Lecture 8

Convexity and Concavity of a Curve. Points of Inflection

Let us consider a plane curve $y = f(x)$, which is the graph of a single-valued differentiable function $f(x)$.

Definition 1. A curve is *convex upwards* (*or just convex curve*) on the interval (a,b) if all points of the curve lie below any tangent to it on the interval. A curve is *convex down* wards (*or just concave*) on the interval (a,b) if all points of the curve lie above any tangent to it on the interval. An important characteristic of a curve shape is its convexity and concavity.

Theorem 1. If the second derivative of the function $y = f(x)$ is positive, i.e. $f''(x) > 0$ at all points of an interval $[a,b]$, then the curve $y = f(x)$ is concave on this interval.

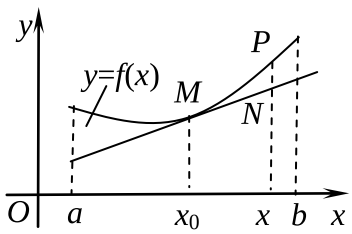


Fig. 5.35

Note. Proved theorem may be illustrated by geometrically. A condition $f''(x) > 0$ may be written as: $[f'(x)]' > 0$. It means that the first derivative $f'(x)$ increases on the interval $[a,b]$, and since $f'(x) = \tan\alpha$, then a value of $\tan\alpha$, and consequently an angle α, increases at increasing of x value, that is, a tangent became more steep than asserts a concavity of the curve, (Fig. 5.36).

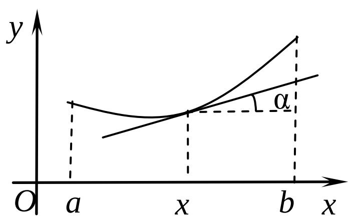


Fig. 5.36

Theorem 2. If a curve $y = f(x)$ is concave on interval $[a,b]$, then value of the second derivative $f''(x)$ of this function is non negative and may be equal to zero in separate points.

In similar way the following theorem may be proved.

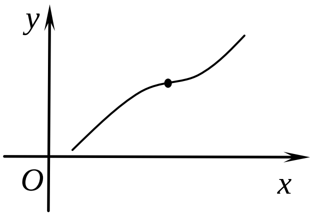


Fig. 5.37

Theorem 3. If $f''(x)<0$ for all $x\in(a,b)$ then a curve $y=f(x)$ is convex on this interval (Fig.°5.37, $x<x_0$).

Theorem 4. If a curve $y=f(x)$ is convex on an interval $[a,b]$, then value of the second derivative $f''(x)$ of this function is negative and may be equal to zero in separate points.

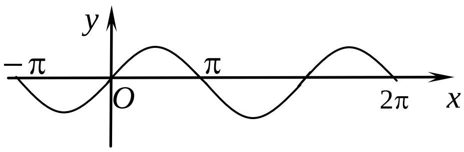


Fig. 5.38

The point which separates the convex part of a continuous curve from the concave part is called the *point of inflection* of the curve (Fig.°5.37). For example, the curve $y=\sin x$ has infinite number of inflection points: $(0,0)$, $(\pi,0)$, $(-\pi,0)$, $(2\pi,0)$,…, (Fig. 5.38).

It is obvious that the tangent line, if it exists, cuts the curve at the point of inflection, because on one side the curve lies below of the tangent and on the other side, above it.

Theorem 5. If the point $M(x_0, y_0)$ is the inflection one of the curve $y=f(x)$, then a value $f''(x_0)$ either vanishes or does not exist at this point x_0.

But if the value $f''(x_0)$ either vanishes or does not exist at a point then it is not mean that the point is inflection one.

For example consider a function $y=x^4$. Find $y''=12x^2$, that is, $y''(0)=0$. However the point $(0,0)$ is not inflection one, because in both case if $x<0$, and if $x>0$ it will be $y''>0$, that is a curve is concave everywhere on interval (Fig. 5.39).

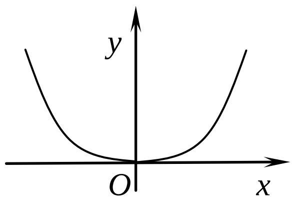


Fig. 5.39

Thus for existence of an inflection point it should be sure that a value $f''(x_0)$ changes its sign when passing through this point.

Example. Find the points of inflection and determine the intervals of convexity and concavity of the curve $y = x^3 - 3x^2 + 5x - 1$.

Solution.

Let us find the first and the second derivative

$$y' = 3x^2 - 6x + 5,$$

passing through the point $x = 1$ the value y'' changes sign from «–» to «+». Thus in the interval $(-\infty, 1)$ the curve is convex, but in the interval $(1, +\infty)$ this curve is concave. The point $(1,2)$ is inflection one.

5.13. Asymptotes

Very frequently it is required to investigate the behavior of a function in the case of unlimited increase (in absolute value) of abscissa or ordinate of a current point of the curve, or of the abscissa and ordinate simultaneously. An important special case is when the curve under study approaches a given straight line without bound as the current point of the curve recedes to infinity.

Definition. A straight line A is called *an asymptote to a curve*, if the distance from the variable point M of the curve to this straight line approaches zero as the point M recedes to infinity along the curve.

In future we shall consider vertical asymptotes (parallel to the axis of ordinates) and inclined asymptotes (not parallel to the axis of ordinates) and horizontal asymptotes (parallel to x-axis)

1. Horizontal asymptotes. The straight line $y = b$ is horizontal asymptotes to a curve $y = f(x)$ (Fig. 5.40), if at least one of the following equalities

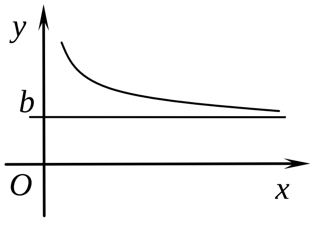


Fig. 5.40

$$\lim_{x\to+\infty} f(x) = b,\ \lim_{x\to-\infty} f(x) = b$$

is fulfilled.

Example. Since $\lim\limits_{x\to-\infty} \operatorname{arctg} x = -\dfrac{\pi}{2}$, $\lim\limits_{x\to+\infty} \operatorname{arctg} x = \dfrac{\pi}{2}$, then curve $y = \operatorname{arctg} x$ has two horizontal asymptotes: $y = -\dfrac{\pi}{2}$ и $y = \dfrac{\pi}{2}$, (Fig.°5.41).

Example. The Ox-axes is horizontal asymptotes to the curve $y = \dfrac{\sin x}{x}$ (Fig. 5.42), as it is obvious that,

$$\lim_{x\to-\infty} \frac{\sin x}{x} = \lim_{x\to+\infty} \frac{\sin x}{x} = 0.$$

2. Vertical asymptotes. From the definition of an asymptote it

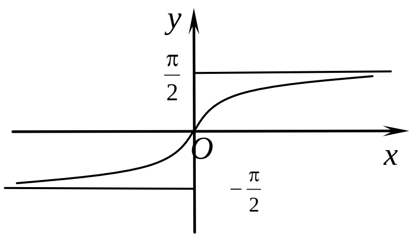


Fig. 5.41

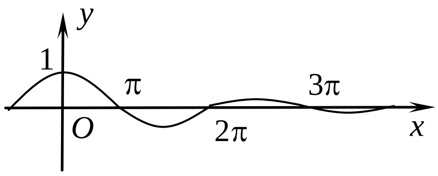


Fig. 5.42

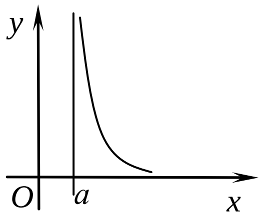


Fig. 5.43

follows that if $\lim\limits_{x\to a-0} f(x) = \infty$, $\lim\limits_{x\to a+0} f(x) = \infty$ or $\lim\limits_{x\to a} f(x) = \infty$ then the straight line $x = a$ is an asymptote to the curve $y = f(x)$ (Fig.°5.43). And conversely, if the straight line $x = a$ is an asymptote, then one of the foregoing equalities is fulfilled. Consequently, to find vertical asymptotes one has to find values of $x = a$ such that when x approaches a the function approaches infinity. As a rule these points are discontinuous points of the second kind.

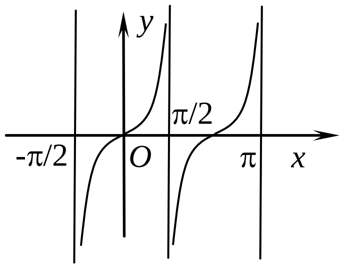


Fig. 5.44

For example, the curve $y = \tan x$ (Fig. 5.44) has an infinity number of vertical asymptotes: $x = \pm\frac{\pi}{2},\ x = \pm\frac{3\pi}{2},\ldots$

Example. The curve $y = e^{\frac{1}{x}}$ has a vertical asymptote $x = 0$, and too horizontal asymptotes $y = 0, y = 1$. Indeed since

$$\lim_{x\to+0} e^{\frac{1}{x}} = \infty;\ \lim_{x\to-0} e^{\frac{1}{x}} = 0;\ \lim_{x\to+\infty} e^{\frac{1}{x}} = 1;\ \lim_{x\to-\infty} e^{\frac{1}{x}} = 1.$$

3. Inclined asymptotes. Let the curve $y = f(x)$ have an inclined asymptote whose equation is

$$y = kx + b\ (k \neq 0). \tag{5.11}$$

$$k = \lim_{x\to\infty} \frac{f(x)}{x}.$$

$$b = \lim_{x\to\infty}[f(x) - kx]. \tag{5.16}$$

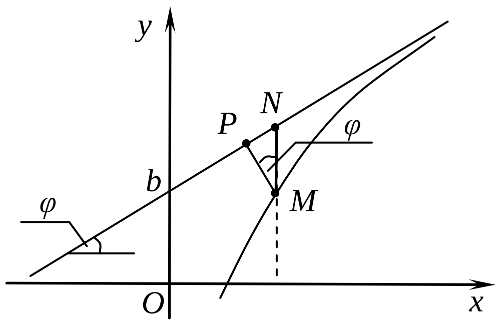


Fig. 5.45

Simultaneously existence of these limits is necessary and sufficiency condition for existence of asymptotes to the given curve. If one of these limits does not exist then the curve has not an asymptote.

Example. Find the asymptotes to the curve $y = \frac{x^2}{x-2}$.

Solution. Since $\lim_{x\to\infty} y = \infty$, then the curve has not horizontal asymptote. It is obvious the straight line $x = 2$ is vertical asymptote because

$$\lim_{x\to 2-0} y = -\infty,\ \lim_{x\to 2+0} y = +\infty.$$

Look for inclined asymptotes: $k = \lim\limits_{x\to\infty} \frac{x}{x-2} = 1$.

Thus

$$b = \lim_{x\to\infty}\left(\frac{x^2}{x-2} - x\right) = \lim_{x\to\infty}\frac{2x}{x-2} = 2.$$

$$k = \lim_{x\to-\infty}\frac{x}{x-2} = 1, \qquad b = \lim_{x\to-\infty}\left(\frac{x^2}{x-2} - x\right) = \lim_{x\to\infty}\frac{2x}{x-2} = 2$$

Therefore, the straight line $y = x + 2$ is an inclined asymptote to the given curve.

To investigate the mutual positions of a curve and asymptotes, let us consider the difference of the ordinate of the curve and the asymptote for one and the same value of x

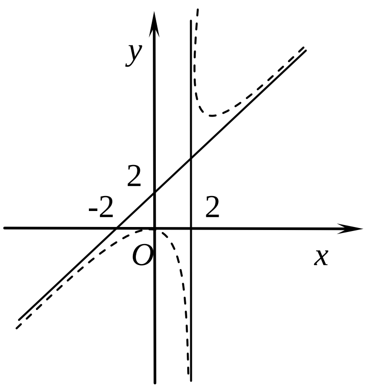


Fig. 5.46

$$\frac{x^2}{x-2} - (x+2) = \frac{x^2 - x^2 + 4}{x-2} = \frac{4}{x-2},$$

The difference is positive if $x > 2$ therefore the curve lies above inclined asymptote. And this difference is negative, if $x < 2$, consequently the curve lies below the asymptote (Fig. 5.46).

Example. Find the asymptotes to the curve $y = x + \sqrt{x}$. We have

$$\lim_{x\to+\infty} y = +\infty,$$

that is, the horizontal asymptote is absent. Further, a function is defined at all $x \geq 0$, it means that the vertical asymptote is absent as well. At last

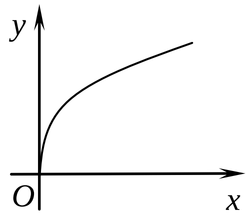


Fig. 5.47

$$k = \lim_{x\to+\infty}\frac{x+\sqrt{x}}{x} = \lim_{x\to+\infty}\frac{x}{x} = 1;$$

$$b = \lim_{x\to+\infty} x + \sqrt{x} - x = \lim_{x\to+\infty}\sqrt{x} = +\infty.$$

Since the last limit does not exist then the curve has not an inclined asymptote too. Thus the curve under consideration has no asymptotes (Fig. 5.47).

5.14. General Plan for Investigating Functions and Constructing Graphs

Complete investigation of a function usually includes the following steps:

1°. Finding the natural domain of the function.

2°. Investigating the even, odd and periodic of the function.

3°. Investigating the discontinuity of the function.

4°. Finding the asymptotes.

5°. Finding intervals of increase and decrease of the function.

6°. Determination of the extremum of the function.

7°. Determination of the intervals of convexity and concavity of the graph, and points of inflection.

8°. Finding roots of the function and other specified points of its plot.

9°. Investigating function on infinity.

Example. Let us investigate the function $y = \frac{x^3}{3-x^2}$

1°.Obvious that the function is defined over numerical axis except of points $x = \pm\sqrt{3}$.

2°. The function is odd because $f(-x) = -f(x)$. Indeed let us calculate $f(-x) = \frac{-x^3}{3-x^2}$. So its plot is symmetrical about the point of origin.

3°. Let us investigate discontinuity of the function.

$$\lim_{x\to\sqrt{3}+0}\frac{x^3}{3-x^2} = -\infty; \quad \lim_{x\to-\sqrt{3}+0}\frac{x^3}{3-x^2} = -\infty;$$

$$\lim_{x\to\sqrt{3}-0}\frac{x^3}{3-x^2} = \infty; \quad \lim_{x\to-\sqrt{3}-0}\frac{x^3}{3-x^2} = \infty.$$

From this it follows that $x = \pm\sqrt{3}$ are vertical asymptotes.

4°. To determine the incline asymptotes let us calculate the following limits:

$$k=\lim_{x\to\pm\infty}\frac{x^3}{\left(3-x^2\right)x}=-1;$$

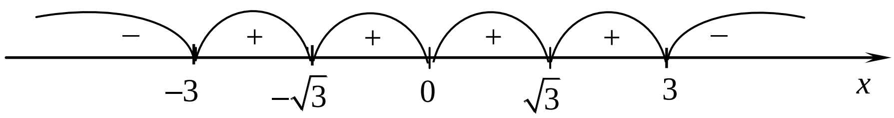
$$b=\lim_{x\to\pm\infty}\left(\frac{x^3}{\left(3-x^2\right)}+x\right)=\lim_{x\to\pm\infty}\left(\frac{x^3+3x-x^3}{\left(3-x^2\right)}\right)=0.$$

So the straight line $y=-x$ is incline asymptote.

5°. Let us investigate the function for extremum.

With this purpose find the first derivative.

$$y'=\frac{3x^2\left(3-x^2\right)+2x^4}{\left(3-x^2\right)^2}=\frac{9x^2-3x^4-2x^4}{\left(3-x^2\right)^2}=\frac{9x^2-x^4}{\left(3-x^2\right)^2}.$$



Behaviar of y'

$$x^2\left(9-x^2\right)=0,\quad x_1=0,\quad x_2=3,\quad x_3=-3,\quad x_{4,5}=\pm\sqrt{3}.$$

Illustrate the obtained results in the table 5.1.

Table 5.1

x	$(-\infty,-3)$	-3	$(-3,-\sqrt{3})$	$-\sqrt{3}$	$(-\sqrt{3},0)$	0	$(0,\sqrt{3})$	$\sqrt{3}$	$(\sqrt{3},3)$	3	$(3,+\infty)$
y'	–	0	+	Does not exist	+	0	+	Does not exist	+	0	–
y	↘	min $y_{\min}=$ 4.5	↗	Does not exist	↗	0	↗	Does not exist	↗	max $y_{\max}=$ $-4{,}5$	↘

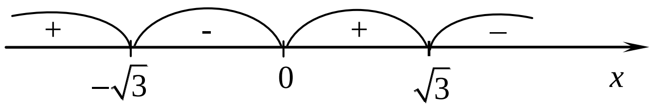
To investigate a curve for convexity and concavity let us calculate the second derivative

$$y'' = \frac{\left(18x-4x^3\right)\left(3-x^2\right)^2 + 2\left(3-x^2\right)2x\left(9x^2-x^4\right)}{\left(3-x^2\right)^4} =$$

$$= \frac{2x\left(\left(9-2x^2\right)\left(3-x^2\right)+2x^2\left(9-x^2\right)\right)}{\left(3-x^2\right)^3} =$$

$$= \frac{2x\left(27-15x^2+2x^4+18x^2-2x^4\right)}{\left(3-x^2\right)^3} = \frac{2x\left(27+3x^2\right)}{\left(3-x^2\right)^3} = \frac{6x\left(9+x^2\right)}{\left(3-x^2\right)^3}.$$

Let us investigate the sign of the second derivative:



Behaviour of y''

The results obtained by the second derivative are illustrated in the table 5.2.

Table 5.2

x	$\left(-\infty,-\sqrt{3}\right)$	$-\sqrt{3}$	$\left(-\sqrt{3},0\right)$	0	$\left(0,\sqrt{3}\right)$	$\sqrt{3}$	$\left(\sqrt{3},\infty\right)$
y''	+	Does not	–	0	+	Does not	–
y	Concave	exist	Convex	Point of inflection $y=0$	Concave	exist	Convex

Using fulfilled investigation we can construct the plot (Fig.°5.48).

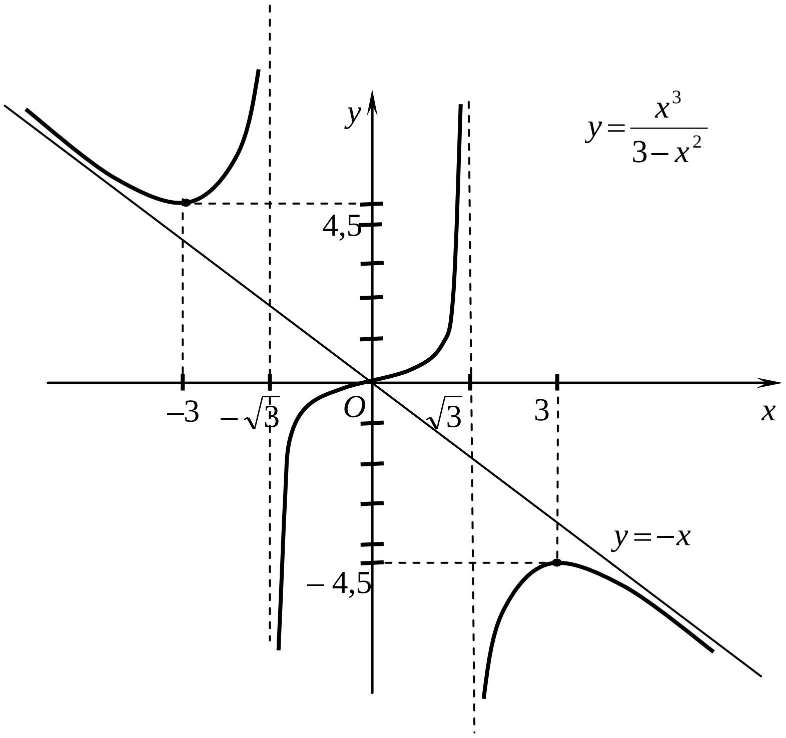


Fig. 5.48