6.7. Integration of Some Irrational Functions

Now we shall consider irrational functions whose integrals are reduced (by means of substitution) to integrals of rational functions and consequently they are integrated completely (or in the closed form).

We consider the integral $\int R\left(x, x^{\frac{m}{n}}, \ldots, x^{\frac{r}{s}}\right)dx$ where R is sign of the rational function of its arguments. Let k be common denominator of the fractions $\frac{m}{n}, \ldots, \frac{r}{s}$.

We make the substitution

$$x = t^k,\ dx = kt^{k-1}dt.$$

Then each fractional power of x will be expressed in terms of an integer power of t and the integrand will be transformed into a rational function of t.

Example 1. It is required to find the integral

$$I = \int \frac{\sqrt{x}}{\sqrt[4]{x^3} + 1}dx.$$

Solution. The common denominator of fractions $\frac{1}{2}, \frac{3}{4}$ is 4. And so we make the substitution $x = t^4,\ dx = 4t^3dt$.

$$I = 4\int \frac{t^2 \cdot t^3 dt}{\left[\left(t^4\right)^3\right]^{\frac{1}{4}} + 1} = 4\int \frac{t^2 \cdot t^3}{t^3 + 1}dt = \frac{4}{3}\int \frac{t^3 \cdot d(t^3)}{t^3 + 1} = \frac{4}{3}\left\| t^3 = z,\ d\left(t^3\right) = dz \right\| =$$

$$\frac{4}{3}\int \frac{zdz}{z+1} = \frac{4}{3}\int \frac{z+1-1}{z+1} = \frac{4}{3}\int \frac{z+1}{z+1}dz - \frac{4}{3}\int \frac{1}{z+1}dz = \frac{4}{3}z - \frac{4}{3}\ln|z+1| = \frac{4}{3}t^3 - \ln\left|t^3+1\right| + C$$

.

Example 2.

$$\int \frac{dx}{3x-4\sqrt{x}} = \left\| \begin{matrix} x=t^2 \\ dx=2tdt \end{matrix} \right\| = 2\int \frac{tdt}{3t^2-4t} = 2\int \frac{tdt}{t(3t-4)} = 2\int \frac{dt}{3t-4} =$$

$$= 2\cdot\frac{1}{3}\int \frac{d(3t-4)}{3t-4} = \frac{2}{3}\ln|3t-4| + C = \frac{2}{3}\ln\left|3\sqrt{x}-4\right| + C.$$

Example 3.

$$\int \frac{\sqrt{x}dx}{\sqrt[3]{x^2}-\sqrt[4]{x}} = \left\| \begin{matrix} x=t^{12} \\ dx=12t^{11}dt \end{matrix} \right\| = 12\int \frac{t^6\cdot t^{11}dt}{t^8-t^3} = 12\int \frac{t^{17}dt}{t^3(t^5-1)} = 12\int \frac{t^{14}dt}{(t^5-1)} =$$

$$\left| \begin{matrix} t^5=u \\ 5t^4dt=du \Rightarrow t^4dt=\frac{1}{5}du \\ t^{14}dt=t^{10}t^4dt=\frac{1}{5}u^2du \end{matrix} \right| = \frac{12}{5}\int \frac{u^2du}{u-1} = \frac{12}{5}\int \frac{u^2-1+1}{u-1}du =$$

$$= \frac{12}{5}\int (u+1)du + \frac{12}{5}\ln|u-1| = \frac{12}{5}\frac{(u+1)^2}{2} + \frac{12}{5}\ln\left|t^5-1\right| + C =$$

$$= \frac{12}{5}\left(\frac{(t^5+1)^2}{2} + \ln\left|t^5-1\right|\right) + C = \left\| t^5 = x^{\frac{5}{12}} \right\| = \frac{12}{5}\left(\frac{\left(x^{\frac{5}{12}}+1\right)^2}{2} + \ln\left|x^{\frac{5}{12}}-1\right|\right) + C$$

II. Let us consider an integral of the form а где **I**

$$\int R\left[x, \left(\frac{ax+b}{cx+d}\right)^{\frac{m}{n}} + \ldots \left(\frac{ax+b}{cx+d}\right)^{\frac{r}{s}}\right]dx.$$

This integral is reduced to the integral of a rational function by means of the substitution

$$\frac{ax+b}{cx+d} = t^k,$$

where k is common denominator of the fractions $\frac{m}{n},\dots,\frac{r}{s}$.

Example 1. It is required to find the integral

$$\int\frac{\sqrt{x+5}}{x}dx=\left\|\begin{matrix}x+5=t^2\\dx=2tdt\end{matrix}\right\|=2\int\frac{t^2dt}{t^2-5}=$$

$$=2\int\frac{t^2-5+5}{t^2-5}dt=2++10\cdot\frac{1}{2\sqrt{5}}\ln\left|\frac{t-\sqrt{5}}{t+\sqrt{5}}\right|+C.$$

Example 2.

$$\int\frac{dx}{\sqrt{3x+4}+2\sqrt[4]{3x+4}}=\left\|\begin{matrix}3x+4=z^4\\3dx=4z^3dz\end{matrix}\right\|=\frac{4}{3}\int\frac{z^3dz}{z^2+2z}=\frac{4}{3}\int\frac{z^3dz}{z(z+2)}=$$

$$=\frac{4}{3}\int\frac{z^3dz}{z+2}=\frac{4}{3}\int\frac{z^2-4+4}{z+2}dz=\frac{4}{3}\left(\int z-2\right)dz+4\int\frac{dz}{z+2}=$$

$$=\frac{4}{3}\left(\frac{z-2}{2}+4\ln|z+2|\right)+C=\frac{4}{3}\left(\frac{\left(\sqrt[4]{3x+4}-2\right)^2}{2}+4\ln\sqrt[4]{3x+4}+2\right)+C$$

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Example 3. $$\int\frac{dx}{\sqrt[4]{(x-1)^3(x+2)^5}}=\int\frac{dx}{\sqrt[4]{\frac{(x-1)^3(x+2)^8}{(x+2)^3}}}=$$

$$=\left\|\begin{matrix}\frac{x-1}{x+2}=t^4\\\frac{x+2-(x-1)}{(x+2)^2}\cdot dx=4t^3dt\Rightarrow\frac{dx}{(x+2)^2}=\frac{4}{3}t^3dt\end{matrix}\right\|=$$

$$=\frac{4}{3}\int\frac{t^3dt}{t^3}=\frac{4}{3}t+C=\frac{4}{3}\sqrt[4]{\frac{x-1}{x+2}}+C.$$