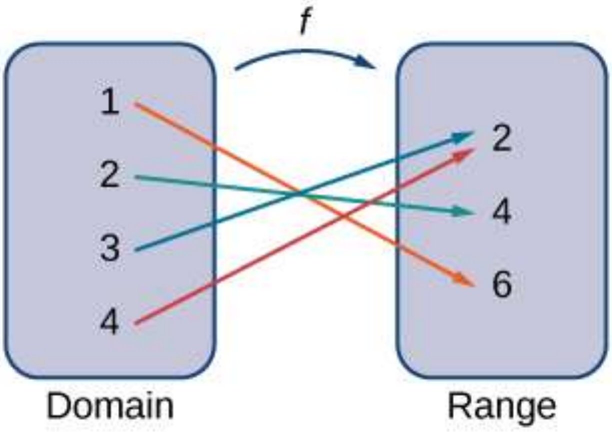
3. FUNCTIONS OF SEVERAL VARIABLES

Up to now we dealt with function of single variable, i.e.

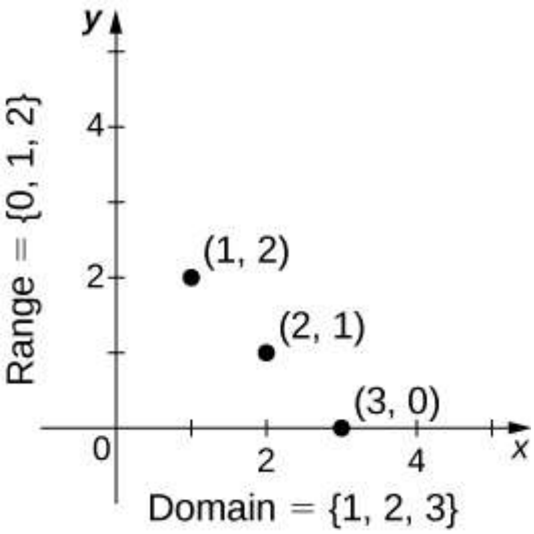
Remainder:

A function of single variable f consists of a *set of inputs*, a set of *outputs*, and a rule for assigning each input to exactly one output. The set of inputs is called *the domain of the function*. The set of outputs is called *the range of the function*.



Note: Every function has a domain. However, sometimes a function is described by an equation, as in $f(x) = x^2$, with no specific domain given. In this case, the domain is taken to be the set of all real numbers x for which $f(x)$ is a real number. For a general function f with domain D, we often use x to denote the input and y to denote the output associated with x. When doing so, we refer to x as the **independent variable** and y as the **dependent variable**, because it depends on x. Using function notation, we write $y = f(x)$, and we read this equation as "y equals f of x". For the squaring function, we write $f(x) = x^2$.

We can also visualize a function by plotting points (x, y) in the coordinate plane where $y = f(x)$. The *graph of a function* is the set of all these points. For example, consider the function f, where the domain is the set $D = \{1,2,3\}$ and the rule is $f(x) = 3 - x$. In Figure, we plot a graph of this function.

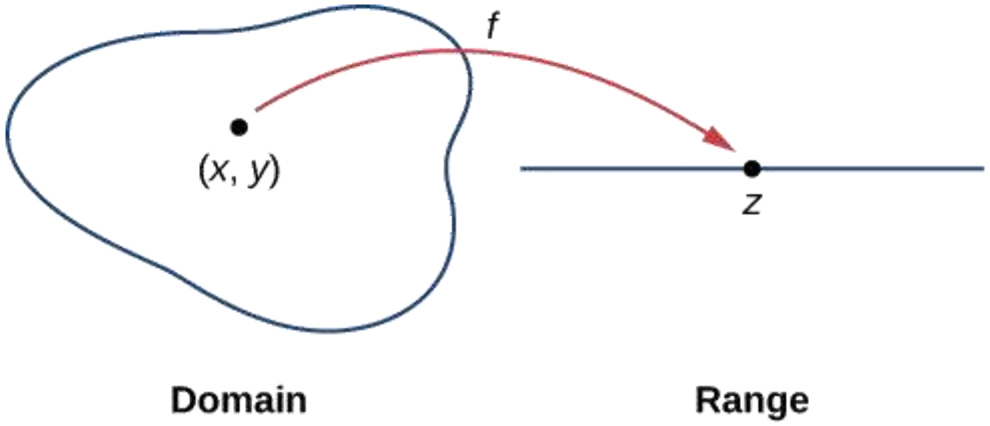


Our next step is to explain what a function of more than one variable is. We start with functions of two independent variables. This step includes identifying the domain and range of such functions and learning how to graph them.

3.1 Functions of Two Variables

The definition of a function of two variables is very similar to the definition for a function of one variable. The main difference is that, instead of mapping values of one variable to values of another variable, we map ordered pairs of variables to another variable.

Definition: A function of two variables $z = f(x, y)$ maps each ordered pair (x, y) in a subset D of the real plane $\mathbf{R}^2$ to a unique real number z. The set D is called *the domain* of the function. *The range* of function f is the set of all real numbers z that has at least one ordered pair $(x, y) \in D$ such that $f(x, y) = z$ as shown in Figure.



This function has two independent variables (x and y) and one dependent variable z.

Determining the domain of a function of two variables involves taking into account any domain restrictions that may exist. Let's take a look.

Example: Find domains and ranges for functions of two variables

(a) $f(x, y) = 3x + 5y + 2$

This is an example of a linear function in two variables. There are no values or combinations of x and y that cause $f(x, y)$ to be undefined, so *the domain* of f is $\mathbf{R}^2$. To determine *the range*, first pick a value for z. We need to find a solution to the equation $f(x, y) = z$, or $3x - 5y + 2 = z$. One such solution can be obtained by first setting $y = 0$, which yields the equation $3x + 2 = z$. The solution to this equation is $x = \frac{z-2}{3}$, which gives the ordered pair $(\frac{z-2}{3}, 0)$ as a solution to the equation $f(x, y) = z$ for any value of z. Therefore, the range of the function is all real numbers, or $\boldsymbol{R}$.

(b) $g(x, y) = \sqrt{9 - x^2 - y^2}$

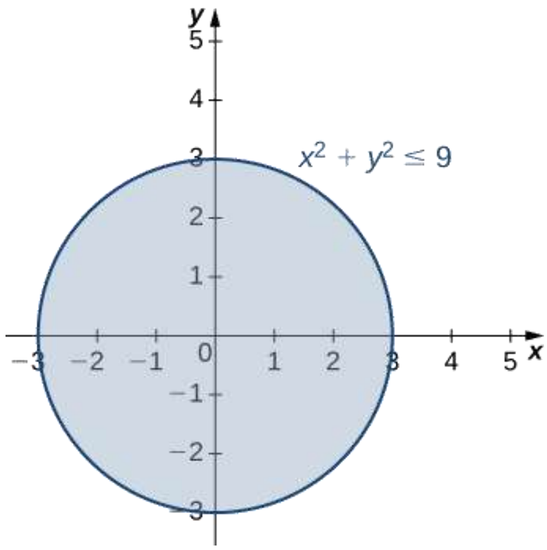
For the function $g(x, y)$ to have a real value, the quantity under the square root must be nonnegative:

$$9 - x^2 - y^2 \geq 0.$$

This inequality can be written in the form

$$x^2 + y^2 \leq 9$$

Therefore, *the domain* of $g(x, y)$ is $\{(x, y) \in R^2 \mid x^2 + y^2 \leq 9\}$. The graph of this set of points can be described as a disk of radius 3 centered at the origin. The domain includes the boundary circle as shown in the following graph.



To determine *the range* of $g(x, y) = \sqrt{9 - x^2 - y^2}$ we start with a point (x_0, y_0) on the boundary of the domain, which is defined by the relation $x^2 + y^2 = 9$:

$$g(x_0, y_0) = \sqrt{9 - x_0^2 - y_0^2} = \sqrt{9 - (x_0^2 + y_0^2)} = \sqrt{9 - 9} = 0.$$

If $x_0^2 + y_0^2 = 0$ (in other words, $x_0 = y_0 = 0$), then

$$g(x_0, y_0) = \sqrt{9 - x_0^2 - y_0^2} = \sqrt{9 - (x_0^2 + y_0^2)} = \sqrt{9 - 0} = 3.$$

Given any value c between 0 and 3, we can find an entire set of points inside the domain of g. Therefore, the range of this function can be written in interval notation as [0,3].

3.1.1 Graphing Functions of Two Variables

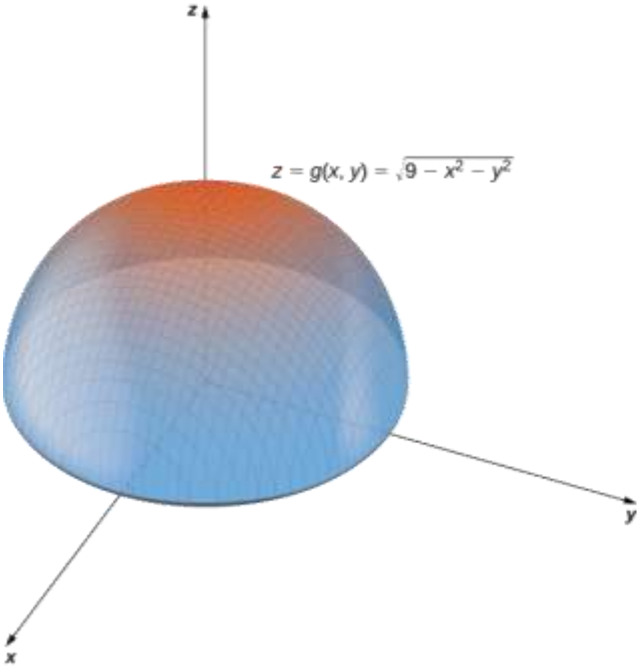
Suppose we are going to graph the function $z = (x, y)$. When graphing a function $y = f(x)$ of one variable, we use the Cartesian plane. We are able to graph any ordered pair (x, y) in the plane, and every point in the plane has an ordered pair (x, y) associated with it. With a function of two variables, each ordered pair (x, y) in the domain of the function is mapped to a real number z . Therefore, the graph of the function f consists of ordered triples (x, y, z) . The graph of a function $z = (x, y)$ of two variables is called *a surface*.

To understand more completely the concept of plotting a set of ordered triples to obtain a surface in three-dimensional space, imagine the (x, y) coordinate system laying flat. Then, every point in the domain of the function f has a unique *z*-value associated with it. If z is positive, then the graphed point is located above the xy-plane, if z is negative, then the graphed point is located below the xy-plane. The set of all the graphed points becomes the two-dimensional surface that is the graph of the function f.

Example: Graphing functions of two variables

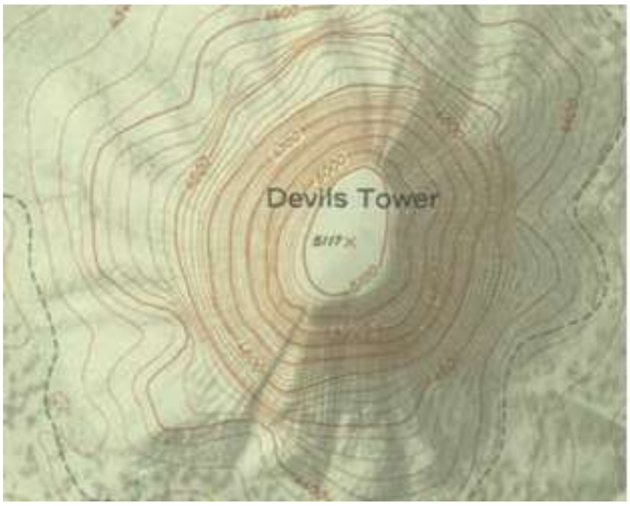
(a) $g(x,y) = \sqrt{9 - x^2 - y^2}$

We determined that the domain of $g(x,y) = \sqrt{9 - x^2 - y^2}$ is $\{x^2 + y^2 \leq 9\}$, and the range is $\{0 \leq z \leq 3\}$. Therefore any point on the circle of radius 3 centered at the origin in the xy-plane maps to $z = 0$ in $\mathbf{R}^3$. When $x^2 + y^2 = 0$ then $g(x,y) = 3$. This is the origin in the xy–plane. If $x^2 + y^2$ is equal to any other value between 0 and 9, then $g(x,y)$ equals some other constant between 0 and 3. The surface described by this function is a hemisphere centered at the origin with radius 3 as shown in the following graph.



3.1.2 Level Curves

A topographical map contains curved lines called *contour lines*. Each contour line corresponds to the points on the map that have equal elevation. A level curve of a function of two variables $f(x,y)$ is completely analogous to a contour line on a topographical map.

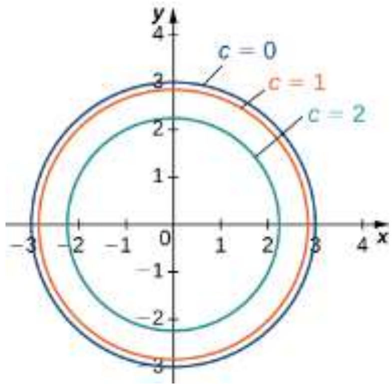


Definition: Given a function $f(x, y)$ and a number c in the range of f , a *level curve* of a function of two variables for the value c is defined to be the set of points satisfying the equation $\mathrm{f}(x, y) = c$.

Returning to the function $g(x, y) = \sqrt{9 - x^2 - y^2}$, we can determine the level curves of this function. The range of g is the closed interval [0,3]. First, we choose any number in this closed interval — say, $c = 2$. The level curve corresponding to $c = 2$ is described by the equation

$$x^2 + y^2 = 5.$$

This equation describes a circle centered at the origin with radius $\sqrt{5}$, etc. a graph of the level curves of this function corresponding to c=0,1,2, and 3 is shown in Figure



3.1.3 Vertical traces

Another useful tool for understanding the graph of a function of two variables is called a vertical trace. Level curves are always graphed in the *xy*–plane , but as their name implies, vertical traces are graphed in the *xz*– or *yz*– planes.

Definition: Consider a function $z = f(x, y)$ with domain $D \subseteq \boldsymbol{R}^2$. A vertical trace of the function can be either the set of points that solves the equation $f(a, y) = z$ for a given constant $x = a$ or $f(x, b) = z$ for a given constant $y = b$.

There are vertical traces parallel to the *xz*–plane for the function $f(x, y)$ and vertical traces parallel to the *yz*–plane for the function $f(x, y)$.

3.2 Functions of More Than Two Variables

So far, we have examined only functions of two variables. However, it is useful to take a brief look at functions of more than two variables. An example is $u = f(x, y, z)$ - (x, y, z) represents a point in space, and the function f maps each point in space to a fourth quantity u.

The method for finding the domain of a function of more than two variables is analogous to the method for functions of one or two variables.

Example: Find the domain of the following function $f(x, y, z) = \frac{3x-4y+2z}{\sqrt{9-x^2-y^2-z^2}}$?

For the function $f(x, y, z)$ to be defined (and be a real value), two conditions must hold: (1) The denominator cannot be zero; (2) The radicand cannot be negative.

Combining these conditions leads to the inequality $9 - x^2 - y^2 - z^2 > 0$.

Moving the variables to the other side and reversing the inequality gives the domain as $(f) = \{(x, y, z) \in R^3 \mid x^2 + y^2 + z^2 < 9\}$, which describes a sphere of radius 3 centered at the origin. (Note: The surface of the sphere is not included in this domain.)

When the function has three variables, the level curves become surfaces, so we can define level surfaces for functions of three variables.

Definition: Given a function $f(x, y, z)$ and a number c in the range of f , a level surface of a function of three variables is defined to be the set of points satisfying the equation $f(x, y, z) = c$.

In general, the variable u is called the function of the independent variables x_1, x_2,…, x_n if for each set of values of these variables $\{x_1, x_2,\ldots, x_n\}$ from the given range of their changes – the domain of the function f is established one value of u according to some the rule or law f. The function of the several variables is denoted by

$$u = f(x_1, x_2,\ldots, x_n)$$

3.3 Limit of a Function of Two Variables

Recall from the definition of a limit of a function of one variable:

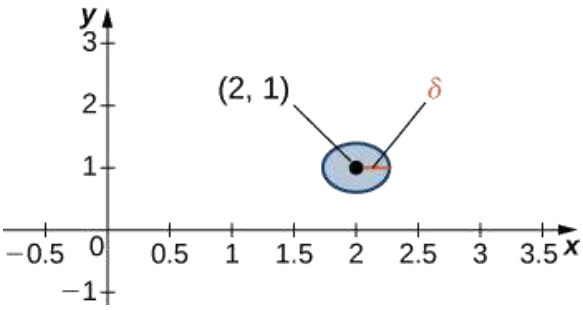
Let $f(x)$ be defined for all $x \neq a$ in an open interval containing a . Let L be a real number. Then take place a limit of the function: $\lim_{x\to a} f(x) = L$ if for every $\varepsilon > 0$, there exists a $\delta > 0$, such that if $0 < |x - a| < \delta$ for all x in the domain of f , then

$$|f(x) - L| < \varepsilon.$$

Before we can adapt this definition to define a limit of a function of two variables, we first need to see how to extend the idea of an open interval in one variable to an open interval in two variables.

Definition of δ-disk: Consider a point $(a, b) \in \boldsymbol{R}^2$. A δ-disk centered at point (a, b) is defined to be an open disk of radius δ centered at point (a, b) —that is,

$$\{(x, y) \in \mathbb{R}^2 \mid (x-a)^2 + (y-b)^2 < \delta^2\}$$



(In one dimension, we express this restriction as $a - \delta < x < a + \delta$. In more than one dimension, we need to use a δ-disk.)

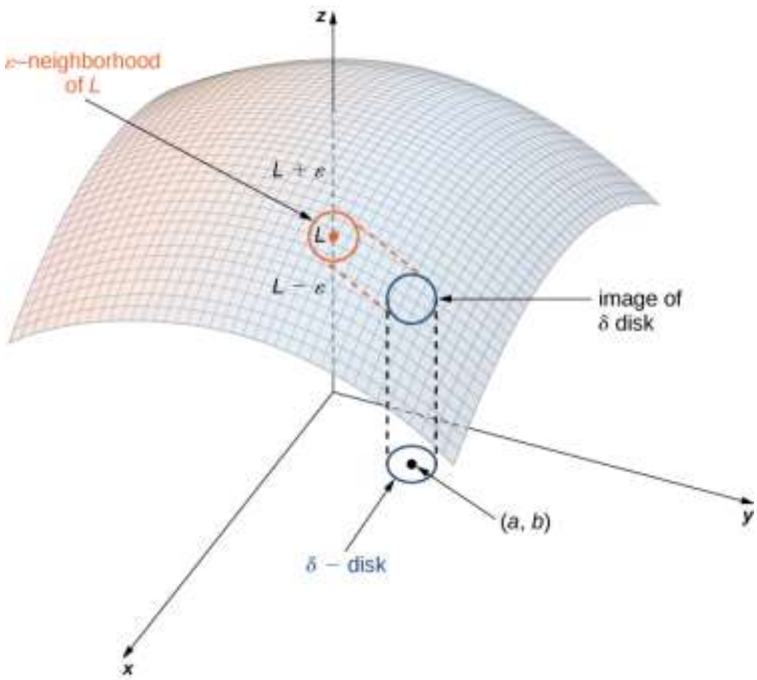
Definition of limit of a function of two variables: Let f be a function of two variables, x and y. The limit of $f(x, y)$ as (x, y) approaches (a, b) is L , written

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

if for each $\varepsilon > 0$ there exists a small enough $\delta > 0$ such that for all points (x, y) in a δ-disk around (a, b) , except possibly for (a, b) itself, the value of $f(x, y)$ is no more than ε away from L, i.e.

$$|f(x, y) - L| < \varepsilon$$

whenever $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$.



3.3.1 Limit laws for functions of two variables

Proving that a limit exists using the definition of a limit of a function of two variables can be challenging. Instead, we use the following theorem, which gives us shortcuts to finding limits. The formulas in this theorem are an extension of the formulas in the limit laws theorem:

Let $f(x, y)$ and $g(x, y)$ be defined for all $(x, y) \neq (a, b)$ in a neighborhood around (a, b), and assume the neighborhood is contained completely inside the domain of f. Assume that L and M are real numbers such that

$$\lim_{(x,y)\to(a,b)} f(x, y) = L \text{ and } \lim_{(x,y)\to(a,b)} g(x, y) = M,$$

and let c be a constant. Then each of the following statements holds:

1. $\lim_{(x,y)\to(a,b)} c = c$ - Constant Law
2. $\lim_{(x,y)\to(a,b)} x = a$ and $\lim_{(x,y)\to(a,b)} y = b$ - Identity Laws
3. $\lim_{(x,y)\to(a,b)} (f(x, y) + g(x, y)) = L + M$ - Sum Law
4. $\lim_{(x,y)\to(a,b)} (f(x, y) - g(x, y)) = L - M$ - Difference Law
5. $\lim_{(x,y)\to(a,b)} (cf(x, y)) = cL$ - Constant Multiple Law
6. $\lim_{(x,y)\to(a,b)} (f(x, y)g(x, y)) = LM$ - Product Law
7. $\lim_{(x,y)\to(a,b)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M}$ for $M \neq 0$ - Quotient Law
8. $\lim_{(x,y)\to(a,b)} (f(x, y))^n = L^n$ - Power Law
9. $\lim_{(x,y)\to(a,b)} \sqrt[n]{f(x, y)} = \sqrt[n]{L}$ - Root Law

The proofs of these properties are similar to those for the limits of functions of one variable. We can apply these laws to finding limits of various functions.

Example

Find the following limit: $\lim_{(x,y)\to(2,-1)} (x^2 - 2xy + 3y^2 - 4x + 3y - 6)$

First use the sum and difference laws to separate the terms:

$$\lim_{(x,y)\to(2,-1)} (x^2 - 2xy + 3y^2 - 4x + 3y - 6)$$
$$= (\lim_{(x,y)\to(2,-1)} x^2) - (\lim_{(x,y)\to(2,-1)} 2xy) + (\lim_{(x,y)\to(2,-1)} 3y^2) - (\lim_{(x,y)\to(2,-1)} 4x)$$
$$+(\lim_{(x,y)\to(2,-1)} 3y) - (\lim_{(x,y)\to(2,-1)} 6).$$

Next, use the constant multiple law on the second, third, fourth, and fifth limits:

$$= (\lim_{(x,y)\to(2,-1)} x^2) - 2(\lim_{(x,y)\to(2,-1)} xy) + 3(\lim_{(x,y)\to(2,-1)} y^2) - 4(\lim_{(x,y)\to(2,-1)} x)$$
$$+3(\lim_{(x,y)\to(2,-1)} y) - \lim_{(x,y)\to(2,-1)} 6.$$

Now, use the power law on the first and third limits, and the product law on the second limit:

$$(\lim_{(x,y)\to(2,-1)} x)^2 - 2(\lim_{(x,y)\to(2,-1)} x)(\lim_{(x,y)\to(2,-1)} y) + 3(\lim_{(x,y)\to(2,-1)} y)^2$$
$$-4(\lim_{(x,y)\to(2,-1)} x) + 3(\lim_{(x,y)\to(2,-1)} y) - \lim_{(x,y)\to(2,-1)} 6.$$

Last, use the identity laws on the first six limits and the constant law on the last limit:

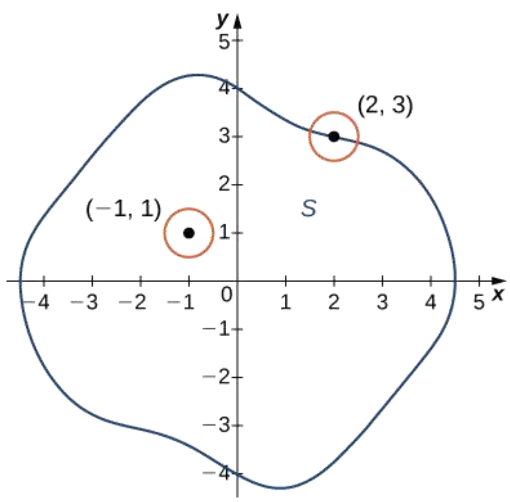
$$\lim_{(x,y)\to(2,-1)} (x^2 - 2xy + 3y^2 - 4x + 3y - 6) = (2)^2 - 2(2)(-1)$$
$$+3(-1)^2 - 4(2) + 3(-1) - 6 = -6.$$

3.3.2 Interior Points and Boundary Points

To study continuity and differentiability of a function of two or more variables, we first need to learn some new terminology.

Definition of interior and boundary points: Let S be a subset of $\mathbb{R}^2$, then we define:

1. A point P_0 is called *an interior point* of S if there is a δ-disk centered around P_0 contained completely in S.
2. A point P_0 is called *a boundary point* of S if every δ-disk centered around P_0 contains points both inside and outside S.



Definition of Open and closed sets: Let S be a subset of $\mathbb{R}^2$, then we define:

1. S is called an *open set* if every point of S is an interior point.
2. S is called a *closed set* if S contains all its boundary points.

3.3.3 Continuity of Functions of Two Variables

In Continuity, we defined the continuity of a function of one variable as three conditions are necessary for $f(x)$ to be continuous at point $x = a$

1. $f(a)$ exists
2. $\lim\limits_{x\to a} f(x)$ exists
3. $\lim\limits_{x\to a} f(x) = f(a)$.

These three conditions are necessary for continuity of a function of two variables as well.

Definition of continuous Functions: A function $f(x, y)$ is continuous at a point (a, b) in its domain if the following conditions are satisfied:

1. $f(a, b)$ exists
2. $\lim\limits_{x\to a} f(x, y)$ exists
3. $\lim\limits_{(x,y)\to(a,b)} f(x, y) = f(a, b)$.

Example: Show that the function $f(x, y) = \frac{3x+2y}{x+y+1}$ is continuous at point $(5, -3)$.

There are three conditions to be satisfied, per the definition of continuity. In this example, $a = 5$ and $b = -3$.

1. $f(a, b)$ exists. This is true because the domain of the function f consists of those ordered pairs for which the denominator is nonzero (i.e., $x + y + 1 \neq 0$). Point (5,–3) satisfies this condition. Furthermore,

$$f(a, b) = f(5, -3) = \frac{3(5) + 2(-3)}{5 + (-3) + 1} = \frac{15 - 6}{2 + 1} = 3.$$

2. $\lim\limits_{(x,y)\to(a,b)} f(x, y)$ exists. This also is true because

$$\lim_{(x,y)\to(a,b)} f(x, y) = \lim_{(x,y)\to(5,-3)} \frac{3x + 2y}{x + y + 1} = \frac{\lim\limits_{(x,y)\to(5,-3)} (3x + 2y)}{\lim\limits_{(x,y)\to(5,-3)} (x + y + 1)}$$

$$= \frac{15-6}{5-3+1} = 3.$$

3. $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$. This is true because we have just shown that both sides of this equation equal three.

The limit of a function of three or more variables can be introduced in the same manner.