}{\sqrt{x^2+4x+10}}dx=\int\frac{5x+3}{\sqrt{(x+2)^2+6}}dx=\left\|\begin{matrix}x+2=u\\dx=du\\x=u-2\end{matrix}\right\|=$

$$\int\frac{5u-10+3}{\sqrt{u^2+6}}du=5\int\frac{udu}{\sqrt{u^2+6}}-7\int\frac{du}{\sqrt{u^2+6}}=\frac{5}{2}\int\frac{d\left(u^2+6\right)}{\sqrt{u^2+6}}-7\ln\left|u+\sqrt{u^2+6}\right|=$$

$$=5\sqrt{x^2+4x+10}-7\ln\left|x+2+\sqrt{x^2+4x+10}+C\right|.$$