An Example of Doing Mandatory Homework **IDZ 3-1**

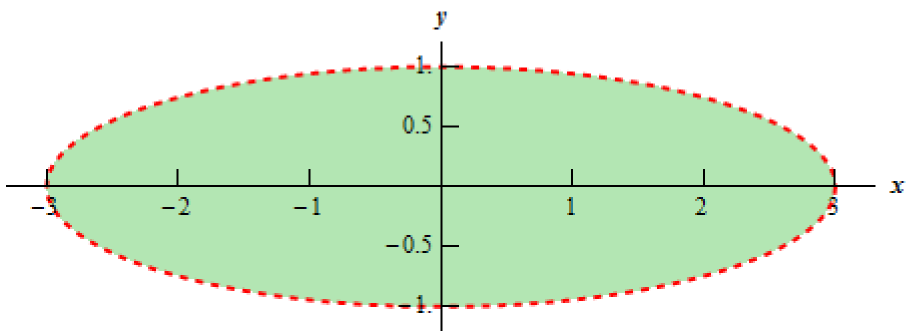
1. Find the Domain of a function $z = f(x, y)$ and Plot this Region.

a) $f(x, y) = \ln(9 - x^2 - 9y^2)$

Solution: In this function we know that we can't take the logarithm of a negative number or zero. Therefore, we need to require that,

$$9 - x^2 - 9y^2 > 0 \Rightarrow \frac{x^2}{9} + y^2 < 1$$

and upon rearranging we see that we need to stay interior to an ellipse for this function. Here is a sketch of this region as follows:

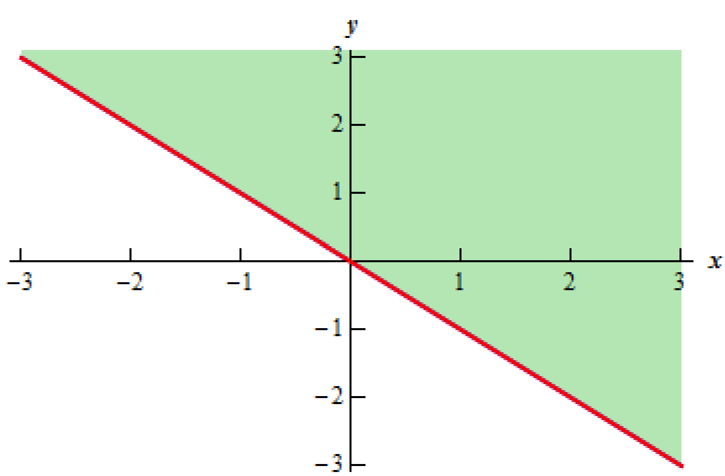


b) $f(x, y) = \sqrt{x + y}$

Solution: In this case we know that we can't take the square root of a negative number so this means that we must require,

$$x + y \geq 0$$

Here is a sketch of the graph of this region,

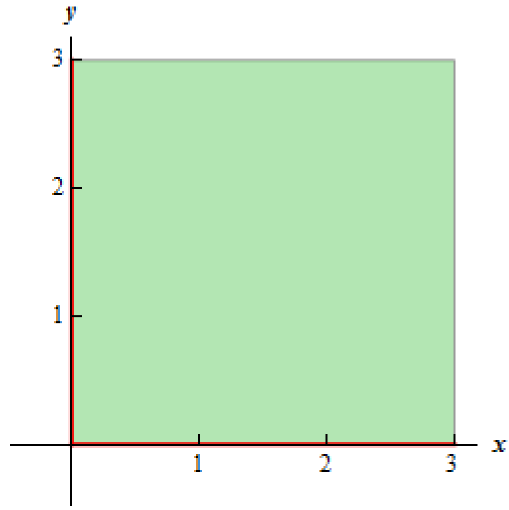


c) $f(x, y) = \sqrt{x} + \sqrt{y}$

Solution: Here we must require for each variable that,

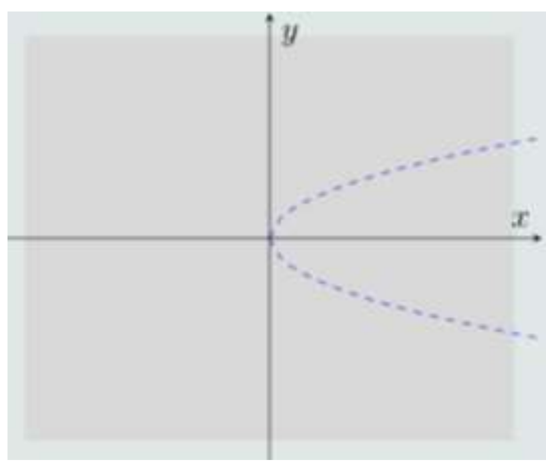
$$x \geq 0 \text{ and } y \geq 0$$

and they really do need to be separate inequalities. There is one for each square root in the function. Here is the sketch of this region,



d) $f(x, y) = \frac{1}{x - y^2}$.

Solution: We see that $f(x, y)$ is undefined for $x = y^2$. The domain of the function therefore consists of all points in the xy-plane except those which satisfy $y = \pm\sqrt{x}$. Here is the sketch of this region,



2. Partial Derivatives

a) Find a total differential du of the function $u(x, y, z) = \frac{x\sin(y)}{z^2}$.

Solution: Let's do the derivatives with respect to x and y first. In both these cases the z's are constants and so the denominator in this is a constant and so we don't really need to worry too much about it. Here are the derivatives for these two cases.

$$u_x(x,y,z) = \frac{\partial u}{\partial x} = \frac{\sin(y)}{z^2} \text{ and } u_y(x,y,z) = \frac{\partial u}{\partial y} = \frac{x\cos(y)}{z^2}$$

Now, in the case of differentiation with respect to z we can avoid the quotient rule with a quick rewrite of the function. Here is the rewrite as well as the derivative with respect to z:

$$u(x,y,z) = x\sin(y)z^{-2},$$

then

$$u_z(x,y,z) = \frac{\partial u}{\partial z} = -2x\sin(y)z^{-3} = -\frac{2x\sin(y)}{z^3}$$

Finally, the total differential of the function is calculated as follows:

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz =$$

$$\frac{\sin(y)}{z^2}dx + \frac{x\cos(y)}{z^2}dy - \frac{2x\sin(y)}{z^3}dz$$

b) Find a total differential df of the function $f(x,y,z) = xyz - xe^x + x\sin y$ at a point $(1, \frac{\pi}{2}, 1)$:

Solution: Let's do the derivatives with respect to x, y and z first.

$$f_x(x,y) = \frac{\partial}{\partial x}(xyz - xe^x + x\sin y) = yz - (e^x + xe^x) + \sin y$$

$$f_y(x,y) = \frac{\partial}{\partial y}(xyz - xe^x + x\sin y) = xz + x\cos y$$

$$f_z(x,y) = \frac{\partial}{\partial z}(xyz - xe^x + x\sin y) = xy$$

Then,

$$df(x,y,z) = (yz - (e^x + xe^x) + \sin y)dx + (xz + x\cos y)dy + (xy)dz$$

Hence,

$$df\left(1, \frac{\pi}{2}, 1\right) = \left(\frac{\pi}{2} \cdot 1 - (e^1 + 1 \cdot e^1) + \sin\frac{\pi}{2}\right)dx + \left(1 \cdot 1 + 1 \cdot \cos\frac{\pi}{2}\right)dy + \left(1 \cdot \frac{\pi}{2}\right)dz =$$

$$\left(\frac{\pi}{2} - 2e + 1\right)dx + dy + \frac{\pi}{2}dz$$

3. The Use of Chain Rule

a) Compute $\frac{dz}{dt}$ if $z = xe^{xy}$, $x = t^2$ and $y = t^{-1}$

Solution: There really isn't all that much to do here other than using the formula.

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Then,

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(xe^{xy}) = e^{xy} + yxe^{xy} \text{ and } \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(xe^{xy}) = x^2e^{xy}$$

$$\frac{dx}{dt} = \frac{d}{dt}(t^2) = 2t \text{ and } \frac{dy}{dt} = \frac{d}{dt}(t^{-1}) = -t^{-2}$$

Hence

$$\frac{dz}{dt} = (e^{xy} + yxe^{xy})(2t) + x^2e^{xy}(-t^{-2})$$

$$= 2t(e^{xy} + yxe^{xy}) - t^{-2}x^2e^{xy}$$

b) Compute $\frac{dz}{dt}$ if $z = x^2y^3 + y\cos x$, $x = \ln(t^2)$ and $y = \sin(4t)$

Solution: In this case it would almost definitely be more work to do the substitution first so we'll use the chain rule first and then substitute.

$$\frac{dz}{dt} = (2xy^3 - y\sin x)(\frac{2}{t}) + (3x^2y^2 + \cos x)(4\cos(4t))$$

$$= \frac{4\sin^3(4t)\ln t^2 - 2\sin(4t)\sin(\ln t^2)}{t} + 4\cos(4t)(3\sin^2(4t)[\ln t^2]^2 + \cos(\ln t^2))$$

c) Compute $\frac{dz}{dx}$ if $z = x\ln(xy) + y^3$ and $y = \cos(x^2 + 1)$

Solution: This is a special case we have taken above. The formula now is

$$\frac{dz}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}$$

Then,

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x\ln(xy) + y^3) = \ln(xy) + x\frac{y}{xy}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x\ln(xy) + y^3) = x\frac{x}{xy} + 3y^2$$

$$\frac{dy}{dx} = \frac{d}{dx}(\cos(x^2 + 1)) = -2x\sin(x^2 + 1)$$

Hence

$$\frac{dz}{dx} = (\ln(xy) + x\frac{y}{xy}) + (x\frac{x}{xy} + 3y^2)(-2x\sin(x^2 + 1))$$

$$= \ln(x\cos(x^2 + 1)) + 1 - 2x\sin(x^2 + 1)(\frac{x}{\cos(x^2 + 1)} + 3\cos^2(x^2 + 1))$$

$$= \ln(x\cos(x^2 + 1)) + 1 - 2x^2\tan(x^2 + 1) - 6x\sin(x^2 + 1)\cos^2(x^2 + 1)$$

d) Compute $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ if $z = e^{2x}\sin(3y)$, $x = u \cdot v - v^2$ and $y = \sqrt{u^2 + v^2}$

Solution: In this case if we were to substitute in for x and y we would get that z is a function of u and v and so it makes sense that we would be computing partial derivatives here and that there would be two of them. Here is the chain rule for both of these cases.

$$\frac{\partial z}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} \text{ and } \frac{\partial z}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}$$

Then,

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(e^{2x}\sin(3y)) = 2e^{2x}\sin(3y)$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(e^{2x}\sin(3y)) = 3e^{2x}\cos(3y)$$

$$\frac{\partial x}{\partial u} = \frac{\partial}{\partial u}(u \cdot v - v^2) = v$$

$$\frac{\partial x}{\partial v} = \frac{\partial}{\partial v}(u \cdot v - v^2) = u - 2v$$

$$\frac{\partial y}{\partial u} = \frac{\partial}{\partial u}\left(\sqrt{u^2 + v^2}\right) = \frac{u}{\sqrt{u^2 + v^2}}$$

$$\frac{\partial y}{\partial v} = \frac{\partial}{\partial v}\left(\sqrt{u^2 + v^2}\right) = \frac{v}{\sqrt{u^2 + v^2}}$$

Here is the chain rule for $\frac{\partial z}{\partial u}$

$$\frac{\partial z}{\partial u} = (2e^{2x}\sin(3y))(v) + (3e^{2x}\cos(3y))\frac{u}{\sqrt{u^2 + v^2}}$$

$$= v(2e^{2(u\cdot v - v^2)}\sin(3\sqrt{u^2 + v^2})) + \frac{3ue^{2(u\cdot v - v^2)}\cos(3\sqrt{u^2 + v^2})}{\sqrt{u^2 + v^2}}$$

Now the chain rule for $\frac{\partial z}{\partial v}$

$$\frac{\partial z}{\partial v} = (2e^{2x}\sin(3y))(u - 2v) + (3e^{2x}\cos(3y))\frac{v}{\sqrt{u^2 + v^2}}$$

$$= (u - 2v)(2e^{2(u\cdot v - v^2)}\sin(3\sqrt{u^2 + v^2})) + \frac{3ve^{2(u\cdot v - v^2)}\cos(3\sqrt{u^2 + v^2})}{\sqrt{u^2 + v^2}}$$

4. The Higher Order Partial Derivatives

a) Compute all four second derivatives of $f(x, y) = x^2y^2$

Solution: To take a derivative,' we must take a partial derivative with respect to x or y, and there are four ways to do it in the following order: x then x, x then y, y then x, y then y.

$$f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}(x^2y^2)\right) = \frac{\partial}{\partial x}(2xy^2) = 2y^2$$

$$f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}(x^2y^2)\right) = \frac{\partial}{\partial y}(2xy^2) = 4xy$$

$$f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}(x^2y^2)\right) = \frac{\partial}{\partial x}(2yx^2) = 4xy$$

$$f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}(x^2y^2)\right) = \frac{\partial}{\partial y}(2yx^2) = 2x^2$$

Note. According to Clairaut's Theorem, the mixed partial derivatives are equal, if they are continuous. That is we might find either f_{yx} or f_{xy}.

b) Compute all second derivatives of $f = \frac{xy}{x^2+y^2}$

Solution:

$$f_x = \frac{\partial}{\partial x}\left(\frac{xy}{x^2+y^2}\right) = \frac{y^3-yx^2}{(x^2+y^2)^2},\ f_{xx} = \frac{\partial}{\partial x}\left(\frac{y^3-yx^2}{(x^2+y^2)^2}\right) = \frac{2x^3y-6xy^3}{(x^2+y^2)^3},$$

$$f_{xy} = \frac{\partial}{\partial y}\left(\frac{y^3-yx^2}{(x^2+y^2)^2}\right) = \frac{6x^2y^2-x^4-y^4}{(x^2+y^2)^3},\ f_{yx} = f_{xy}$$

$$f_y = \frac{\partial}{\partial y}\left(\frac{xy}{x^2+y^2}\right) = \frac{x^3-xy^2}{(x^2+y^2)^2},\ f_{yy} = \frac{\partial}{\partial y}\left(\frac{x^3-xy^2}{(x^2+y^2)^2}\right) = \frac{2xy^3-6x^3y}{(x^2+y^2)^3}$$

c)

5. Implicit Differentiation Problems for a function $z = f(x, y)$: find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

a) $x^3z^2 - 5xy^5z = x^2 + y^3$

Solution: Let's start with finding $\frac{\partial z}{\partial x}$. We first will differentiate both sides with respect to x and remember to add on a $\frac{\partial z}{\partial x}$ whenever we differentiate a z from the chain rule.

Remember that since we are assuming $z = z(x, y)$ then any product of x's and z's will be a product and so will need the product rule!

$$3x^2z^2 + 2x^3z\frac{\partial z}{\partial x} - 5y^5z - 5xy^5\frac{\partial z}{\partial x} = 2x$$

Now, solve for $\frac{\partial z}{\partial x}$

$$
\begin{aligned}
(2x^3z - 5xy^5)\frac{\partial z}{\partial x} &= 2x - 3x^2z^2 + 5y^5z \\
\frac{\partial z}{\partial x} &= \frac{2x - 3x^2z^2 + 5y^5z}{2x^3z - 5xy^5}
\end{aligned}
$$

Now we'll do the same thing for $\frac{\partial z}{\partial y}$ except this time we'll need to remember to add on a $\frac{\partial z}{\partial y}$ whenever we differentiate a z from the chain rule.

$$
\begin{aligned}
2x^3z\frac{\partial z}{\partial y} - 25xy^4z - 5xy^5\frac{\partial z}{\partial y} &= 3y^2 \\
(2x^3z - 5xy^5)\frac{\partial z}{\partial y} &= 3y^2 + 25xy^4z \\
\frac{\partial z}{\partial y} &= \frac{3y^2 + 25xy^4z}{2x^3z - 5xy^5}
\end{aligned}
$$

The second way for solving this task is the use of the rule,

$$\frac{\partial z}{\partial x} = -\frac{\partial F/\partial x}{\partial F/\partial z} \text{ and } \frac{\partial z}{\partial y} = -\frac{\partial F/\partial y}{\partial F/\partial z},$$

where

$$F(x, y, z) = 0$$

We first rewrite the given function in the form:

$$F(x, y, z) = x^3z^2 - 5xy^5z - x^2 - y^3 = 0$$

Then, we differentiate $F(x, y, z)$ with respect to x, that is

$$\frac{\partial F}{\partial x} = 3x^2z^2 - 5y^5z - 2x$$

and with respect to y

$$\frac{\partial F}{\partial y} = -25xy^4z - 3y^2$$

and, finally, and with respect to z

$$\frac{\partial F}{\partial z} = 2x^3z - 5xy^5$$

Hence,

$$\frac{\partial z}{\partial x} = -\frac{3x^2z^2 - 5y^5z - 2x}{2x^3z - 5xy^5} = \frac{2x - 3x^2z^2 + 5y^5z}{2x^3z - 5xy^5}$$

and

$$\frac{\partial z}{\partial y} = -\frac{-25xy^4z - 3y^2}{2x^3z - 5xy^5} = \frac{3y^2 + 25xy^4z}{2x^3z - 5xy^5}$$

b) $x^2\sin(2y - 5z) = 1 + y\cos(6zx)$

We'll do the same thing for this function as we did in the previous part. First let's find $\frac{\partial z}{\partial x}$. Don't forget to do the chain rule on each of the trig functions and when we are differentiating the inside function on the cosine we will need to also use the product rule.

$$2x\sin(2y-5z)+x^2\cos(2y-5z)(-5\frac{\partial z}{\partial x})=-y\sin(6zx)(6z+6x\frac{\partial z}{\partial x})$$

Now, solve for $\frac{\partial z}{\partial x}$

$$\begin{aligned} 2x\sin(2y-5z)-5\frac{\partial z}{\partial x}x^2\cos(2y-5z) &= -6zy\sin(6zx)-6yx\sin(6zx)\frac{\partial z}{\partial x} \\ 2x\sin(2y-5z)+6zy\sin(6zx) &= (5x^2\cos(2y-5z)-6yx\sin(6zx))\frac{\partial z}{\partial x} \\ \frac{\partial z}{\partial x} &= \frac{2x\sin(2y-5z)+6zy\sin(6zx)}{5x^2\cos(2y-5z)-6yx\sin(6zx)} \end{aligned}$$

Now let's take care of $\frac{\partial z}{\partial y}$. This one will be slightly easier than the first one.

$$\begin{aligned} x^2\cos(2y-5z)(2-5\frac{\partial z}{\partial y}) &= \cos(6zx)-y\sin(6zx)(6x\frac{\partial z}{\partial y}) \\ 2x^2\cos(2y-5z)-5x^2\cos(2y-5z)\frac{\partial z}{\partial y} &= \cos(6zx)-6xy\sin(6zx)\frac{\partial z}{\partial y} \\ (6xy\sin(6zx)-5x^2\cos(2y-5z))\frac{\partial z}{\partial y} &= \cos(6zx)-2x^2\cos(2y-5z) \\ \frac{\partial z}{\partial y} &= \frac{\cos(6zx)-2x^2\cos(2y-5z)}{6xy\sin(6zx)-5x^2\cos(2y-5z)} \end{aligned}$$

The second way is a bit easier.

First let's get everything on one side to form the function:

$$F(x,y,z)=x^2\sin(2y-5z)-1-y\cos(6zx)=0$$

Now, the function on the left is $F(x,y,z)$ and so all that we need to do is use the formulas developed above to find the derivatives.

$$\frac{\partial z}{\partial x}=-\frac{2x\sin(2y-5z)+6yz\sin(6zx)}{-5x^2\cos(2y-5z)+6yx\sin(6zx)}$$

and

$$\frac{\partial z}{\partial y}=-\frac{2x^2\cos(2y-5z)-\cos(6zx)}{-5x^2\cos(2y-5z)+6yx\sin(6zx)}$$

6. Find the plane tangent to a multi-variable function at a point.

   a) Find the plane tangent to and a straight line normal to $x^2+y^2+z^2=4$ at $(1,1,\sqrt{2})$.

Solution: For a surface given by a differentiable multivariable function $z = f(x, y)$ the equation of the tangent plane at (x_0, y_0, z_0) is given as

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0$$

whereas the canonical equations of the straight line normal to the tangent plane at (x_0, y_0, z_0) is given as

$$\frac{(x - x_0)}{f_x(x_0, y_0)} = \frac{(y - y_0)}{f_y(x_0, y_0)} = \frac{(z - z_0)}{-1}$$

As the point $(1,1,\sqrt{2})$. is on the upper hemisphere, so we can use $f(x, y) = \sqrt{4 - x^2 - y^2}$. Then,

$$f_x(x, y) = -x(4 - x^2 - y^2)^{-1/2} \text{ and } f_y(x, y) = -y(4 - x^2 - y^2)^{-1/2},$$

so

$$f_x(1,1) = f_y(1,1) = -1/\sqrt{2}$$

and the equation of the plane is

$$z = -\frac{1}{\sqrt{2}}(x - 1) - \frac{1}{\sqrt{2}}(y - 1) + \sqrt{2}.$$

The equation of the straight line is

$$\frac{(x - 1)}{-\frac{1}{\sqrt{2}}} = \frac{(y - 1)}{-\frac{1}{\sqrt{2}}} = \frac{z - \sqrt{2}}{-1}$$

b) Find the equation of the tangent plane to and a straight line normal to $z = \ln(2x + y)$ at $M_0(-1,3)$.

Solution: We first find the applicate of the point at which the tangent plane and a normal straight line to the surface should be found: $z_0 = f(-1,3) = \ln(2(-1) + 3) = \ln(1) = 0$

Then,

$$f_x(x, y) = \frac{2}{2x+y}, \; f_x(-1,3) = 2$$

$$f_y(x, y) = \frac{1}{2x+y}, \; f_y(-1,3) = 1$$

So, the equation of the plane is then,

$$z - 0 = 2(x + 1) + (1)(y - 3)$$

$$z = 2x + y - 1$$

The equation of the straight line is

$$\frac{(x+1)}{2} = \frac{(y-3)}{1} = \frac{z-0}{-1}$$

c) Find an equation for the plane tangent to $2x^2 + 3y^2 - z^2 = 4$ at $(1,1,-1)$.

Solution: We find the partial derivatives by using implicit differentiation. First let's get everything on one side to form the function:

$$F(x,y,z) = 2x^2 + 3y^2 - z^2 - 4 = 0$$

Then,

$$f_x(x,y) = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}, \; f_y(x,y) = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$$

$$\frac{\partial F}{\partial x} = \frac{\partial}{\partial x}(2x^2 + 3y^2 - z^2 - 4) = 4x$$

$$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}(2x^2 + 3y^2 - z^2 - 4) = 6y$$

$$\frac{\partial F}{\partial z} = \frac{\partial}{\partial z}(2x^2 + 3y^2 - z^2 - 4) = -2z$$

So,

$$f_x(x,y) = -\frac{4x}{-2z} = \frac{2x}{z}, \; f_x(1,1) = 2$$

and

$$f_y(x,y) = -\frac{6y}{-2z} = \frac{3y}{z}, \; f_y(1,1) = 3$$

So, the equation of the plane is then,

$$z + 1 = 2(x-1) + 3(y-1)$$

$$z = 2x + 3y - 6$$

The equation of the straight line is

$$\frac{(x-1)}{2} = \frac{(y-1)}{3} = \frac{z+1}{-1}$$

7. Find the gradient of the function $w = w(x,y,z) = 5x^2 - 2xy + y^2 - 4yz + z^2 + 3xz$ and its directional derivative in the direction of vector $\vec{u} = -\vec{i} + 2\vec{j} + 2\vec{k}$ at a point $M_0(1,-2,3)$

We first calculate the partial derivatives of f

$$\begin{aligned} f_x(x,y,z) &= 10x - 2y + 3z \\ f_y(x,y,z) &= -2x + 2y - 4z \\ f_z(x,y,z) &= -4y + 2z + 3x, \end{aligned}$$

then substitute them into Equation

$$\vec{\nabla} f(x,y,z) = f_x(x,y,z)\vec{i} + f_y(x,y,z)\vec{j} + f_z(x,y,z)\vec{k} =$$
$$= (10x - 2y + 3z)\vec{i} + (-2x + 2y - 4z)\vec{j} + (-4y + 2z + 3x)\vec{k}$$

Then,

$$\vec{\nabla} f(1,-2,3) = (10 \cdot 1 - 2 \cdot (-2) + 3 \cdot 3)\vec{i} + (-2 \cdot 1 + 2 \cdot (-2) - 4 \cdot 3)\vec{j}$$
$$+(-4 \cdot (-2) + 2 \cdot 3 + 3 \cdot 1)\vec{k} = 23\vec{i} - 18\vec{j} + 17\vec{k}$$

The directional derivative of f in the direction of $\vec{u}$ is given by

$$\frac{\partial f}{\partial \vec{u}}(x,y,z) = \vec{\nabla} f(x,y,z) \cdot \vec{u} = f_x(x,y,z)\cos\alpha + f_y(x,y,z)\cos\beta + f_z(x,y,z)\cos\gamma.$$

Since the vector is $\vec{u}$ not unit, we find the magnitude of $\vec{u}$

$$\left\|\vec{u}\right\| = \sqrt{(-1)^2 + (2)^2 + (2)^2} = \sqrt{9} = 3$$

Therefore, a unit vector in the direction of $\vec{u}$ is

$$\frac{\vec{u}}{\|\vec{u}\|} = \frac{-\vec{i} + 2\vec{j} + 2\vec{k}}{3} = -\frac{1}{3}\vec{i} + \frac{2}{3}\vec{j} + \frac{2}{3}\vec{k}$$

so

$$\cos\alpha = -\frac{1}{3}, \cos\beta = \frac{2}{3}, \text{ and } \cos\gamma = \frac{2}{3}$$

Hence,

$$\frac{\partial f}{\partial \vec{u}}(1,-2,3) = 23 \cdot \left(-\frac{1}{3}\right) - 18 \cdot \left(\frac{2}{3}\right) + 17 \cdot \left(\frac{2}{3}\right) = -\frac{25}{3}$$

8. Find the Local Extrema of the Function

$$f(x,y) = x^3 + 2xy - 6x - 4y^2.$$

We first calculate $f_x(x,y)$ and $f_y(x,y)$:

$$f_x(x,y) = 3x^2 + 2y - 6$$
$$f_y(x,y) = 2x - 8y$$

Equate the equations to zero yielding the system of equations:

$$3x^2 + 2y - 6 = 0$$
$$2x - 8y = 0$$

The second equation gives $2y = \frac{x}{4}$. Substituting it into the first equation, we have the solutions to this system are $\left(-\frac{3}{2}, -\frac{3}{8}\right)$ and $\left(\frac{4}{3}, \frac{1}{3}\right)$ Therefore they are critical points of f .

Next, we calculate the second partial derivatives of f:

$$f_{xx}(x,y) = 6x,\ f_{xy}(x,y) = 2,\ f_{yy}(x,y) = -8$$

Therefore,

$$D\left(-\frac{3}{2},-\frac{3}{8}\right) = f_{xx}\left(-\frac{3}{2},-\frac{3}{8}\right) \cdot f_{yy}\left(-\frac{3}{2},-\frac{3}{8}\right) - f^2{}_{xy}\left(-\frac{3}{2},-\frac{3}{8}\right) =$$
$$6\left(-\frac{3}{2}\right) \cdot (-8) - 2^2 = 78 > 0$$
$$D\left(\frac{4}{3},\frac{1}{3}\right) = f_{xx}\left(\frac{4}{3},\frac{1}{3}\right) \cdot f_{yy}\left(\frac{4}{3},\frac{1}{3}\right) - f^2{}_{xy}\left(\frac{4}{3},\frac{1}{3}\right) =$$
$$6\left(\frac{4}{3}\right) \cdot (-8) - 2^2 = -68 < 0$$

Applying the theorem to point $\left(-\frac{3}{2},-\frac{3}{8}\right)$, where $f_{xx}(x,y) = 6x = 6\left(-\frac{3}{2}\right) = -9 < 0$ leads to case 2, which means that the point corresponds to a *local maximum*.
Applying the theorem to point $(\frac{4}{3},\frac{1}{3})$ leads to case 3, which means that the point is a saddle point.

Example. Use the second derivative test to find the local extrema of the function:

$$g(x,y) = \frac{1}{3}x^3 + y^2 + 2xy - 6x - 3y + 4$$

Setting $g_x(x,y)$ and $g_y(x,y)$ equal to zero yields the system of equations

$$\begin{aligned} x^2 + 2y - 6 &= 0 \\ 2y + 2x - 3 &= 0. \end{aligned}$$

To solve this system, first solve the second equation for y. This gives $y = \frac{3-2x}{2}$. Substituting this into the first equation gives

$$\begin{aligned} x^2 + 3 - 2x - 6 &= 0 \\ x^2 - 2x - 3 &= 0 \\ (x-3)(x+1) &= 0. \end{aligned}$$

Therefore, $x = -1$ or $x = 3$. Substituting these values into the equation $y = \frac{3-2x}{2}$, it yields the critical points $(-1,\frac{5}{2})$ and $(3,-\frac{3}{2})$.

Calculate the second partial derivatives of g:

$$\begin{aligned} g_{xx}(x,y) &= 2x \\ g_{xy}(x,y) &= 2 \\ g_{yy}(x,y) &= 2. \end{aligned}$$

Then, we find a general formula for D:

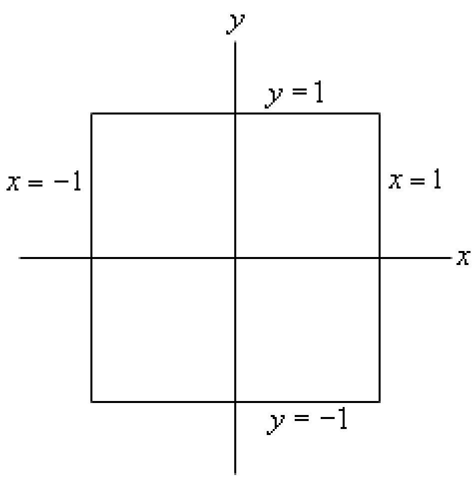
$$\begin{aligned} D(x_0,y_0) &= g_{xx}(x_0,y_0)g_{yy}(x_0,y_0) - (g_{xy}(x_0,y_0))^2 \\ &= (2x_0)(2) - 2^2 = 4x_0 - 4 \end{aligned}$$

Next, we substitute each critical point into this formula:

$$D(-1,\frac{5}{2}) \quad = (2(-1))(2)-(2)^2 = -4-4 = -8$$
$$D(3,-\frac{3}{2}) \quad = (2(3))(2)-(2)^2 = 12-4 = 8.$$

We note that, applying Note to point $(-1,\frac{5}{2})$ leads to case 3, which means that $(-1,\frac{5}{2})$ is a saddle point. Applying the theorem to point $(3,-\frac{3}{2})$ leads to case 1, which means that $(3,-\frac{3}{2})$ corresponds to a local minimum.

9. *Example* 1. Find the absolute minimum and absolute maximum of $f(x,y) = x^2+4y^2-2x^2y+4$ on the rectangle given by $-1 \le x \le 1$ and $-1 \le y \le 1$.



Let's first get a quick picture of the rectangle for reference purposes. The boundary of this rectangle is given by the following conditions.

right side : $x = 1, -1 \le y \le 1$
left side : $x = -1, -1 \le y \le 1$
upper side : $y = 1, -1 \le x \le 1$
lower side : $y = -1, -1 \le x \le 1$

We'll start this off by finding all the critical points that lie inside the given rectangle. To do this we'll need the two first order derivatives.

$$f_x = 2x-4xy,\ f_y = 8y-2x^2$$

To find the critical points we will need to solve the system,

$$2x-4xy \quad = 0$$
$$8y-2x^2 \quad = 0$$

We can solve the second equation for y to get, $y = \frac{x^2}{4}$. Plugging this into the first equation gives us, $2x-4x(\frac{x^2}{4}) = 2x-x^3 = x(2-x^2) = 0$.

This tells us that we must have roots: $x = 0$ and $x = \pm\sqrt{2} = \pm 1.414...$

Now, recall that we only want critical points in the region that we're given. That means that we only want critical points for which $-1 \le x \le 1$. The only value of x that will satisfy this is the first one so we can ignore the last two for this problem.

Plugging $x = 0$ into the equation for y gives us, $y = \frac{0^2}{4} = 0$. The single critical point, in the region (and again, that's important), is (0,0). We now need to get the value of the function at the critical point: $f(0,0) = 4$.

Now, we're going to look at what the function is doing along each of the sides of the rectangle listed above.

Let's first take a look at the right side. As noted above the right side is defined by $x = 1, -1 \leq y \leq 1$. Let's take advantage of this by defining a new function as follows,

$$g(y) = f(1, y) = 1^2 + 4y^2 - 2(1^2)y + 4 = 5 + 4y^2 - 2y$$

Now, finding the absolute extrema of $f(x, y)$ along the right side will be equivalent to finding the absolute extrema of $g(y)$ in the range $-1 \leq y \leq 1$.

Let's do that for this problem:

$$g'(y) = 8y - 2 \Rightarrow y = \frac{1}{4}$$

In this range we will need the following function evaluations:

$$g(-1) = 11,\ g(1) = 7,\ g(\tfrac{1}{4}) = \tfrac{19}{4} = 4.75$$

Notice that, using the definition of $g(y)$ these are also function values for $f(x, y)$ at

$$\begin{aligned} g(-1) &= f(1,-1) = 11 \\ g(1) &= f(1,1) = 7 \\ g(\frac{1}{4}) &= f(1,\frac{1}{4}) = \frac{19}{4} = 4.75 \end{aligned}$$

We can now do the left side of the rectangle which is defined by, $x = -1, -1 \leq y \leq 1$ Again, we'll define a new function as follows,

$$g(y) = f(-1, y) = (-1)^2 + 4y^2 - 2(-1)^2y + 4 = 5 + 4y^2 - 2y$$

We will find the critical point $y = \frac{1}{4} \Rightarrow (-1, \frac{1}{4})$ and on the boundary $(-1, -1)$ and $(-1,1)$

$$\begin{aligned} g(-1) &= f(-1,-1) = 11 \\ g(1) &= f(-1,1) = 7 \\ g(\frac{1}{4}) &= f(-1,\frac{1}{4}) = \frac{19}{4} = 4.75 \end{aligned}$$

Next, we can now look at the upper side defined by, $y = 1, -1 \leq x \leq 1$

Define a new function except this time it will be a function of x:

$$h(x) = f(x, 1) = x^2 + 4(1^2) - 2x^2(1) + 4 = 8 - x^2$$

Hence, we need to find the absolute extrema of $h(x)$ on the range $-1 \leq x \leq 1$

$$h'(x) = -2x \Rightarrow x = 0$$

The value of this function $f(x, y)$ at the critical point and the end points is,

$$\begin{aligned} h(-1) &= f(-1,1) = 7 \\ h(1) &= f(1,1) = 7 \\ h(0) &= f(0,1) = 8 \end{aligned}$$

Last, we need to take care of the lower side. This side is defined by, $y = -1, -1 \le x \le 1$ The new function we'll define in this case is,

$$h(x) = f(x,-1) = x^2 + 4(-1)^2 - 2x^2(-1) + 4 = 8 + 3x^2$$

The critical point for this function is,

$$h'(x) = 6x \Rightarrow x = 0$$

The function values $f(x,y)$ at the critical point and the endpoint are,

$$\begin{aligned} h(-1) &= f(-1,-1) = 11 \\ h(1) &= f(1,-1) = 11 \\ h(0) &= f(0,-1) = 8 \end{aligned}$$

Finally, we need compare all values of the function found in all the steps and take the largest and smallest as the absolute extrema of the function in the rectangle.

$$\begin{array}{llllll} f(0,0) & = 4 & f(1,-1) & = 11 & f(1,1) & = 7 \\ f(1,\frac{1}{4}) & = 4.75 & f(-1,1) & = 7 & f(-1,-1) & = 11 \\ f(-1,\frac{1}{4}) & = 4.75 & f(0,1) & = 8 & f(0,-1) & = 8 \end{array}$$

The absolute minimum is at (0,0) since gives the smallest function value and the absolute maximum occurs at (1,–1) and (–1,–1) since these two points give the largest function value.

Example 2. Find the absolute minimum and absolute maximum of $f(x,y) = 2x^2 - y^2 + 6y$ on the disk of radius 4, $x^2 + y^2 \le 16$.

Let's first find the critical points of the function that lies inside the disk. This will require the following two first order partial derivatives.

$$f_x = 4x,\ f_y = -2y + 6$$

To find the critical points we'll need to solve the following system.

$$\begin{aligned} 4x &= 0 \\ -2y + 6 &= 0 \end{aligned}$$

So, the only critical point for this function is (0,3). The function value at this critical point is, $f(0,3) = 9$

Now we need to look at the boundary. We can solve this for x^2 and plug this into the x^2 in $f(x,y)$ to get a function of y as follows: $x^2 = 16 - y^2$,

$$g(y) = 2(16 - y^2) - y^2 + 6y = 32 - 3y^2 + 6y$$

We will need to find the absolute extrema of this function on the range $-4 \leq y \leq 4$. We'll first need the critical points of this function.

$$g'(y) = -6y + 6 \Rightarrow y = 1$$

To find the points, we can do this by plugging the value of y into our equation for the circle and solving for x as

$$y = -4: \quad x^2 = 16 - 16 = 0 \Rightarrow x = 0$$
$$y = 4: \quad x^2 = 16 - 16 = 0 \Rightarrow x = 0$$
$$y = 1: \quad x^2 = 16 - 1 = 15 \Rightarrow x = \pm\sqrt{15} = \pm 3.87$$

The function values for $f(x, y)$

$$g(-4) = -40 \quad \Rightarrow \quad f(0, -4) = -40$$
$$g(4) = 8 \quad \Rightarrow \quad f(0,4) = 8$$
$$g(1) = 35 \quad \Rightarrow \quad f\left(-\sqrt{15}, 1\right) = 35 \quad \text{and} \quad f(\sqrt{15}, 1) = 35$$

So, comparing these values to the value of the function at the critical point of $f(x, y)$ that we found earlier we can see that the absolute minimum occurs at $(0, -4)$ while the absolute maximum occurs twice at $(-\sqrt{15}, 1)$ and $(\sqrt{15}, 1)$.