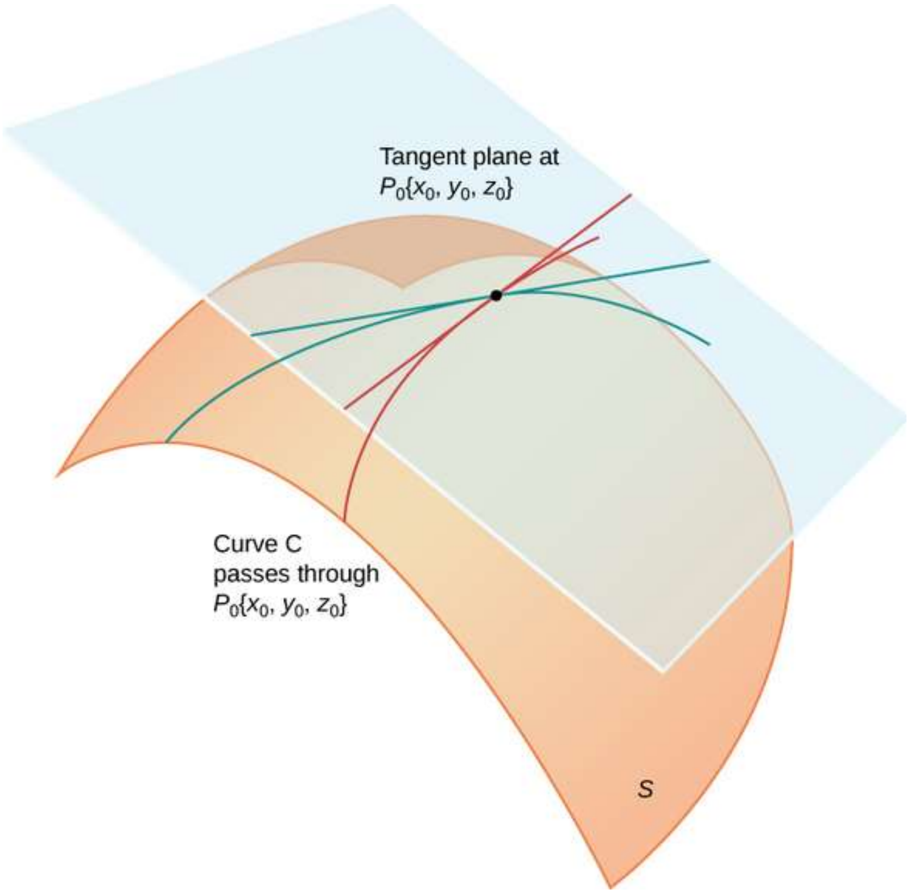
3.9 Geometrical Applications of Derivatives

3.9.1 Tangent Planes and Normal Lines

Intuitively, it seems clear that, in a plane, only one line can be tangent to a curve at a point. However, in three-dimensional space, many lines can be tangent to a given point. If these lines lie in the same plane, they determine the tangent plane at that point. A more intuitive way to think of a tangent plane is to assume the surface is smooth at that point (no corners). Then, a tangent line to the surface at that point in any direction does not have any abrupt changes in slope because the direction changes smoothly. Therefore, in a small-enough neighborhood around the point, a tangent plane touches the surface at that point only.



Definition: Let $P_0 = (x_0, y_0, z_0)$ be a point on a surface S, and let C be any curve passing through P_0 and lying entirely in S. If the *tangent lines* to all such curves C at P_0 lie in the same plane, then this plane is called the *tangent plane* to S at P_0 as shown in Figure.

Note: For a tangent plane to a surface to exist at a point on that surface, it is sufficient for the function that defines the surface to be differentiable at that point. We define the term tangent plane here and then explore the idea intuitively.

Definition: Let S be a surface defined by a differentiable function $z = f(x, y)$, and let $P_0 = (x_0, y_0)$ be a point in the domain of f. Then, the equation of the *tangent plane* to S at P_0 is given by

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Proof: To see why this formula is correct, let's first find two tangent lines to the surface S. The equation of the tangent line to the curve that is represented by the intersection of S with the vertical trace given by $x = x_0$ is

$$z = f(x_0, y_0) + f_y(x_0, y_0)(y - y_0)$$

Similarly, the equation of the tangent line to the curve that is represented by the intersection of S with the vertical trace given by $y = y_0$ is

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0)$$

A parallel vector to the first tangent line is

$$\vec{a} = \vec{j} + f_y(x_0, y_0)\vec{k}$$

and a parallel vector to the second tangent line is

$$\vec{b} = \vec{i} + f_x(x_0, y_0)\vec{k}$$

We can compose a vector by taking the cross product of these two vectors:

$$\vec{a} \times \vec{b} = \left(\vec{j} + f_y(x_0, y_0)\vec{k}\right) \times \left(\vec{i} + f_x(x_0, y_0)\vec{k}\right) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & f_y(x_0, y_0) \\ 1 & 0 & f_x(x_0, y_0) \end{vmatrix} =$$
$$= f_x(x_0, y_0)\vec{i} + f_y(x_0, y_0)\vec{j} - \vec{k}$$

which is perpendicular to both lines and is therefore perpendicular to the tangent plane. We can use this vector as a normal vector $\vec{n}$ to the tangent plane, along with the point $P_0 = (x_0, y_0, f(x_0, y_0))$ in the equation for a plane:

$$\vec{n} \cdot ((x - x_0)\vec{i} + (y - y_0)\vec{j} + (z - f(x_0, y_0))\vec{k}) = 0,$$
$$(f_x(x_0, y_0)\vec{i} + f_y(x_0, y_0)\vec{j} - \vec{k}) \cdot ((x - x_0)\vec{i} + (y - y_0)\vec{j} + (z - f(x_0, y_0))\vec{k}) = 0,$$
$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - f(x_0, y_0)) = 0.$$

Definition: The vector $\vec{n}$ perpendicular to the tangent plane defines also a line normal to the surface S, i.e. the equation of the line normal to the surface S is

$$\frac{(x-x_0)}{f_x(x_0,y_0)}=\frac{(y-y_0)}{f_y(x_0,y_0)}=\frac{(z-f(x_0,y_0))}{-1}$$

Example 1: Find the equation of the tangent plane to the surface defined by the function $f(x,y)=2x^2-3xy+8y^2+2x-4y+4$ at point $(2,-1)$.

First, we must calculate $f_x(x,y)$ and $f_y(\mathrm{x},y)$, then use the Equation with $x_0=2$ and $y_0=-1$:

$$f_x(x,y)=4x-3y+2$$
$$f_y(x,y)=-3x+16y-4$$
$$f(2,-1)=2(2)^2-3(2)(-1)+8(-1)^2+2(2)-4(-1)+4=34$$
$$f_x(2,-1)=4(2)-3(-1)+2=13$$
$$f_y(2,-1)=-3(2)+16(-1)-4=-26.$$

Then the Equation of a tangent plane becomes

$$z=f(x_0,y_0)+f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)$$
$$z=34+13(x-2)-26(y-(-1))$$
$$z=34+13x-26-26y-26$$
$$z=13x-26y-18.$$

The Equation of a normal line is

$$\frac{(x-2)}{13}=\frac{(y+1)}{-26}=\frac{(z-34)}{-1}$$

Example 2: Find the equation of the tangent plane to the surface defined by the function $f(x,y)=x^3-x^2y+y^2-2x+3y-2$ at point $(-1{,}3)$
Answer: $z=7x+8y-3$

Note: A tangent plane to a surface does not always exist at every point on the surface.

3.9.2 Differentiability

When working with a function $y=f(x)$ of one variable, the function is said to be differentiable at a point $x=a$ if $f'(a)$ exists. Furthermore, if a function of one variable is differentiable at a point, the graph is "smooth" at that point (i.e., no corners exist) and a tangent line is well-defined at that point.

The idea behind differentiability of a function of two variables is connected to the idea of smoothness at that point. In this case, a surface is considered to be smooth at point $P_0(x_0, y_0)$, if a tangent plane to the surface exists at that point. If a function is differentiable at a point, then a tangent plane to the surface exists at that point. Since the Equation for a tangent plane at a point (x_0, y_0) is given by

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

for a tangent plane to exist at the point (x_0, y_0), the partial derivatives must therefore exist at that point. However, this is not a sufficient condition for smoothness.

Definition: A function $f(x, y)$ is differentiable at a point $P_0(x_0, y_0)$, if, for all points (x, y) in a δ-disk around P_0, we can write

$$f(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + E(x, y),$$

where the error term E satisfies

$$\lim_{(x,y)\to(x_0,y_0)} \frac{E(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0.$$

The last term in Eq. is to as the *error term* and it represents how closely the tangent plane comes to the surface in a small neighborhood (δ-disk) of point P_0. For the function $f(x, y)$ to be differentiable at P_0, the function must be smooth—that is, the graph of $f(x, y)$ must be close to the tangent plane for points near P_0.

3.9.3 Continuity

Differentiability and continuity for functions of two or more variables are connected, the same as for functions of one variable. In fact, with some adjustments of notation, the basic theorem is the same.

Theorem: Let $z = f(x, y)$ be a function of two variables with (x_0, y_0) in the domain of $f(x, y)$. If $f(x, y)$ is differentiable at (x_0, y_0), then $f(x, y)$ is continuous at (x_0, y_0) (differentiability implies continuity).

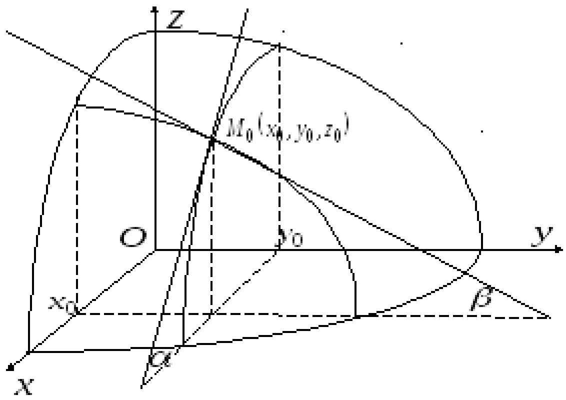
Note: if a function is differentiable at a point, then it is continuous there. However, if a function is continuous at a point, then it is not necessarily differentiable at that point.

We can further explore the connection between continuity and differentiability at a point. This next theorem says that if the function and its partial derivatives are continuous at a point, the function is *differentiable*.

Theorem: Let $z = f(x, y)$ be a function of two variables with (x_0, y_0) in the domain of $f(x,y)$. If $f(x,y)$, $f_x(x,y)$, and $f_y(x,y)$ all exist in a neighborhood of (x_0,y_0) and are continuous at (x_0,y_0), then $f(x,y)$ is *differentiable* there (continuity of first partials implies differentiability).

3.10 Directional Derivatives and the Gradient

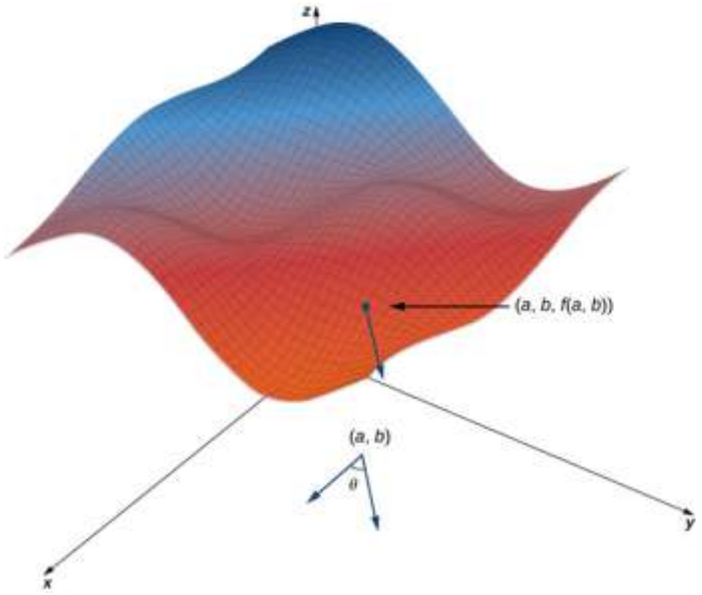
A function $z=f(x,y)$ has two partial derivatives: $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$. These derivatives correspond to each of the independent variables and can be interpreted as instantaneous rates of change (that is, as slopes of a tangent line). For example, $\frac{\partial z}{\partial x}$ represents the slope of a tangent line passing through a given point on the surface defined by $z=f(x,y)$, assuming the tangent line is parallel to the x-axis. Similarly, $\frac{\partial z}{\partial y}$ represents the slope of the tangent line parallel to the y-axis.



Now we consider the possibility of a tangent line parallel to neither axis.

Directional Derivatives

We start with the graph of a surface defined by the equation $z=f(x,y)$. Given a point (a,b) in the domain of f, we choose a direction to travel from that point. We measure the direction using an angle θ, which is measured counterclockwise in the xy-plane, starting at zero from the positive x-axis (Figure). The distance we travel is h and the direction we travel is given by the unit vector $\vec{u}=(\cos\theta)\vec{\imath}+(\sin\theta)\vec{\jmath}$. Therefore, the z-coordinate of the second point on the graph is given by $z=f(a+h\cos\theta, b+h\sin\theta)$.



Definition: Suppose $z = f(x, y)$ is a function of two variables with a domain of D. Let $(a, b) \in D$ and define $\vec{u} = (\cos\theta)\vec{\imath} + (\sin\theta)\vec{\jmath}$. Then the *directional derivative* of f in the direction of $\vec{u}$ is given by

$$\frac{\partial f}{\partial \vec{u}}(a, b) = \lim_{h\to 0}\frac{f(a + h\cos\theta, b + h\sin\theta) - f(a, b)}{h}$$

provided the limit exists.

Note that since the point (a, b) is chosen randomly from the domain D of the function f, we can use this definition to find the directional derivative as a function of x and y. That is,

$$\frac{\partial f}{\partial \vec{u}}(x, y) = f'_{\vec{u}}(x, y) = \lim_{h\to 0}\frac{f(x + h\cos\theta, y + h\sin\theta) - f(x, y)}{h}$$

An easier approach to calculating directional derivatives involves partial derivatives. This is outlined in the following theorem. Since

$$\Delta z = f_x(x, y)\Delta x + f_y(x, y)\Delta y + o\left(\sqrt{\Delta x^2 + \Delta y^2}\right)$$

then

$$\frac{\partial f}{\partial \vec{u}}(x, y) = \lim_{h\to 0}\frac{\Delta z}{h} = \lim_{h\to 0}\left\{f_x(x, y)\underbrace{\frac{\Delta x}{h}}_{=\cos\theta} + f_y(x, y)\underbrace{\frac{\Delta y}{h}}_{=\sin\theta} + \underbrace{\frac{o\left(\sqrt{\Delta x^2 + \Delta y^2}\right)}{h}}_{\to 0}\right\} =$$

$$\frac{\partial f}{\partial \vec{u}}(x, y) = f'_{\vec{u}}(x, y) = f_x(x, y)\cdot\cos\theta + f_y(x, y)\cdot\sin\theta$$

Theorem. Let $z = f(x, y)$ be a function of two variables x and y, and assume that f_x and f_y exist. Then the directional derivative of f in the direction of $\vec{u} = (\cos\theta)\vec{\imath} +$

$(\sin\theta)\vec{j}$ is given by

$$\frac{\partial f}{\partial \vec{u}}(x,y) = f'_{\vec{u}}(x,y) = f_x(x,y)\cdot\cos\theta + f_y(x,y)\cdot\sin\theta$$

Example: Finding a directional derivative of $f(x,y) = x^2 - xy + 3y^2$ in the direction of $\vec{u} = (\cos\theta)\vec{\imath} + (\sin\theta)\vec{\jmath}$ if $\theta = \arccos(3/5)$.

First, we must calculate the partial derivatives of f:

$$f_x(x,y) \quad = 2x - y$$
$$f_y(x,y) \quad = -x + 6y,$$

Then we use Equation with $\theta = \arccos(3/5)$, where

$$\cos\theta = \frac{3}{5}, \sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\left(\frac{3}{5}\right)^2} = \frac{4}{5}$$

$$\frac{\partial f}{\partial \vec{u}}(x,y) \quad = f_x(x,y)\cos\theta + f_y(x,y)\sin\theta = (2x-y)\frac{3}{5} +$$

$$(-x+6y)\frac{4}{5} = \frac{6x}{5} - \frac{3y}{5} - \frac{4x}{5} + \frac{24y}{5} = \frac{2x+21y}{5}.$$

To calculate $\frac{\partial f}{\partial \vec{u}}(-1,2)$, let $x = -1$ and $y = 2$:

$$\frac{\partial f}{\partial \vec{u}}(-1,2) = \frac{2(-1)+21\cdot 2}{5} = 8$$

Example: Finding a directional derivative of $f(x,y) = 3x^2y - 4xy^3 + 3y^2 - 4x$ in the direction of $\vec{u} = (\cos\frac{\pi}{3})\vec{\imath} + (\sin\frac{\pi}{3})\vec{\jmath}$ at the point (3,4).

$$\frac{\partial f}{\partial \vec{u}}(x,y) = \frac{(6xy-4y^3-4)(1)}{2} + \frac{(3x^2-12xy^2+6y)\sqrt{3}}{2}$$

$$\frac{\partial f}{\partial \vec{u}}(3,4) = \frac{72-256-4}{2} + \frac{(27-576+24)\sqrt{3}}{2} = -94 - \frac{525\sqrt{3}}{2}$$

Note. If the vector that is given for the direction of the derivative is not a unit vector, then it is only necessary to divide by the norm of the vector. For example, if we wished to find the directional derivative of the function in the direction of the vector $(-5,12)$, we would first divide by its magnitude to get $\vec{u}$. This gives us $\vec{u} = \left(-\frac{5}{13},\frac{12}{13}\right)$.

Gradient

The right-hand side of Equation for a directional derivative is equal to $f_x(x,y)\cos\theta + f_y(x,y)\sin\theta$, which can be written as the dot product of two vectors. Define the first vector as $\vec{\nabla} f(x,y) = f_x(x,y)\vec{\imath} + f_y(x,y)\vec{\jmath}$ and the second vector as $\vec{u} = (\cos\theta)\vec{\imath} + (\sin\theta)\vec{\jmath}$. Then the right-hand side of the equation can be written as the dot product of these two vectors:

$$\frac{\partial f}{\partial \vec{u}}(x,y) = f'_{\vec{u}}(x,y) = \vec{\nabla} f(x,y) \cdot \vec{u}$$

The first vector in this Equation has a special name: the *gradient of the function* f. The symbol ∇ is called *nabla* and the vector $\vec{\nabla} f$ is read "del f ."

Definition. Let $z = f(x,y)$ be a function of x and y such that f_x and f_y exist. The vector $\vec{\nabla} f(x,y)$ is called the *gradient* of f and is defined as

$$\vec{\nabla} f(x,y) = f_x(x,y)\vec{\imath} + f_y(x,y)\vec{\jmath}$$

The vector $\vec{\nabla} f(x,y)$ is also written as "grad f ."

Example. Find the gradient $\vec{\nabla} f(x,y)$ of each of the following functions: $f(x,y) = x^2 - xy + 3y^2$

$$f_x(x,y) = 2x - y \text{ and } f_y(x,y) = -x + 6y$$

$$\vec{\nabla} f(x,y) \quad = f_x(x,y)\vec{\imath} + f_y(x,y)\vec{\jmath} = (2x - y)\vec{\imath} + (-x + 6y)\vec{\jmath}$$

Properties of the Gradient

Following the definition of the dot product we can write:

$$\frac{\partial f}{\partial \vec{u}}(x,y) = \vec{\nabla} f(x,y) \cdot \vec{u} = \left\|\vec{\nabla} f(x,y)\right\| \underbrace{\left\|\vec{u}\right\|}_{=1} \cos\varphi = \left\|\vec{\nabla} f(x,y)\right\| \cos\varphi$$

Recall that $cos\varphi$ ranges from -1 to 1. Therefore, at (x_0, y_0)

If $\varphi = 0$, then $cos\varphi = 1$ and $\vec{\nabla} f(x,y)$ and both point in the same direction;

If $\varphi = \pi$, then $cos\varphi = -1$ and $\vec{\nabla} f(x,y)$ and both point in opposite directions;

In the first case, the value of $\frac{\partial f}{\partial \vec{u}}(x_0, y_0)$ is *maximized*; in the second case, the value of $\frac{\partial f}{\partial \vec{u}}(x_0, y_0))$ is *minimized*.

Also see that if $\vec{\nabla} f(x_0, y_0) = 0$, then $\frac{\partial f}{\partial \vec{u}}(x_0, y_0) = \vec{\nabla} f(x_0, y_0) \cdot \vec{u} = 0$ for any vector $\vec{u}$.

Example. Find the direction for which the directional derivative of $f(x,y) = 3x^2 - 4xy + 2y^2$ at $(-2,3)$ is a maximum. What is the maximum value?

We start by calculating $\vec{\nabla} f(x,y)$

$$f_x(x,y) = 6x - 4y \text{ and } f_y(x,y) = -4x + 4y$$

so

$$\vec{\nabla} f(x,y) = (6x - 4y)\vec{\imath} + (-4x + 4y)\vec{\jmath}$$

Next, we evaluate the gradient at $(-2,3)$:

$$\vec{\nabla} f(-2,3) = (6 \cdot (-2) - 4 \cdot 3)\vec{\imath} + (-4 \cdot (-2) + 4 \cdot 3)\vec{\jmath} = -24\vec{\imath} + 20\vec{\jmath}$$

We need to find a unit vector that points in the same direction as $\vec{\nabla} f(-2,3)$ so the next step is to divide $\vec{\nabla} f(-2,3)$ by its magnitude, $\left\|\vec{\nabla} f(-2,3)\right\| = 4\sqrt{61}$,

$$\frac{\vec{\nabla} f(-2,3)}{\|\vec{\nabla} f(-2,3)\|} = \frac{-24}{4\sqrt{61}}\vec{\imath} + \frac{20}{4\sqrt{61}}\vec{\jmath} = -\frac{6\sqrt{61}}{61}\vec{\imath} + \frac{5\sqrt{61}}{61}\vec{\jmath}$$

This is the unit vector that points in the same direction as $\vec{\nabla} f(-2,3)$. To find the angle corresponding to this unit vector, we solve the equations

$$\cos\theta = \frac{-6\sqrt{61}}{61} \text{ and } \sin\theta = \frac{5\sqrt{61}}{61}$$

for θ. Since cosine is negative and sine is positive, the angle must be in the second quadrant. Therefore, $\theta = \pi - \arcsin((5\sqrt{61})/61) \approx 2.45$ rad

Suppose the function $z = f(x,y)$ has continuous first-order partial derivatives in an open disk centered at a point (x_0, y_0). If $\vec{\nabla} f(x_0, y_0) \neq 0$ then $\vec{\nabla} f(x_0, y_0)$ is normal to the level curve of f at (x_0, y_0).

We can use this theorem to find tangent and normal vectors to level curves of a function.

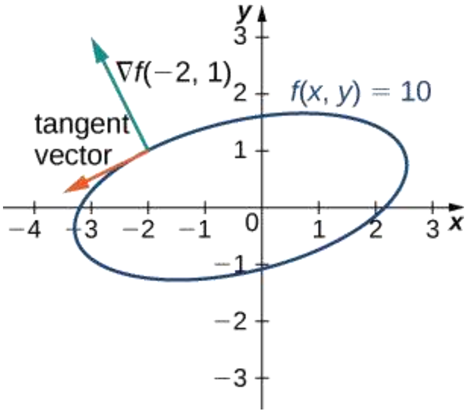
Example. Finding tangents to level curves for the function $f(x,y) = 2x^2 - 3xy + 8y^2 + 2x - 4y + 4$, find a tangent vector to the level curve at point $(-2,1)$. Graph the level curve corresponding to $f(x,y) = 10$ and draw in $\vec{\nabla} f(-2,1)$ and a tangent vector.

First, we must calculate $\vec{\nabla} f(-2,1)$:

$$f_x(x,y) = 4x - 3y + 2 \text{ and } f_y(x,y) = -3x + 16y - 4 \text{ so}$$

$$\vec{\nabla} f(x,y) = (4x - 3y + 2)\vec{\imath} + (-3x + 16y - 4)\vec{\jmath},\ \vec{\nabla} f(-2,1) = -9\vec{\imath} + 18\vec{\jmath}$$

This vector is orthogonal to the curve at point $(-2,1)$. We can obtain a tangent vector by reversing the components and multiplying either one by -1. Thus, for example, $-9\vec{\imath} + 18\vec{\jmath}$ is a tangent vector.



Three-Dimensional Gradients and Directional Derivatives

The definition of a gradient can be extended to functions of more than two variables.

Definition. Let $w = f(x, y, z)$ be a function of three variables such that f_x, f_y, and f_z exist. The vector $\vec{\nabla} f(x, y, z)$ is called the gradient of f and is defined as

$$\vec{\nabla} f(x, y, z) = f_x(x, y, z)\vec{\imath} + f_y(x, y, z)\vec{\jmath} + f_z(x, y, z)\vec{k}$$

$\vec{\nabla} f(x, y, z)$ can also be written as $grad\ f(x, y, z)$.

Calculating the gradient of a function in three variables is very similar to calculating the gradient of a function in two variables. First, we calculate the partial derivatives f_x, f_y, and f_z.

Example. Find the gradient of the function $f(x, y, z) = 5x^2 - 2xy + y^2 - 4yz + z^2 + 3xz$.

We first calculate the partial derivatives f_x, f_y, and f_z , then

$$f_x(x, y, z) = 10x - 2y + 3z,$$
$$f_y(x, y, z) = -2x + 2y - 4z,$$
$$f_z(x, y, z) = 3x - 4y + 2z$$

So,

$$\vec{\nabla} f(x, y, z) = (10x - 2y + 3z)\vec{\imath} + (-2x + 2y - 4z)\vec{\jmath} + (3x - 4y + 2z)\vec{k}$$

The directional derivative can also be generalized to functions of three variables. To determine a direction in three dimensions, a vector with three components is needed. This vector is a unit vector, and the components of the unit vector are called directional cosines. Given a three-dimensional unit vector $\vec{u}$ in standard form (i.e., the initial point is at the origin), this vector forms three different angles with the positive x-, y-, and z-axes. Let's call these angles α, β, and γ. Then the directional cosines are given by $cos\alpha$, $cos\beta$, and $cos\gamma$.

Definition. Let $f(x,y,z)$ be a differentiable function of three variables and let $\vec{u} = \cos\alpha\vec{\imath} + \cos\beta\vec{\jmath} + \cos\gamma\vec{k}$ be a unit vector. Then, the directional derivative of f in the direction of $\vec{u}$ is given by

$$\frac{\partial f}{\partial \vec{u}}(x,y,z) = \vec{\nabla} f(x,y,z) \cdot \vec{u} = f_x(x,y,z)\cos\alpha + f_y(x,y,z)\cos\beta + f_z(x,y,z)\cos\gamma.$$

Note. The three angles α, β, and γ determine the unit vector $\vec{u}$. In practice, we can use an arbitrary (nonunit) vector, then divide by its magnitude to obtain a unit vector in the desired direction.

Example: Finding a directional derivative in three dimensions $\frac{\partial f}{\partial \vec{u}}(1,-2,3)$ in direction $\vec{u} = -\vec{\imath} + 2\vec{\jmath} + 2\vec{k}$ for the function $f(x,y,z) = 5x^2 - 2xy + y^2 - 4yz + z^2 + 3xz$.

First, we find the magnitude of $\vec{u}$,

$$\left\|\vec{u}\right\| = \sqrt{(-1)^2 + (2)^2 + (2)^2} = \sqrt{9} = 3$$

Therefore, a unit vector in the direction of $\vec{u}$ is

$$\frac{\vec{u}}{\|\vec{u}\|} = \frac{-\vec{\imath} + 2\vec{\jmath} + 2\vec{k}}{3} = -\frac{1}{3}\vec{\imath} + \frac{2}{3}\vec{\jmath} + \frac{2}{3}\vec{k}$$

so

$$\cos\alpha = -\frac{1}{3}, \cos\beta = \frac{2}{3}, \text{ and } \cos\gamma = \frac{2}{3}$$

Next, we calculate the partial derivatives of f :

$$\begin{aligned} f_x(x,y,z) &= 10x - 2y + 3z \\ f_y(x,y,z) &= -2x + 2y - 4z \\ f_z(x,y,z) &= -4y + 2z + 3x, \end{aligned}$$

then substitute them into Equation

$$\frac{\partial f}{\partial \vec{u}}(x,y,z) = (10x - 2y + 3z)\left(-\frac{1}{3}\right) + (-2x + 2y - 4z)\left(\frac{2}{3}\right)$$

$$+(-4y + 2z + 3x)\left(\frac{2}{3}\right) = -\frac{10x}{3} + \frac{2y}{3} - \frac{3z}{3} - \frac{4x}{3} + \frac{4y}{3} - \frac{8z}{3} - \frac{8y}{3} + \frac{4z}{3} + \frac{6x}{3}$$

$$= -\frac{8x}{3} - \frac{2y}{3} - \frac{7z}{3}$$

Last, to find $\frac{\partial f}{\partial \vec{u}}(1,-2,3)$ we substitute $x=1$, $y=-2$, and $z=3$

$$\frac{\partial f}{\partial \vec{u}}(1,-2,3)=-\frac{8(1)}{3}-\frac{2(-2)}{3}-\frac{7(3)}{3}=-\frac{8}{3}+\frac{4}{3}-\frac{21}{3}=-\frac{25}{3}$$