3.4 Partial Derivatives

3.4.1 Derivatives of a Function of Two Variables

We have already examined limits and continuity of functions of two variables, we can proceed to study derivatives.

When studying derivatives of functions of one variable, we found that one interpretation of the derivative is an instantaneous rate of change of y – dependent variable as a function of x – independent variable. *Leibniz notation* for the derivative is $\frac{dy}{dx}$.

For a function $z = f(x, y)$ of two variables, x and y are the independent variables and z is the dependent variable. How do we adapt Leibniz notation for functions of two variables? The answer lies in introducing of partial derivatives.

Definition of partial derivatives: Let $f(x, y)$ be a function of two variables. Then the partial derivative of f with respect to x, written as $\frac{\partial f}{\partial x}$, , or f_x, is defined as

$$\frac{\partial f}{\partial x} = f_x(x, y) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

The partial derivative of f with respect to y, written as $\frac{\partial f}{\partial y}$, or f_y, is defined as

$$\frac{\partial f}{\partial y} = f_y(x, y) = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}.$$

This definition shows two differences already. First, the notation changes, in the sense that we still use a version of Leibniz notation, but the d in the original notation is replaced with the symbol ∂. (This rounded "d" is usually called "partial," so $\partial f / \partial x$ is spoken as the "partial of f with respect to x.")

Second, we now have two different derivatives we can take, since there are two different independent variables. Depending on which variable we choose, we can come up with different partial derivatives altogether, and often do.

As a practical hint, when calculating partial derivatives is to treat all independent variables, other than the variable with respect to which we are differentiating, as constants. Then proceed to differentiate as with a function of a single variable. In doing so all differentiation rules for differentiating functions of a single variable apply.

Example: Calculate $\partial f/\partial x$ and $\partial f/\partial y$ for the following functions:

(a) $f(x,y) = x^2 - 3xy + 2y^2 - 4x + 5y - 12$

To calculate $\partial f/\partial x$, treat the variable y as a constant. Then differentiate $f(x,y)$ with respect to x using the sum, difference, and power rules:

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}[x^2 - 3xy + 2y^2 - 4x + 5y - 12] \\ &= \frac{\partial}{\partial x}[x^2] - \frac{\partial}{\partial x}[3xy] + \frac{\partial}{\partial x}[2y^2] - \frac{\partial}{\partial x}[4x] + \frac{\partial}{\partial x}[5y] - \frac{\partial}{\partial x}[12] \\ &= 2x - 3y + 0 - 4 + 0 - 0 = 2x - 3y - 4.\end{aligned}$$

Calculating $\partial f/\partial y$:

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}[x^2 - 3xy + 2y^2 - 4x + 5y - 12] \\ &= \frac{\partial}{\partial y}[x^2] - \frac{\partial}{\partial y}[3xy] + \frac{\partial}{\partial y}[2y^2] - \frac{\partial}{\partial y}[4x] + \frac{\partial}{\partial y}[5y] - \frac{\partial}{\partial y}[12] \\ &= -3x + 4y - 0 + 5 - 0 = -3x + 4y + 5.\end{aligned}$$

(b) $g(x,y) = \sin(x^2y - 2x + 4)$

To calculate $\partial g/\partial x$, treat the variable y as a constant. Then differentiate $g(x,y)$ with respect to x using the chain rule and power rule:

$$\begin{aligned}\frac{\partial g}{\partial x} &= \frac{\partial}{\partial x}[\sin(x^2y - 2x + 4)] \\ &= \cos(x^2y - 2x + 4)\frac{\partial}{\partial x}[x^2y - 2x + 4] \\ &= (2xy - 2)\cos(x^2y - 2x + 4).\end{aligned}$$

To calculate $\partial g/\partial y$, treat the variable x as a constant. Then differentiate $g(x,y)$ with respect to y using the chain rule and power rule:

$$\frac{\partial g}{\partial y} = \frac{\partial}{\partial y}[\sin(x^2y - 2x + 4)]$$

$$= \cos(x^2y - 2x + 4)\frac{\partial}{\partial y}[x^2y - 2x + 4] = x^2\cos(x^2y - 2x + 4).$$

(c) $f(x,y) = \tan(x^3 - 3x^2y^2 + 2y^4)$

$$\frac{\partial f}{\partial x} = (3x^2 - 6xy^2)\sec^2(x^3 - 3x^2y^2 + 2y^4)$$

$$\frac{\partial f}{\partial y} = (-6x^2y + 8y^3)\sec^2(x^3 - 3x^2y^2 + 2y^4)$$

We can interpret these partial derivatives as follows. If we graph $f(x,y)$ and $f(x + \Delta x, y)$ for an arbitrary point (x,y), then the slope of the secant line passing through these two points is given by

$$\frac{f(x + \Delta x, y) - f(x,y)}{\Delta x}$$

This line is parallel to the x-axis. Therefore, the slope of the secant line represents an average rate of change of the function f as we travel parallel to the x-axis. As Δx approaches zero, the slope of the secant line approaches the slope of the tangent line.

If we choose to change y instead of x by the same incremental value Δy, then the secant line is parallel to the y-axis and so is the tangent line. Therefore, $\partial f/\partial x$ represents the slope of the tangent line passing through the point $(x, y, f(x,y))$ parallel to the x-axis and $\partial f/\partial y$ represents the slope of the tangent line passing through the point $(x, y, f(x,y))$ parallel to the y-axis.

If we wish to find the slope of a tangent line passing through the same point in any other direction, then we need what are called *directional derivatives*.

3.4.2 Functions of More Than Two Variables

Suppose we have a function of three variables, such as $w = f(x,y,z)$. We can calculate partial derivatives of w with respect to any of the independent variables, simply as extensions of the definitions for partial derivatives of functions of two variables.

Definition of Partial Derivatives: Let $f(x,y,z)$ be a function of three variables. Then, the partial derivative of f with respect to x, written as $\partial f/\partial x$, or f_x, is defined to be

$$\frac{\partial f}{\partial x} = f_x(x,y,z) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y, z) - f(x,y,z)}{\Delta x}.$$

The partial derivative of f with respect to y, written as $\partial f/\partial y$, or f_y, is defined to be

$$\frac{\partial f}{\partial y} = f_y(x,y,z) = \lim_{\Delta y \to 0} \frac{f(x,y+\Delta y,z) - f(x,y,z)}{\Delta y}$$

The partial derivative of f with respect to z, written as $\partial f/\partial z$, or f_z, is defined to be

$$\frac{\partial f}{\partial z} = f_z(x,y,z) = \lim_{\Delta z \to 0} \frac{f(x,y,z+\Delta z) - f(x,y,z)}{\Delta z}.$$

We can calculate a partial derivative of a function of three variables using the same idea we used for a function of two variables. For example, if we have a function f of x, y, and z, and we wish to calculate $\partial f/\partial x$, then we treat the other two independent variables as if they are constants, then differentiate with respect to x.

Example: Calculate the three partial derivatives of the following functions.

(a) $f(x,y,z) = \frac{(x^2y-4xz+y^2)}{(x-3yz)}$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[\frac{x^2y-4xz+y^2}{x-3yz}\right]$$

$$= \frac{\frac{\partial}{\partial x}(x^2y-4xz+y^2)(x-3yz) - (x^2y-4xz+y^2)\frac{\partial}{\partial x}(x-3yz)}{(x-3yz)^2}$$

$$= \frac{(2xy-4z)(x-3yz) - (x^2y-4xz+y^2)(1)}{(x-3yz)^2}$$

$$= \frac{2x^2y-6xy^2z-4xz+12yz^2-x^2y+4xz-y^2}{(x-3yz)^2}$$

$$= \frac{x^2y-6xy^2z-4xz+12yz^2+4xz-y^2}{(x-3yz)^2}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[\frac{x^2y-4xz+y^2}{x-3yz}\right]$$

$$= \frac{\frac{\partial}{\partial y}(x^2y-4xz+y^2)(x-3yz) - (x^2y-4xz+y^2)\frac{\partial}{\partial y}(x-3yz)}{(x-3yz)^2}$$

$$= \frac{(x^2+2y)(x-3yz) - (x^2y-4xz+y^2)(-3z)}{(x-3yz)^2}$$

$$= \frac{x^3-3x^2yz+2xy-6y^2z+3x^2yz-12xz^2+3y^2z}{(x-3yz)^2}$$

$$= \frac{x^3+2xy-3y^2z-12xz^2}{(x-3yz)^2}$$

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}[\frac{x^2y - 4xz + y^2}{x - 3yz}]$$

$$= \frac{\frac{\partial}{\partial z}(x^2y - 4xz + y^2)(x - 3yz) - (x^2y - 4xz + y^2)\frac{\partial}{\partial z}(x - 3yz)}{(x - 3yz)^2}$$

$$= \frac{(-4x)(x - 3yz) - (x^2y - 4xz + y^2)(-3y)}{(x - 3yz)^2}$$

$$= \frac{-4x^2 + 12xyz + 3x^2y^2 - 12xyz + 3y^3}{(x - 3yz)^2}$$

$$= \frac{-4x^2 + 3x^2y^2 + 3y^3}{(x - 3yz)^2}$$

(b) $g(x, y, z) = \sin(x^2y - z) + \cos(x^2 - yz)$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}[\sin(x^2y - z) + \cos(x^2 - yz)]$$

$$= (\cos(x^2y - z))\frac{\partial}{\partial x}(x^2y - z) - (\sin(x^2 - yz))\frac{\partial}{\partial x}(x^2 - yz)$$

$$= 2xy\cos(x^2y - z) - 2x\sin(x^2 - yz)$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}[\sin(x^2y - z) + \cos(x^2 - yz)]$$

$$= (\cos(x^2y - z))\frac{\partial}{\partial y}(x^2y - z) - (\sin(x^2 - yz))\frac{\partial}{\partial y}(x^2 - yz)$$

$$= x^2\cos(x^2y - z) + z\sin(x^2 - yz)$$

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}[\sin(x^2y - z) + \cos(x^2 - yz)]$$

$$= (\cos(x^2y - z))\frac{\partial}{\partial z}(x^2y - z) - (\sin(x^2 - yz))\frac{\partial}{\partial z}(x^2 - yz)$$

$$= -\cos(x^2y - z) + y\sin(x^2 - yz)$$

3.5 Higher-Order Partial Derivatives

Consider the function

$$f(x, y) = 2x^3 - 4xy^2 + 5y^3 - 6xy + 5x - 4y + 12.$$

Its partial derivatives are

$$\frac{\partial f}{\partial x} = 6x^2 - 4y^2 - 6y + 5 \text{ and } \frac{\partial f}{\partial y} = -8xy + 15y^2 - 6x - 4.$$

Each of these partial derivatives is a function of two variables, so we can calculate partial derivatives of these functions. Just as with derivatives of single-variable functions, we can call these second-order derivatives, third-order derivatives, and so on. In general, they are referred to as *higher-order partial derivatives*. There are four second-order partial derivatives for any function (provided they all exist):

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right]\\ \frac{\partial^2 f}{\partial y \partial x} &= \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right]\\ \frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right]\\ \frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right].\end{aligned}$$

An alternative notation for each is f_{xx}, f_{xy}, f_{yx}, and f_{yy}, respectively. Higher-order partial derivatives calculated with respect to different variables, such as f_{xy} and f_{yx}, are commonly called *mixed partial derivatives*.

Example: Calculate all four second partial derivatives for the function.

(a) $f(x,y) = xe^{-3y} + \sin(2x - 5y)$. We first calculate $\partial f/\partial x$: $\frac{\partial f}{\partial x} = e^{-3y} + 2\cos(2x - 5y)$.

Then,

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right] = \frac{\partial}{\partial x}\left[e^{-3y} + 2\cos(2x - 5y)\right] = -4\sin(2x - 5y)$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right] = \frac{\partial}{\partial y}\left[e^{-3y} + 2\cos(2x - 5y)\right] = -3e^{-3y} + 10\sin(2x - 5y)$$

Also, calculating $\partial f/\partial y$: $\frac{\partial f}{\partial y} = -3xe^{-3y} - 5\cos(2x - 5y)$.

We can calculate

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right] = \frac{\partial}{\partial y}\left[-3xe^{-3y} - 5\cos(2x - 5y)\right] = 9xe^{-3y} - 25\sin(2x - 5y)$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right] = \frac{\partial}{\partial x}\left[-3xe^{-3y} - 5\cos(2x - 5y)\right] = -3e^{-3y} + 10\sin(2x - 5y)$$

(b) $f(x,y) = \sin(3x - 2y) + \cos(x + 4y)$.

$$\frac{\partial^2 f}{\partial x^2} = -9\sin(3x - 2y) - \cos(x + 4y)$$
$$\frac{\partial^2 f}{\partial y \partial x} = 6\sin(3x - 2y) - 4\cos(x + 4y)$$
$$\frac{\partial^2 f}{\partial x \partial y} = 6\sin(3x - 2y) - 4\cos(x + 4y)$$
$$\frac{\partial^2 f}{\partial y^2} = -4\sin(3x - 2y) - 16\cos(x + 4y)$$

At this point we should notice that, in both Example and the checkpoint, it was true that $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$ Under certain conditions, this is always true. In fact, it is a direct consequence of the following theorem.

Equality of Mixed Partial Derivatives (Clairaut's Theorem) Suppose that $f(x,y)$ is defined on an open disk D that contains the point (a,b). If the functions f_{xy} and f_{yx} are continuous on D, then $f_{xy} = f_{yx}$.

Clairaut's theorem guarantees that as long as mixed second-order derivatives are continuous, the order in which we choose to differentiate the functions (i.e., which variable goes first, then second, and so on) does not matter. It can be extended to higher-order derivatives as well. Two other second-order partial derivatives can be calculated for any function $f(x,y)$. The partial derivative f_{xx} is equal to the partial derivative of f_x with respect to x, and f_{yy} is equal to the partial derivative of f_y with respect to y.

Example. $u = e^{xyz}$. Find $\frac{\partial^3 u}{\partial x^2 \partial y}$.

$$\frac{\partial u}{\partial x} = yz \cdot e^{xyz}. \qquad \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(yz \cdot e^{xyz}) = y^2 z^2 e^{xyz}.$$

$$\frac{\partial^3 u}{\partial x^2 \partial y} = \frac{\partial}{\partial y}(y^2 z^2 e^{xyz}) = 2yz^2 e^{xyz} + xy^2 z^3 e^{xyz} = yz^2 e^{xyz}(2 + xyz).$$

3.6 Differentials

The differential of a function of one variable $y = f(x)$, written as dy, is defined as $f'(x)dx$. The differential is used to approximate $\Delta y = f(x + \Delta x) - f(x)$, where

$\Delta x = dx$. Extending this idea to the linear approximation of a function of two variables at the point (x_0, y_0) yields the formula for *the total differential* for a function of two variables.

Definition: Let $z = f(x, y)$ be a function of two variables with (x_0, y_0) in the domain of $f(x, y)$, and let Δx and Δy be chosen so that $(x_0 + \Delta x, y_0 + \Delta y)$ is also in the domain of $f(x, y)$. If $f(x, y)$ is differentiable at the point (x_0, y_0), then the differentials dx and dy are defined as

$$dx = \Delta x \text{ and } dy = \Delta y$$

The differential dz, also called the *total differential* of $z = f(x, y)$ at (x_0, y_0), is defined as

$$dz = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy.$$

Notice that the symbol ∂ is not used to denote the total differential; rather, d appears in front of z.

Now, let's define $\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$. We use dz to approximate Δz, so

$$\Delta z \approx dz = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy.$$

Therefore, the differential is used to approximate the change in the function $z = f(x_0, y_0)$ at the point (x_0, y_0) for given values of Δx and Δy. Since $\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$ this can be used further to approximate $f(x + \Delta x, y + \Delta y)$:

$$f(x + \Delta x, y + \Delta y) \approx f(x, y) + f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y.$$

Example: Find the differential dz of the function $f(x, y) = 3x^2 - 2xy + y^2$ and use it to approximate Δz at point $(2, -3)$. Use $\Delta x = 0.1$ and $\Delta y = -0.05$. What is the exact value of Δz ?

First, we must calculate $f(x_0, y_0)$, $f_x(x_0, y_0)$, and $f_y(x_0, y_0)$ using $x_0 = 2$ and $x_0 = -3$.

$$f(x_0, y_0) = f(2, -3) = 3(2)^2 - 2(2)(-3) + (-3)^2 = 12 + 12 + 9 = 33$$

$f_x(x, y) = 6x - 2y$, $f_x(x_0, y_0) = f_x(2, -3) = 6(2) - 2(-3) = 12 + 6 = 18$

$f_y(x, y) = -2x + 2y$, $f_y(x_0, y_0) = f_y(2, -3) = -2(2) + 2(-3) = -4 - 6 = -10$

Then, we substitute these quantities into Equation of the differential:

$$dz = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy$$
$$dz = 18(0.1) - 10(-0.05) = 1.8 + 0.5 = 2.3.$$

This is the approximation to $\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$. The exact value of Δz is given by

$$\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0) = f(2 + 0.1, -3 - 0.05) - f(2, -3)$$
$$= f(2.1, -3.05) - f(2, -3) = 2.3425$$

Example: Find an approximate value of $(0.98)^{3.03}$.

The required value can be considered as a value of the function $z = x^y$ at $x = x_0 + \Delta x$, $y = y_0 + \Delta y$, where $x_0 = 1$, $y_0 = 3$, $\Delta x = 0.02$, $\Delta y = 0.03$.

$$z(1,3) = 1^3 = 1; \qquad \left.\frac{\partial z}{\partial x}\right|_{(1,3)} = \left. yx^{y-1}\right|_{(1,3)} = 3$$
$$\left.\frac{\partial z}{\partial y}\right|_{(1,3)} = \left. x^y \ln x\right|_{1,3)} = 0$$

According to the formula we have $(0.98)^{3.03} \approx 1 + 3 \cdot (-0.02) = 0.94$.

Differentials of a Function of more than Two Variables

The differential du , also called the *total differential* of $u = f(x, y, z)$ at (x_0, y_0, z_0), is defined as

$$du = f_x(x_0, y_0, z_0)dx + f_y(x_0, y_0, z_0)dy + f_z(x_0, y_0, z_0)dz$$

Example. Find total differential of the function $u = ln(z + \sqrt{x^2 + y^2})$. Let's find the partial derivatives first:

$$\frac{\partial u}{\partial x} = \frac{x}{(z + \sqrt{x^2 + y^2}) \cdot \sqrt{x^2 + y^2}}$$
$$\frac{\partial u}{\partial y} = \frac{y}{(z + \sqrt{x^2 + y^2}) \cdot \sqrt{x^2 + y^2}}$$
$$\frac{\partial u}{\partial z} = \frac{1}{z + \sqrt{x^2 + y^2}}$$

Then,

$$du = \frac{x}{(z+\sqrt{x^2+y^2})\cdot\sqrt{x^2+y^2}}dx + \frac{y}{(z+\sqrt{x^2+y^2})\cdot\sqrt{x^2+y^2}}dy +$$

$$+\frac{1}{z+\sqrt{x^2+y^2}}dz = \frac{xdx+ydy+\sqrt{x^2+y^2}dz}{(z+\sqrt{x^2+y^2})\cdot\sqrt{x^2+y^2}}$$

In general case, the total differential of function $u = f(x_1, x_2, ..., x_n)$ of n variables is defined by

$$du = \frac{\partial u}{\partial x_1}dx_1 + \frac{\partial u}{\partial x_2}dx_2 + ... + \frac{\partial u}{\partial x_n}dx_n$$

Differentials of the higher orders

Let the function $u = f(x_1, x_2, ..., x_n)$ have continuous partial derivatives of the 1st order. Obviously du is also the function of n variables $x_1, x_2, ..., x_n$ and therefore we can find the total differential of this function, i.e. $d(du)$, which is called the *differential of the 2nd order* and is designated as d^2u.

Similarly we can find $d^3u = d(d^2u)$ – *differential of the 3rd order*, etc. Thus, the *differential of the nth order* takes the form:

$$d^n u = d(d^{n-1}u)$$

Note: Thus the increments $dx_1, dx_2, ..., dx_n$ of independent variables are considered constant and while passing from one differential to another they remain the same.

Let's consider a function of two variables $z = f(x, y)$. Then,

$$d^2z = d(dz) = d\left(\frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy\right)dx +$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy\right)dy = \frac{\partial^2 z}{\partial x^2}dx^2 + \frac{\partial^2 z}{\partial y\partial x}dydx + \frac{\partial^2 z}{\partial x\partial y}dxdy + \frac{\partial^2 z}{\partial y^2}dy^2 =$$

$$\frac{\partial^2 z}{\partial x^2}dx^2 + 2\frac{\partial^2 z}{\partial x\partial y}dxdy + \frac{\partial^2 z}{\partial y^2}dy^2 = \left(\frac{\partial^2}{\partial x^2}dx^2 + 2\frac{\partial^2}{\partial x\partial y}dxdy + \frac{\partial^2}{\partial y^2}dy^2\right)z =$$

$$\left(\frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy\right)^2 z$$

Hence,

$$d^2z = (\frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy)^2 z = \frac{\partial^2 z}{\partial x^2}dx^2 + 2\frac{\partial^2 z}{\partial x \partial y}dxdy + \frac{\partial^2 z}{\partial y^2}dy^2$$

$$d^3z = (\frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy)^3 z = \frac{\partial^3 z}{\partial x^3}dx^3 + 3\frac{\partial^3 z}{\partial x^2 \partial y}dx^2dy +$$

$$+3\frac{\partial^3 z}{\partial x \partial y^2}dxdy^2 + \frac{\partial^3 z}{\partial y^3}dy^3$$

For differentials of higher orders there takes place the following symbolical formula:

$$d^m u = (\frac{\partial}{\partial x_1}dx_1 + \frac{\partial}{\partial x_2}dx_2 + ... + \frac{\partial}{\partial x_n}dx_n)^m u$$

Example. $z = arctan\, x/y$. Find d^2z.At first let's find the partial derivatives of the 1st order of the given function

$$\frac{\partial u}{\partial x} = \frac{1}{1 + x^2/y^2} \cdot \frac{1}{y} = \frac{y}{x^2 + y^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{1 + x^2/y^2} \cdot (-\frac{x}{y^2}) = -\frac{x}{x^2 + y^2}$$

Then let's find partial derivatives of the 2nd order

$$\frac{\partial^2 u}{\partial x^2} = -\frac{2xy}{(x^2 + y^2)^2}. \quad \frac{\partial^2 u}{\partial y^2} = \frac{2xy}{(x^2 + y^2)^2}. \quad \frac{\partial^2 u}{\partial x \partial y} = \frac{x^2 - y^2}{(x^2 + y^2)^2}.$$

Substituting found derivatives in the formula we obtain

$$d^2z = \frac{2xy(dy^2 - dx^2) + 2(x^2 - y^2)dxdy}{(x^2 + y^2)^2}.$$

3.7 The Chain Rule for Multivariable Functions

In single-variable calculus, we found that one of the most useful differentiation rules is the chain rule, which allows us to find the derivative of the composition of two functions. The same thing is true for multivariable calculus, but this time we have to deal with more than one form of the chain rule. In this section, we study extensions of the chain rule and learn how to take derivatives of compositions of functions of more than one variable.

Recall that the chain rule for the derivative of a composite of two functions can be written in the form

$$\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x).$$

In this equation, both $f(x)$ and $g(x)$ are functions of one variable. Now suppose that f is a function of two variables and $g(x)$ is a function of one variable. Or perhaps they are both functions of two variables, or even more. How would we calculate the derivative in these cases? The following theorem gives us the answer for the case of one independent variable.

3.7.1 Chain Rule for One Independent Variables

Suppose that $x = g(t)$ and $y = h(t)$ are differentiable functions of t and $z = f(x, y)$ is a differentiable function of x and y. Then $z = f(x(t), y(t))$ is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt},$$

where the ordinary derivatives are evaluated at t and the partial derivatives are evaluated at (x, y).

Two terms appear on the right-hand side of the formula, and f is a function of two variables. This pattern works with functions of more than two variables as well.

Example. Calculate $\frac{dz}{dt}$ for each of the following functions:

1) $z = f(x, y) = 4x^2 + 3y^2, x = x(t) = \sin t, y = y(t) = \cos t$

To use the chain rule, we need four quantities: $\frac{\partial z}{\partial x} = 8x$, $\frac{dx}{dt} = \cos t$, $\frac{\partial z}{\partial y} = 6y$, $\frac{dy}{dt} = -\sin t$

Thus, $\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt} = (8x)(\cos t) + (6y)(-\sin t) = 8x\cos t - 6y\sin t.$

This answer has three variables in it. To reduce it to one variable, use the fact that $x(t) = \sin t$ and $y(t) = \cos t$. We obtain

$$\frac{dz}{dt} = 8x\cos t - 6y\sin t = 8(\sin t)\cos t - 6(\cos t)\sin t = 2\sin t\cos t.$$

This derivative can also be calculated by first substituting $x(t)$ and $y(t)$ into $f(x, y)$, then differentiating with respect to t :

$$z = f(x, y) = f(x(t), y(t)) = 4(x(t))^2 + 3(y(t))^2 = 4\sin^2 t + 3\cos^2 t.$$

Then,

$$\begin{aligned}\frac{dz}{dt} &= 2(4\sin t)(\cos t) + 2(3\cos t)(-\sin t) = 8\sin t\cos t - 6\sin t\cos t \\ &= 2\sin t\cos t,\end{aligned}$$

which is the same solution. However, it may not always be this easy to differentiate in this form.

2) $z = f(x, y) = \sqrt{x^2 - y^2}, x = x(t) = e^{2t}, y = y(t) = e^{-t}$

$$\frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2-y^2}}, \frac{dx}{dt} = 2e^{2t}, \frac{\partial z}{\partial y} = \frac{-y}{\sqrt{x^2-y^2}}, \frac{dx}{dt} = -e^{-t}.$$

$$\begin{aligned}\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt} &= (\frac{x}{\sqrt{x^2 - y^2}})(2e^{2t}) + (\frac{-y}{\sqrt{x^2 - y^2}})(-e^{-t}) \\ &= \frac{2xe^{2t} - ye^{-t}}{\sqrt{x^2 - y^2}}\end{aligned}$$

To reduce this to one variable, we use the fact that $x(t) = e^{2t}$ and $y(t) = e^{-t}$

$$\frac{dz}{dt} = \frac{2xe^2t + ye^{-t}}{\sqrt{x^2 - y^2}} = \frac{2(e^{2t})e^{2t} + (e^{-t})e^{-t}}{\sqrt{e^{4t} - e^{-2t}}} = \frac{2e^{4t} + e^{-2t}}{\sqrt{e^{4t} - e^{-2t}}}$$

3) $z = f(x, y) = x^2 - 3xy + 2y^2$, $x = x(t) = 3\sin 2t, y = y(t) = 4\cos 2t$

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = (2x - 3y)(6\cos 2t) + (-3x + 4y)(-8\sin 2t)$$

$$= -92\sin 2t\cos 2t - 72(\cos^2 2t - \sin^2 2t) = -46\sin 4t - 72\cos 4t.$$

In case, $z = f(x, y)$ is a function of x nd y, and both $x = g(u, v)$ and $y =$

$h(u, v)$ are functions of the independent variables u and v.

3.7.2 Chain Rule for Two Independent Variables:

Suppose $x = g(u, v)$ and $y = h(u, v)$ are differentiable functions of u and v, and $z = f(x, y)$ is a differentiable function of x and y. Then, $z = f(g(u, v), h(u, v))$ is a differentiable function of u and v, and

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \text{ and } \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}.$$

Example: Using the chain rule for two variables calculate $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ of the following functions:

1) $z = f(x, y) = 3x^2 - 2xy + y^2, x = x(u, v) = 3u + 2v, y = y(u, v) = 4u - v.$

To implement the chain rule for two variables, we need six partial derivatives:

$$\frac{\partial z}{\partial x} = 6x - 2y, \frac{\partial z}{\partial y} = -2x + 2y, \frac{\partial x}{\partial u} = 3\,\frac{\partial x}{\partial v} = 2,\ \frac{\partial y}{\partial u} = 4\,\frac{\partial y}{\partial v} = -1.$$

To find $\frac{\partial z}{\partial u}$, we use Equation

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} = 3(6x - 2y) + 4(-2x + 2y) = 10x + 2y$$

Next, we substitute $x(u, v) = 3u + 2v$ and $y(u, v) = 4u - v$:

$$\frac{\partial z}{\partial u} = 10x + 2y = 10(3u + 2v) + 2(4u - v) = 38u + 18v$$

To find $\frac{\partial z}{\partial v}$, we use Equation

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = 2(6x - 2y) + (-1)(-2x + 2y) = 14x - 6y$$

Then we substitute $x(u, v) = 3u + 2v$ and $y(u, v) = 4u - v$:

$$\frac{\partial z}{\partial v} = 14x - 6y = 14(3u + 2v) - 6(4u - v) = 18u + 34v$$

Example: Using the chain rule for two variables calculate $\partial z/\partial u$ and $\partial z/\partial v$ given the following functions:

$$z = f(x,y) = \frac{2x-y}{x+3y}, x(u,v) = e^{2u}\cos 3v, y(u,v) = e^{2u}\sin 3v.$$

$$\frac{\partial z}{\partial u} = 0, \frac{\partial z}{\partial v} = \frac{-21}{(3\sin 3v + \cos 3v)^2}$$

3.7.3 The Generalized Chain Rule

Now that we've see how to extend the original chain rule to functions of two variables, it is natural to ask: Can we extend the rule to more than two variables? The answer is yes, as the generalized chain rule states.

Let $w = f(x_1, x_2, \ldots, x_m)$ be a differentiable function of m independent variables, and for each $i \in 1, \ldots, m$, let $x_i = x_i(t_1, t_2, \ldots, t_n)$ be a differentiable function of n independent variables. Then

$$\frac{\partial w}{\partial t_j} = \frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial t_j} + \frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial t_j} + \cdots + \frac{\partial w}{\partial x_m}\frac{\partial x_m}{\partial t_j}$$

for any $j \in 1,2, \ldots, n$.

Example: Using the generalized chain rule

$w = f(x,y,z) = 3x^2 - 2xy + 4z^2$,

$x = x(u,v) = e^u \sin v, y = y(u,v) = e^u \cos v, z = z(u,v) = e^u$

The formulas for $\partial w/\partial u$ and $\partial w/\partial v$ are

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial u}$$
$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial v}.$$

Therefore, there are nine different partial derivatives that need to be calculated and substituted. We need to calculate each of them:

$$\frac{\partial w}{\partial x} = 6x - 2y, \frac{\partial w}{\partial y} = -2x, \frac{\partial w}{\partial z} = 8z$$

$$\frac{\partial x}{\partial u} = e^u \sin v, \frac{\partial y}{\partial u} = e^u \cos v, \frac{\partial z}{\partial u} = e^u,$$

$$\frac{\partial x}{\partial v} = e^u \cos v, \frac{\partial y}{\partial v} = -e^u \sin v, \frac{\partial z}{\partial v} = 0$$

Now, we substitute each of them into the first formula to calculate $\partial w/\partial u$:

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial u} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial u} = (6x - 2y)e^u \sin v - 2xe^u \cos v + 8ze^u$$

then substitute $x(u,v) = e^u \sin v$, $y(u,v) = e^u \cos v$, $z(u,v) = e^u$ into this equation:

$$\begin{gathered} \frac{\partial w}{\partial u} = (6x - 2y)e^u \sin v - 2xe^u \cos v + 8ze^u \\ = (6e^u \sin v - 2eu \cos v)e^u \sin v - 2(e^u \sin v)e^u \cos v + 8e^{2u} \\ = 6e^{2u} \sin^2 v - 4e^{2u} \sin v \cos v + 8e^{2u} \\ = 2e^{2u}(3\sin^2 v - 2\sin v \cos v + 4). \end{gathered}$$

Next, we calculate $\partial w/\partial v$:

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial v} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial v} = (6x - 2y)e^u \cos v - 2x(-e^u \sin v) + 8z(0)$$

then substitute $x(u,v) = e^u \sin v$, $y(u,v) = e^u \cos v$, $z(u,v) = e^u$ into this equation:

$$\begin{gathered} \frac{\partial w}{\partial v} = (6x - 2y)e^u \cos v - 2x(-e^u \sin v) \\ = (6e^u \sin v - 2e^u \cos v)e^u \cos v + 2(e^u \sin v)(e^u \sin v) \\ = 2e^{2u} \sin^2 v + 6e^{2u} \sin v \cos v - 2e^{2u} \cos^2 v \\ = 2e^{2u}(\sin^2 v + \sin v \cos v - \cos^2 v). \end{gathered}$$

3.8 Implicit Differentiation of a Function of Two or More Variables

Theorem: Suppose the function $z = f(x,y)$ defines y implicitly as a function $y = g(x)$ of x via the equation $f(x,y) = 0$. Then

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}$$

provided $f_y(x,y) \neq 0$.

If the equation $f(x,y,z) = 0$ defines z implicitly as a differentiable function of x and y, then

$$\frac{dz}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial z} \text{ and } \frac{dz}{dy} = -\frac{\partial f/\partial y}{\partial f/\partial z}$$

as long as $f_z(x,y,z) \neq 0$.

Example. $f(x,y) = x^2 + 3y^2 + 4y - 4$,

$\frac{\partial f}{\partial x} = 2x$ and $\frac{\partial f}{\partial y} = 6y + 4$

Then,

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{2x}{6y+4} = -\frac{x}{3y+2},$$

Example. $z = u^2 \ln v$, where $u = y/x$, $v = x^2 + y^2$.Find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$.

$$\frac{\partial z}{\partial x} = 2u\,ln\,v(-\frac{y}{x^2}) + \frac{u^2}{v}2x = 2\frac{y^2}{x}(\frac{1}{x^2+y^2} - \frac{ln(x^2+y^2)}{x^2})$$

$$\frac{\partial z}{\partial y} = 2u\,ln\,v\frac{1}{x} + \frac{u^2}{v}2y = 2\frac{y}{x^2}(ln(x^2+y^2) + \frac{y^2}{x^2+y^2})$$

Example. $f(x,y,z) = x^2e^y - yze^x$.

$\frac{\partial f}{\partial x} = 2xe^y - yze^x$, and $\frac{\partial f}{\partial y} = x^2e^y - ze^x$ and $\frac{\partial f}{\partial z} = -ye^x$

$$\frac{\partial z}{\partial x} = -\frac{\partial f/\partial x}{\partial f/\partial z} = -\frac{2xe^y - yze^x}{-ye^x} = \frac{2xe^y - yze^x}{ye^x}$$

$$\frac{\partial z}{\partial y} = -\frac{\partial f/\partial y}{\partial f/\partial z} = -\frac{x^2e^y - ze^x}{-ye^x} = \frac{x^2e^y - ze^x}{ye^x}$$