## 3.11 Maximum and Minimum Values

One of the most useful applications for derivatives of a function of one variable is the determination of maximum and/or minimum values. This application is also important for functions of two or more variables.

The main ideas of finding critical points and using derivative tests are still valid, but new wrinkles appear when assessing the results.

### *3.11.1 Critical Points*

For functions of a single variable, we defined critical points as the values of the variable at which the function's derivative equals zero or does not exist. For functions of two or more variables, the concept is essentially the same, except for the fact that we are now working with partial derivatives.

*Definition*. (The first derivative test) Let $z = f(x,y)$ be a function of two variables that is differentiable on an open set containing the point $(x_0, y_0)$. The point $(x_0, y_0)$ is called a *critical point* of a function of two variables $f$ if one of the two following conditions holds:

1. $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$
2. Either $f_x(x_0, y_0)$ or $f_y(x_0, y_0)$ does not exist.

*Example*. Find the critical points of each of the following functions:

*a)* $f(x,y) = \sqrt{4y^2 - 9x^2 + 24y + 36x + 36}$

First, we calculate $f_x(x,y)$ and $(x,y)$

$$f_x(x,y) = \frac{1}{2}(-18x + 36)(4y^2 - 9x^2 + 24y + 36x + 36)^{-1/2}$$
$$= \frac{-9x + 18}{\sqrt{4y^2 - 9x^2 + 24y + 36x + 36}}$$

$$f_y(x,y) = \frac{1}{2}(8y + 24)(4y^2 - 9x^2 + 24y + 36x + 36)^{-1/2}$$
$$= \frac{4y + 12}{\sqrt{4y^2 - 9x^2 + 24y + 36x + 36}}.$$

Next, we set each of these expressions equal to zero:

$$\frac{-9x + 18}{\sqrt{4y^2 - 9x^2 + 24y + 36x + 36}} = 0$$
$$\frac{4y + 12}{\sqrt{4y^2 - 9x^2 + 24y + 36x + 36}} = 0.$$

Then, multiply each equation by its common denominator:

$$-9x + 18 = 0$$
$$4y + 12 = 0.$$

Therefore, $x = 2$ and $y = -3$, so $(2, -3)$ is a critical point of $f$.

We must also check for the possibility that the denominator of each partial derivative can equal zero, thus causing the partial derivative not to exist. Since the denominator is the same in each partial derivative, we need only do this once:

$$4y^2 - 9x^2 + 24y + 36x + 36 = 0.$$

To put the hyperbola in standard form, we use the method of completing the square:

$$\frac{(x-2)^2}{4} - \frac{(y+3)^2}{9} = 1$$

Thus, the critical points of the function $f$ are $(2, -3)$ and all points on the hyperbola.

*b)* $g(x, y) = x^2 + 2xy - 4y^2 + 4x - 6y + 4$

$$g_x(x, y) = 2x + 2y + 4$$
$$g_y(x, y) = 2x - 8y - 6.$$

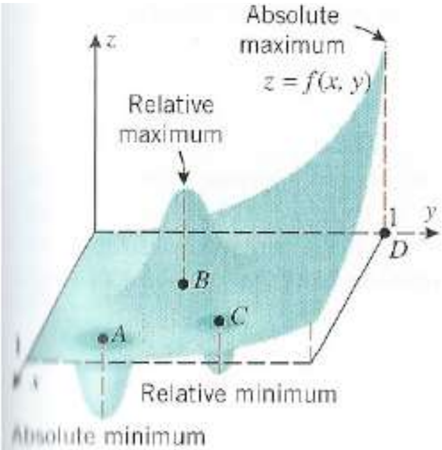
Next, we set each of these expressions equal to zero, which gives a system of equations in $x$ and $y$:

$$2x + 2y + 4 = 0$$
$$2x - 8y - 6 = 0.$$

Therefore $(-1, -1)$ is a critical point of $g$. There are no points in $\mathbf{R}^2$ that make either partial derivative not exist.

*3.11.2 Global and Local Extrema*

The main purpose for determining critical points is to locate relative maxima and minima, as in single-variable calculus. When working with a function of one variable, the definition of a local extremum involves finding an interval around the critical point such that the function value is either greater than or less than all the other function values in that interval. When working with a function of two or more variables, we work with an open disk around the point.



*Definition*. Let $z = f(x,y)$ be a function of two variables that is defined and continuous on an open set containing the point $(x_0, y_0)$. Then $f$ has *a local maximum* at $(x_0, y_0)$ if

$$f(x_0, y_0) \geq f(x,y)$$

for all points $(x,y)$ within some disk centered at $(x_0, y_0)$. The number $f(x_0, y_0)$ is called a *local maximum value*. If the preceding inequality holds for every point $(x,y)$ in the domain of $f$ , then $f$ has a *global maximum* (also called an *absolute maximum*) at $(x_0, y_0)$.

The function $f$ has *a local minimum* at $(x_0, y_0)$ if

$$f(x_0, y_0) \leq f(x,y)$$

for all points $(x,y)$ within some disk centered at $(x_0, y_0)$. The number $f(x_0, y_0)$ is called a *local minimum value*. If the preceding inequality holds for every point $(x,y)$ in the domain of $f$ , then $f$ has a *global minimum* (also called an *absolute minimum*) at $(x_0, y_0)$.

If $f(x_0, y_0)$ is either a local maximum or local minimum value, then it is called a local extremum

In Calculus 1, we showed that extrema of functions of one variable occur at critical points. The same is true for functions of more than one variable, as stated in the following theorem.

*Theorem*. Let $z = f(x,y)$ be a function of two variables that is defined and continuous on an open set containing the point $(x_0, y_0)$. Suppose $f_x$ and $f_y$ each exist at $(x_0, y_0)$. If $f$ has a local extremum at $(x_0, y_0)$, them $(x_0, y_0)$ is a critical point of $f$.

However, the existence of a critical value at $x = x_0$ does not guarantee a local extremum at $(x_0, y_0)$. One way this can happen is at a *saddle point*.

*Definition*: *Saddle Point*. Given the function $z = f(x,y)$ the point $(x_0, y_0, f(x_0, y_0))$ is a saddle point if both $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$ but $f$ does not have a local extremum at $(x_0, y_0)$.

The second derivative test for a function of one variable provides a method for determining whether an extremum occurs at a critical point of a function. The second derivative test for a function of two variables, stated in the following theorem:

*Theorem*. (The second derivative test) Let $z = f(x,y)$ be a function of two variables for which the first- and second-order partial derivatives are continuous on some disk containing the point $(x_0, y_0)$. Suppose $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$. Define the quantity of the discriminant:

$$D = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2.$$

Then

1. If $D > 0$ and $f_{xx}(x_0, y_0) > 0$ then $f$ has a local minimum at $(x_0, y_0)$;
2. If $D > 0$ and $f_{xx}(x_0, y_0) < 0$ then $f$ has a local maximum at $(x_0, y_0)$;
3. If $D < 0$ then $f$ has a saddle point at $(x_0, y_0)$;
4. If $D = 0$ then the test is inconclusive.

To apply the second derivative test, it is necessary that we first find the critical points of the function. There are several steps involved in the entire procedure, which are outlined in a problem-solving strategy:

Let $z = f(x, y)$ be a function of two variables for which the first- and second-order partial derivatives are continuous on some disk containing the point $(x_0, y_0)$. To apply the second derivative test to find local extrema, use the following steps:

1. Determine the critical points $(x_0, y_0)$ of the function $f$ where $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$. Discard any points where at least one of the partial derivatives does not exist.
2. Calculate the discriminant $D = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2$ for each critical point of $f$.
3. Apply the four cases of the test to determine whether each critical point is a local maximum, local minimum, or saddle point, or whether the theorem is inconclusive.

*Example*. Use the second derivative test to find the local extrema of the function: $f(x, y) = 4x^2 + 9y^2 + 8x - 36y + 24$

1) Find the critical points of $f$, we first calculate $f_x(x, y)$ and $f_y(x, y)$, then set each of them equal to zero:

$$\begin{aligned} f_x(x, y) &= 8x + 8 \\ f_y(x, y) &= 18y - 36. \end{aligned}$$

Setting them equal to zero yields the system of equations

$$\begin{aligned} 8x + 8 &= 0 \\ 18y - 36 &= 0. \end{aligned}$$

The solution to this system is $x = -1$ and $y = 2$, i.e. the critical point.

2) Calculate $D$, we first calculate the second partial derivatives of $f$:

$$
\begin{aligned}
f_{xx}(x,y) &= 8\\
f_{xy}(x,y) &= 0\\
f_{yy}(x,y) &= 18.
\end{aligned}
$$

Therefore, $D = f_{xx}(-1,2)f_{yy}(-1,2) - (f_{xy}(-1,2))^2 = (8)(18) - (0)^2 = 144.$

3) Make conclusion to apply the four cases of the test to classify the function's behavior at this critical point.

Since $D > 0$ and $f_{xx}(-1,2) > 0$ this corresponds to case 1. Therefore, $f$ has a local minimum at $(-1,2)$.

The function $f(x,y)$ has a local minimum at $(-1,2)$, then $f(-1,2) = -16.$

*Example*. Use the second derivative test to find the local extrema of the function:

$$g(x,y) = \frac{1}{3}x^3 + y^2 + 2xy - 6x - 3y + 4$$

Setting $g_x(x,y)$ and $g_y(x,y)$ equal to zero yields the system of equations

$$
\begin{aligned}
x^2 + 2y - 6 &= 0\\
2y + 2x - 3 &= 0.
\end{aligned}
$$

To solve this system, first solve the second equation for $y$. This gives $y = \frac{3-2x}{2}$ Substituting this into the first equation gives

$$
\begin{aligned}
x^2 + 3 - 2x - 6 &= 0\\
x^2 - 2x - 3 &= 0\\
(x-3)(x+1) &= 0.
\end{aligned}
$$

Therefore, $x = -1$ or $x = 3$. Substituting these values into the equation $y = \frac{3-2x}{2}$, it yields the critical points $(-1, \frac{5}{2})$ and $(3, -\frac{3}{2})$.

Calculate the second partial derivatives of $g$ :

$$
\begin{aligned}
g_{xx}(x,y) &= 2x\\
g_{xy}(x,y) &= 2\\
g_{yy}(x,y) &= 2.
\end{aligned}
$$

Then, we find a general formula for $D$ :

$$
\begin{aligned}
D(x_0,y_0) &= g_{xx}(x_0,y_0)g_{yy}(x_0,y_0) - (g_{xy}(x_0,y_0))^2\\
&= (2x_0)(2) - 2^2 = 4x_0 - 4
\end{aligned}
$$

Next, we substitute each critical point into this formula:

$$
\begin{aligned}
D(-1,\frac{5}{2}) &= (2(-1))(2) - (2)^2 = -4 - 4 = -8\\
D(3,-\frac{3}{2}) &= (2(3))(2) - (2)^2 = 12 - 4 = 8.
\end{aligned}
$$

We note that, applying Note to point $(-1, \frac{5}{2})$ leads to case 3, which means that $(-1, \frac{5}{2})$

is a saddle point. Applying the theorem to point $(3, -\frac{3}{2})$ leads to case 1, which means that $(3, -\frac{3}{2})$ corresponds to a local minimum.

### *3.11.3 Absolute Maxima and Minima*

When finding global extrema of functions of one variable on a closed interval, we start by checking the critical values over that interval and then evaluate the function at the endpoints of the interval. When working with a function of two variables, the closed interval is replaced by a closed, bounded set. A set is bounded if all the points in that set can be contained within a ball (or disk) of finite radius. First, we need to find the critical points inside the set and calculate the corresponding critical values. Then, it is necessary to find the maximum and minimum value of the function on the boundary of the set. When we have all these values, the largest function value corresponds to the global maximum and the smallest function value corresponds to the absolute minimum. First, however, we need to be assured that such values exist. The following theorem does this.

*Extreme Value Theorem*. A continuous function $f(x, y)$ on a closed and bounded set $D$ in the plane attains an absolute maximum value at some point of $D$ and an absolute minimum value at some point of $D$.

Now that we know any continuous function $f$ defined on a closed, bounded set attains its extreme values, we need to know how to find them.

*Finding Extreme Values of a Function of Two Variables*. Assume $z = f(x, y)$ is a differentiable function of two variables defined on a closed, bounded set $D$. Then $f$ will attain the absolute maximum value and the absolute minimum value, which are, respectively, the largest and smallest values found among the following:

1. The values of $f$ at the critical points of $f$ in $D$.
2. The values of $f$ on the boundary of $D$.

*Finding Absolute Maximum and Minimum Values*. Let $z = f(x, y)$ be a continuous function of two variables defined on a closed, bounded set $D$, and assume $f$ is differentiable on $D$. To find the absolute maximum and minimum values of $f$ on $D$, do the following:

1. Determine the critical points of $f$ in $D$.
2. Calculate $f$ at each of these critical points.
3. Determine the maximum and minimum values of $f$ on the boundary of its

domain.

4. The maximum and minimum values of $f$ will occur at one of the values obtained in steps 2 and 3 .

Finding the maximum and minimum values of $f$ on the boundary of $D$ can be challenging. If the boundary is a rectangle or set of straight lines, then it is possible to parameterize the line segments and determine the maxima on each of these segments.

*Example* 1. Find absolute extrema of the function: $f(x,y) = x^2 - 2xy + 4y^2 - 4x - 2y + 24$ on the domain defined by $0 \le x \le 4$ and $0 \le y \le 2$.

We first calculate $f_x(x,y)$ and $f_y(x,y)$ then set them each equal to zero:

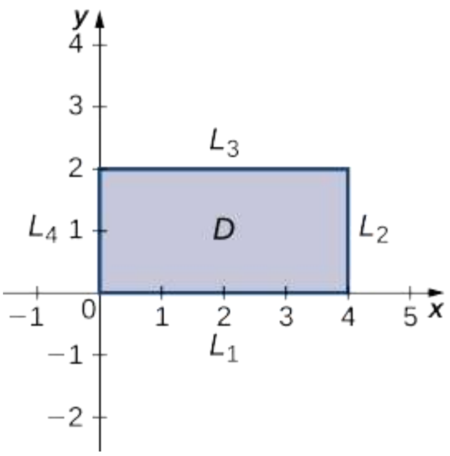
$$f_x(x,y) = 2x - 2y - 4$$
$$f_y(x,y) = -2x + 8y - 2.$$

Setting them equal to zero yields the system of equations

$$2x - 2y - 4 = 0$$
$$-2x + 8y - 2 = 0.$$

The solution to this system is $x = 3$ and $y = 1$ . Therefore $(3,1)$ is a critical point of $f$ . Calculating $f(3,1)$ gives $f(3,1) = 17$.

The next step involves finding the extrema of $f$ on the boundary of its domain. The boundary of its domain consists of four line segments as shown in the following graph:




$L_1$ is the line segment connecting (0,0) and (4,0) , and it can be parameterized by the equations $x(t) = t, y(t) = 0$ for $0 \le t \le 4$. Define a function $g(t) = f(x(t), y(t))$. This gives $g(t) = t^2 - 4t + 24$. Differentiating $g$ leads to $g'(t) = 2t - 4, \Rightarrow 2t - 4 = 0$. Therefore, $g$ has a critical value at $t = 2$, which corresponds to the point $x(2) = 2$ and $y(2) = 0$, i.e. (2,0). Calculating $f(2,0)$ gives the $z$ - value 20.

$L_2$ is the line segment connecting (4,0) and (4,2), and it can be parameterized by the equations $x(t) = 4, y(t) = t$ for $0 \le t \le 2$. Define a function $g(t) = f(x(t), y(t))$.

This gives $g(t) = 4t^2 - 10t + 24$. Differentiating $g$ leads to $g'(t) = 8t - 10$, $\Rightarrow$ $8t - 10 = 0$. Therefore, $g$ has a critical value at $t = \frac{5}{4}$, which corresponds to the point $x(\frac{5}{4}) = 4$ and $y(\frac{5}{4}) = \frac{5}{4}$, i.e. $(4, \frac{5}{4})$. Calculating $f(4, \frac{5}{4})$ gives the $z$ - value 27.75.

$L_3$ is the line segment connecting (0,2) and (4,2) , and it can be parameterized by the equations $x(t) = t$, $y(t) = 2$ for $0 \le t \le 4$. Define a function $g(t) = f(x(t), y(t))$. This gives $g(t) = t^2 - 8t + 36$. Differentiating $g$ leads to $g'(t) = 2t - 8$, $\Rightarrow 2t - 8 = 0$. Therefore, $g$ has a critical value at $t = 4$, which corresponds to the point $x(4) = 4$ and $y(4) = 2$, i.e. $(4,2)$. Calculating $f(4,2)$ gives the $z$ - value 20.

$L_4$ is the line segment connecting (0,0) and (0,2), and it can be parameterized by the equations $x(t) = 0$, $y(t) = t$ for $0 \le t \le 2$. Define a function $g(t) = f(x(t), y(t))$. This gives $g(t) = 4t^2 - 2t + 24$. Differentiating $g$ leads to $g'(t) = 8t - 2$, $\Rightarrow 8t - 2 = 0$. Therefore, $g$ has a critical value at $t = \frac{1}{4}$, which corresponds to the point $x(\frac{1}{4}) = 0$ and $y(\frac{1}{4}) = \frac{1}{4}$, i.e. $(0, \frac{1}{4})$. Calculating $f(0, \frac{1}{4})$ gives the $z$ - value 23.75.

We also need to find the values of $f(x,y)$ at the corners of its domain. These corners are located at (0,0), (4,0), (4,2) and (0,2):

$$\begin{aligned}
f(0,0) &= (0)^2 - 2(0)(0) + 4(0)^2 - 4(0) - 2(0) + 24 = 24 \\
f(4,0) &= (4)^2 - 2(4)(0) + 4(0)^2 - 4(4) - 2(0) + 24 = 24 \\
f(4,2) &= (4)^2 - 2(4)(2) + 4(2)^2 - 4(4) - 2(2) + 24 = 20 \\
f(0,2) &= (0)^2 - 2(0)(2) + 4(2)^2 - 4(0) - 2(2) + 24 = 36.
\end{aligned}$$

Among all the points we have inside the domain *D* and on the boundary of the domain *D*, the absolute maximum value is **36** , which occurs at (0,2) , and the global minimum value is **20** , which occurs at both (4,2) and (2,0).

*Example* 2. Find the absolute extrema of the function: $g(x,y) = x^2 + y^2 + 4x - 6y$ on the domain defined by $x^2 + y^2 \le 16$.

We first calculate $g_x(x,y)$ and $g_y(x,y)$ , then set them each equal to zero:

$$\begin{aligned}
g_x(x,y) &= 2x + 4 \\
g_y(x,y) &= 2y - 6.
\end{aligned}$$

Setting them equal to zero yields the system of equations

$$\begin{aligned}
2x + 4 &= 0 \\
2y - 6 &= 0.
\end{aligned}$$

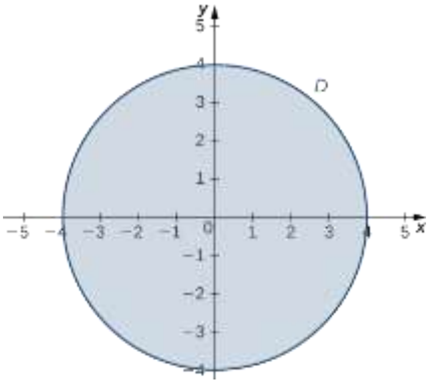
The solution to this system is $x = -2$ and $y = 3$. Therefore, $(-2,3)$ is a critical point of g . Calculating $g(-2,3)$, we get

$$g(-2,3) = (-2)^2 + 3^2 + 4(-2) - 6(3) = 4 + 9 - 8 - 18 = -13.$$

The next step involves finding the extrema of g on the boundary of its domain. The boundary of its domain consists of a circle of radius 4 centered at the origin as shown in the following graph.

The boundary of the domain of $g$ can be parameterized using the functions $x(t) = 4\cos t, y(t) = 4\sin t$ for $0 \leq t \leq 2\pi$. Define $h(t) = g(x(t), y(t))$:

$$h(t) = g(x(t), y(t)) = (4\cos t)^2 + (4\sin t)^2 + 4(4\cos t) - 6(4\sin t)$$
$$= 16\cos^2 t + 16\sin^2 t + 16\cos t - 24\sin t = 16 + 16\cos t - 24\sin t$$




Setting $h'(t) = 0$ leads to $-16\sin t - 24\cos t = 0$, $-16\sin t = 24\cos t$, $\tan t = -\frac{3}{2}$.

This equation has two solutions over the interval $0 \leq t \leq 2\pi$. One is $t = \pi - \arctan(\frac{3}{2})$ and the other is $t = 2\pi - \arctan(\frac{3}{2})$.

For the first angle,

$$\sin t \quad = \sin(\pi - \arctan(\frac{3}{2})) = \sin(\arctan(\frac{3}{2})) = \frac{3\sqrt{13}}{13}$$
$$\cos t \quad = \cos(\pi - \arctan(\frac{3}{2})) = -\cos(\arctan(\frac{3}{2})) = -\frac{2\sqrt{13}}{13}.$$

Therefore, $x(t) = 4\cos t = -\frac{8\sqrt{13}}{13}$ and $y(t) = 4\sin t = \frac{12\sqrt{13}}{13}$, so $(-\frac{8\sqrt{13}}{13}, \frac{12\sqrt{13}}{13})$ is a critical point on the boundary and

$$g(-\frac{8\sqrt{13}}{13}, \frac{12\sqrt{13}}{13}) \quad = (-\frac{8\sqrt{13}}{13})^2 + (\frac{12\sqrt{13}}{13})^2 + 4(-\frac{8\sqrt{13}}{13}) - 6(\frac{12\sqrt{13}}{13})$$
$$= \frac{144}{13} + \frac{64}{13} - \frac{32\sqrt{13}}{13} - \frac{72\sqrt{13}}{13} = \frac{208 - 104\sqrt{13}}{13} \approx -12.844$$

For the second angle,

$$\sin t \quad = \sin(2\pi - \arctan(\frac{3}{2})) = -\sin(\arctan(\frac{3}{2})) = -\frac{3\sqrt{13}}{13}$$
$$\cos t \quad = \cos(2\pi - \arctan(\frac{3}{2})) = \cos(\arctan(\frac{3}{2})) = \frac{2\sqrt{13}}{13}.$$

Therefore, $x(t) = 4\cos t = \frac{8\sqrt{13}}{13}$ and $y(t) = 4\sin t = -\frac{12\sqrt{13}}{13}$, so $(\frac{8\sqrt{13}}{13}, -\frac{12\sqrt{13}}{13})$ is a critical point on the boundary and

$$g(\frac{8\sqrt{13}}{13}, -\frac{12\sqrt{13}}{13}) \quad = (\frac{8\sqrt{13}}{13})^2 + (-\frac{12\sqrt{13}}{13})^2 + 4(\frac{8\sqrt{13}}{13}) - 6(-\frac{12\sqrt{13}}{13})$$
$$= \frac{144}{13} + \frac{64}{13} + \frac{32\sqrt{13}}{13} + \frac{72\sqrt{13}}{13} = \frac{208 + 104\sqrt{13}}{13} \approx 44.844$$

The absolute minimum of $g$ is $-13$, which is attained at the point $(-2,3)$, which is an interior point of D . The absolute maximum of g is approximately equal to 44.844, which is attained at the boundary point $(\frac{8\sqrt{13}}{13}, -\frac{12\sqrt{13}}{13})$.

*Lagrange Multipliers*

In the previous section we optimized (i.e. found the absolute extrema) a function on a region that contained its boundary. Finding potential optimal points in the interior of the region isn't too bad in general, all that we needed to do was find the critical points and plug them into the function. However, as we saw in the examples finding potential optimal points on the boundary was often a fairly long and messy process.

In this section we are going to take a look at another way of optimizing a function subject to given constraint(s). The constraint(s) may be the equation(s) that describe the boundary of a region although in this section we won't concentrate on those types of problems since this method just requires a general constraint and doesn't really care where the constraint came from.

So, let's get things set up. We want to optimize (i.e. find the minimum and maximum value of) a function, $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ (the constraint may be the equation that describes the boundary of a region or it may not be).

From a theoretical standpoint, at the point where the curve of the studied function is tangent to the constraint line, the gradient of both of the functions evaluated at that point must point in the same (or opposite) direction. Recall that the gradient of a function of more than one variable is a vector. If two vectors point in the same (or opposite) directions, then one must be a constant multiple of the other. This idea is the basis of the method of Lagrange multipliers.

*Theorem*. Let $f$ and $g$ be functions of three variables with continuous partial derivatives at every point of some open set containing the smooth curve $g(x, y, z) = k$. Suppose that $f$ , when restricted to points on the curve $g(x, y, z) = k$ has a local extremum at the point $(x_0, y_0, z_0)$ and that $\vec{\nabla} g(x_0, y_0, z_0) \neq 0$ Then there is a number λ called a Lagrange multiplier, for which

$$\vec{\nabla} f(x_0, y_0, z_0) = \lambda \vec{\nabla} g(x_0, y_0, z_0)$$

*Method of Lagrange Multipliers*

1. Solve the following system of equations:

$$\begin{aligned} \nabla f(x, y, z) &= \lambda \nabla g(x, y, z) \\ g(x, y, z) &= k \end{aligned}$$

2. Plug in all solutions, $(x, y, z)$ from the first step into $f(x, y, z)$ and identify the

minimum and maximum values, provided they exist and $\nabla g \neq \vec{0}$ at the point.

*Notice* that the system of equations from the method actually has four equations, we just wrote the system in a simpler form. To see this let's take the first equation and put in the definition of the gradient vector to see what we get:

$$\langle f_x, f_y, f_z \rangle = \lambda \langle g_x, g_y, g_z \rangle = \langle \lambda g_x, \lambda g_y, \lambda g_z \rangle$$

So, we actually have three equations here

$$f_x = \lambda g_x f_y = \lambda g_y f_z = \lambda g_z$$

These three equations along with the constraint $g(x, y, z) = k$, give four equations with four unknowns x, y, z, and λ.

*Note* 1. If we only have functions of two variables then we won't have the third component of the gradient and so will only have three equations in three unknowns x, y, and λ.

*Note* 2. We also need to be careful with the fact that in some cases minimums and maximums won't exist even though the method will seem to imply that they do. In every problem we'll need to make sure that minimums and maximums will exist before we start the problem.

*Example* 1. Find the maximum and minimum of $f(x, y) = 5x - 3y$ subject to the constraint $x^2 + y^2 = 136$.

This one is going to be a little easier than the previous one since it only has two variables. Also, note that it's clear from the constraint that region of possible solutions lies on a disk of radius $\sqrt{136}$ which is a closed and bounded region. Then, by the Extreme Value Theorem we know that a minimum and maximum value must exist.

Here is the system that we need to solve:

$$\begin{aligned} 5 &= 2\lambda x \\ -3 &= 2\lambda y \\ x^2 + y^2 &= 136 \end{aligned}$$

Notice that, as with the last example, we can't have $\lambda = 0$ since that would not satisfy the first two equations. So, since we know that $\lambda \neq 0$ we can solve the first two equations for x and y respectively. This gives,

$$x = \frac{5}{2\lambda}, y = -\frac{3}{2\lambda}$$

Plugging these into the constraint gives,

$$\frac{25}{4\lambda^2} + \frac{9}{4\lambda^2} = \frac{17}{2\lambda^2} = 136$$

We can solve this for λ:

$$\lambda^2 = \frac{1}{16} \Rightarrow \lambda = \pm\frac{1}{4}$$

Now, that we know λ we can find the points that will be potential maximums and/or minimums:

If $\lambda = -\frac{1}{4}$ we get $x = -10$, $y = 6$ and if $\lambda = \frac{1}{4}$ we get $x = 10$, $y = -6$

To determine if we have maximums or minimums we just need to plug these into the function. Here are the minimum and maximum values of the function:

| | | |
|---|---|---|
| $f(-10,6)$ | $= -68$ | Minimum at $(-10,6)$ |
| $f(10,-6)$ | $= 68$ | Maximum at $(10,-6)$ |

*Example* 2. Find the maximum and minimum of $f(x,y,z) = xyz$ subject to the constraint $x + y + z = 1$, $x,y,z \geq 0$.

Here is the system of equation that we need to solve:

$$yz = \lambda,\ xz = \lambda,\ xy = \lambda,\ x + y + z = 1$$

Let's start this solution process off by noticing that since the first three equations all have λ they are all equal. So, let's start off by setting equations

$$yz = xz \Rightarrow z(y - x) = 0 \Rightarrow z = 0, \text{ or } y = x$$

So, we've got two possibilities here. Let's start off with by assuming that $z = 0$. In this case we can see from either equations that we must then have $\lambda = 0$. From equation we see that this means that $xy = 0$. This in turn means that either $x = 0$ or $y = 0$.

So, we've got two possible cases to deal with there. In each case two of the variables must be zero. Once we know this we can plug into the constraint, equation, to find the remaining value:

$$z = 0, x = 0 \quad : \quad \Rightarrow y = 1$$
$$z = 0, y = 0 \quad : \quad \Rightarrow x = 1$$

So, we've got two possible solutions (0,1,0) and (1,0,0).

In the case $x = y = 0$ we can see from the constraint that we must have $z = 1$ and so we now have a third solution (0,0,1).

So, the next solution is $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

So, we have four solutions that we need to check in the function to see whether we have minimums or maximums.

| | | |
|---|---|---|
| $f(0,0,1)$ | $= 0 f(0,1,0) = 0 f(1,0,0) = 0$ | All Minimums |
| $f(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ | $= \frac{1}{27}$ | Maximum |