4. ORDINARY DIFFERENTIAL EQUATIONS

4.1 Prelude to Differential Equations

Many real-world phenomena can be modeled mathematically by using differential equations. Our goal is to develop solution techniques for different types of ordinary differential equations we will consider.

Definition 1. A *differential equation* is an equation with one or more derivatives of a function or functions. In other words, it is defined as the equation that contains derivatives of one or more dependent variables with respect to one or more independent variables.

Definition 2. An *ordinary differential equation* (ODE) is an equation containing an unknown function of one real or complex variable x, its derivatives, and some given functions of x. The unknown function is generally represented by a variable (often denoted y), which, therefore, depends on x. Thus x is often called the independent variable of the equation.

$$F\left(x; y, y', y'', y''', \dots, y^{(n)}\right) = 0$$

The term "*ordinary*" is used in contrast with the term *partial differential equation*, which may be with respect to more than one independent variable.

One of the most common types of differential equations is an equation containing an unknown function $y = f(x)$ and its derivative, given by $\frac{dy}{dx} = y'$, it is known as an *ordinary differential equation*.

Consider an example of a differential equation:

$$\frac{dy}{dx} = y' = 2x,$$

As seen the equation includes a derivative. There is a relationship between the variables x and y is an unknown function $y(x)$. Furthermore, the left-hand side of the equation is the derivative of $y(x)$. Therefore, we can interpret this equation as follows: start with some function $y = f(x)$ and take its derivative. The answer must be equal to $2x$. What function has a derivative that is equal to $2x$? One such function is $y = x^2$, so this function is considered a solution to a differential equation.

Definition. A solution to a differential equation is a function $y = f(x)$ that satisfies the differential equation when $f(x)$ and its derivatives are substituted into the equation.

Example of verifying solutions of differential equation:

Equation	Solution
$y' = 2x$	$y = x^2$
$y' + 3y = 6x + 11$	$y = e^{-3x} + 2x + 3$

To verify the solution, we first calculate y' using the chain rule for derivatives. This gives $y' = -3e^{-3x} + 2$ Next we substitute y and y' into the left-hand side of the differential equation:

$$(-3e^{-2x} + 2) + 3(e^{-2x} + 2x + 3).$$

the resulting expression can be simplified by first distributing to eliminate the parentheses, giving

$$-3e^{-2x} + 2 + 3e^{-2x} + 6x + 9.$$

Combining like terms leads to the expression $6x + 11$, which is equal to the right-hand side of the differential equation. This result verifies that $y = e^{-3x} + 2x + 3$ is a solution of the differential equation.

It is convenient to define characteristics of differential equations that make it easier to talk about them and categorize them. The most basic characteristic of a differential equation is its *orde*r.

Definition. The *order* of a differential equation is the **highest order** of any derivative of the unknown function that appears in the equation.

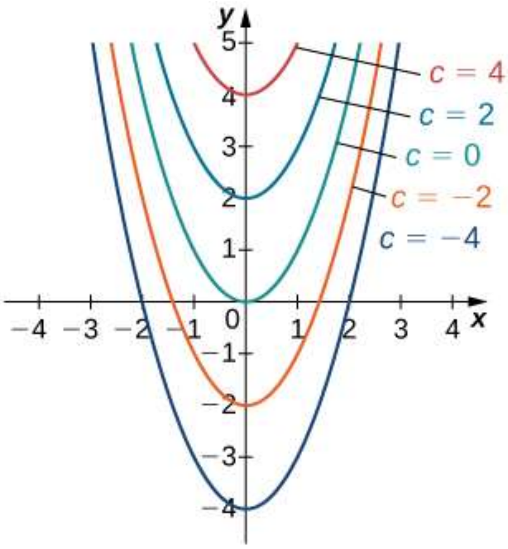
Example:

- ✓ $y' - 4y = x^2 - 3x + 4$ - The highest derivative in the equation is y' ,so the order is 1. It is called a first order ordinary differential equation;
- ✓ $x^2y''' - 3xy'' + xy' - 3y = \sin x$ - The highest derivative in the equation is y''' , so the order is 3. It is called the third order ordinary differential equation;
- ✓ $\frac{4}{x}y^{(4)} - \frac{6}{x^2}y'' + \frac{12}{x^4}y = x^3 - 3x^2 + 4x - 12$ - The highest derivative in the equation is $y^{(4)}$, so the order is 4. It is called the fourth order ordinary differential equation.

4.2 General and Particular Solutions of First-Order Ordinary Differential Equation (FODE)

We can note that the differential equation $y' = 2x$ has not only the solution $y = x^2$, but the other $y = x^2 + 4$ is a solution too. The only difference between these two solutions is the last term, which is a constant. In fact, any function of the form $y = x^2 + C$ where C represents any constant, is a solution as well. The reason is that the derivative of $x^2 + C$ is $2x$, regardless of the value of C. It can be shown that any solution of this differential equation must be of the form $y = x^2 + 4$. This is an example of a *general solution* to a differential equation.

If we will graph some of these solutions $y = x^2 + C$ (Note: in this graph we use even integer values for C ranging between -4 and 4 . In fact, there is no restriction on the value of C ; it can be an integer or not.), we get a family (a set) of the parabolas that presents a *general solution* to the differential equation $y' = 2x$ (i.e. the family of solutions to the differential equation)



We are free to choose any solution we wish; for example, $y = x^2 + 3$ is a member of the family of solutions to this differential equation. This is called a *particular solution* to the differential equation.

As seen, a particular solution can often be uniquely identified if we are given *additional information about the problem*.

Example. Find the particular solution to the differential equation $y' = 2x$ passing through the point $(2,7)$.

Any function of the form $y = x^2 + C$ is a solution to this differential equation. To determine the value of C , we substitute the values $x = 2$ and $y = 7$ into this equation and solve for C:

$$y = x^2 + C, 7 = 2^2 + C, \Rightarrow 7 = 4 + C, \Rightarrow C = 3$$

Therefore the particular solution passing through the point (2,7) is $y = x^2 + 3$.

4.3 Initial-Value Problems

As shown, a given differential equation has *an infinite number of solutions*, so it is natural to ask which one we want to use. To choose one solution, more information is needed. Some specific information that can be useful is *an initial value*, which is an ordered pair that is used to find a particular solution.

Definition. A differential equation together with one or more initial values is called *an initial-value problem*.

Thumb rule. The general rule is that the number of initial values needed for an initial-value problem is equal to the order of the differential equation.

For example,

- ✓ if we have the differential equation $y' = 2x$ then $y(3) = 7$ is an initial value, and when taken together, these equations form an initial-value problem.
- ✓ if the differential equation $y'' - 3y' + 2y = 4e^x$ is second order, so we need two initial values. With initial-value problems of order greater than one, the same value should be used for the independent variable. An example of initial values for this second-order equation would be $y(0) = 2$ and $y'(0) = -1$. These two initial values together with the differential equation form an initial-value problem.

Example. Verify that the function $y = 2e^{-2x} + e^x$ is a solution to the initial-value problem

$$y' + 2y = 3e^x, y(0) = 3.$$

We start by calculating y'. This gives $y' = -4e^{-2x} + e^x$.

Next we substitute both y and y' into the left-hand side of the differential equation and simplify:

$$\begin{aligned} y' + 2y &= (-4e^{-2x} + e^x) + 2(2e^{-2x} + e^x) \\ &= -4e^{-2x} + e^x + 4e^{-2x} + 2e^x = 3e^x. \end{aligned}$$

Next we calculate $y(0)$:

$$y(0) = 2e^{-2(0)} + e^0 = 2 + 1 = 3.$$

This result verifies the initial value. Therefore, the given function satisfies the initial-value problem.

Geometrical interpretation of FODE

In general, a first-order differential equation can be written in the form

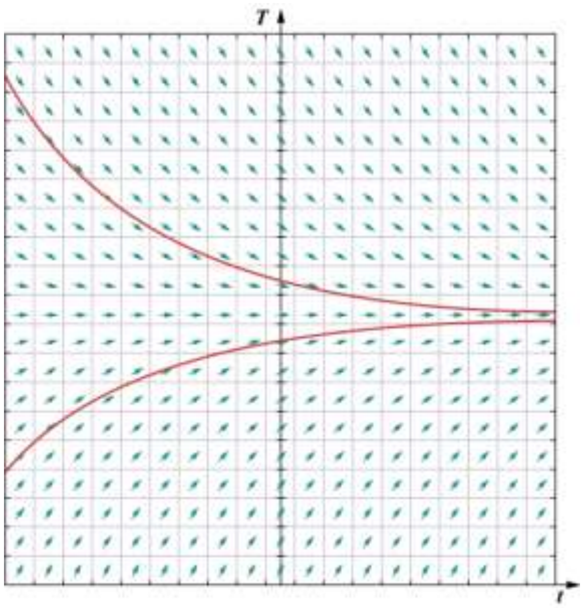
$$y' = f(x, y).$$

Since the derivative of a function evaluated at a given point is the slope of the tangent line to the graph of that function at the same point, we get a *direction field* (also called a *slope field*) on the *xy*-plane.

As example, if $T(t)$ represents the temperature of an object at time t, and the ambient temperature is 72°F, then of cooling we have a differential equation in accordance with the Newton's law as follows:

$$T'(t) = -0.4(T - 72).$$

Figure shows the direction field for this equation, where the red lines show two solutions: one with initial temperature less than 72°F and the other with initial temperature greater than 72°F:



We can generate a direction field of this type for any differential equation of the form $y' = f(x, y)$:

Definition. A direction field (slope field) is a mathematical object used to graphically represent solutions to a first-order differential equation. At each point in a direction field, a line segment appears whose slope is equal to the slope of a solution to the differential equation passing through that point.

Note. We can use a direction field to predict the behavior of solutions to a differential equation without knowing the actual solution.

4.4 Solving first-order ordinary differential equations

Now we will focus on methods of the solutions of various first-order ordinary differential equations. First we examine a solution technique for finding exact solutions to a class of differential equations known as *separable differential equations*.

1. *Separable Equations*

Definition. A separable differential equation is any equation that can be written in the form:

$$y' = f(x)g(y).$$

The term '*separable*' refers to the fact that the right-hand side of Equation can be separated into a function of x times a function of y. Examples of separable differential equations include

$$y' = (x^2 - 4)(3y + 2)$$
$$y' = 6x^2 + 4x$$
$$y' = \sec y + \tan y$$
$$y' = xy + 3x - 2y - 6 = (x + 3)(y - 2).$$

If a differential equation is separable, then it is possible to solve the equation using the *method of separation of variables* as follows:

1. Check for any values of y hat make $g(y) = 0$. These correspond to constant solutions[1].
2. Rewrite the differential equation in the form

$$\frac{dy}{g(y)} = f(x)dx.$$

3. Integrate both sides of the equation.
4. Solve the resulting equation for y if possible.
5. If an initial condition exists, substitute the appropriate values for x and y into

[1] We need to make sure that $g(y) \neq 0$. If there's a number y_0 such that $g(y_0) = 0$, then this number will also be a solution of the differential equation. Division by $g(y)$ causes loss of this solution.

the equation and solve for the constant.

Note that Step 4 states "Solve the resulting equation for y if possible." It is not always possible to obtain y as an explicit function of x. Quite often we have to be satisfied with finding y as an implicit function of x.

Example 1. Find a general solution to the differential equation

$$y' = (x^2 - 4)(3y + 2)$$

Follow the five-step method of separation of variables.

In this example, $f(x) = x^2 - 4$ and $g(y) = 3y + 2$. Setting $g(y) = 0$ gives $y = -\frac{2}{3}$ as a constant solution.

Rewrite the differential equation in the separable form: $\frac{dy}{3y+2} = (x^2 - 4)dx$.

Integrate both sides of the equation: $\int \frac{dy}{3y+2} = \int (x^2 - 4)dx$. So, the solution becomes: $\frac{1}{3}\ln|3y + 2| = \frac{1}{3}x^3 - 4x + C$.

To solve this equation for y, first multiply both sides of the equation by 3, $\ln|3y + 2| = x^3 - 12x + 3C$. If we call the second arbitrary constant C_1, the equation becomes $\ln|3y + 2| = x^3 - 12x + C_1$. Now exponentiate both sides of the equation

$$e^{\ln|3y+2|} = e^{x^3-12x+C_1}$$
$$|3y + 2| = e^{C_1}e^{x^3-12x}$$

Again define a new constant $C_2 = e^{C_1}$: $|3y + 2| = C_2e^{x^3-12x}$. Finally, the solution can be written in the form: $y = \frac{-2+Ce^{x^3-12x}}{3}$.

No initial condition is imposed, so we are finished.

Example 2. Solving an initial-value problem

$$y' = (2x + 3)(y^2 - 4), y(0) = -1.$$

In this example: $f(x) = 2x + 3$ and $g(y) = y^2 - 4$ then, setting $g(y) = 0$ we get $y = \pm 2$ as constant solutions.

Separate the variables: $\frac{dy}{y^2-4} = (2x+3)dx$.

Next integrate both sides: $\int \frac{1}{y^2-4} dy = \int (2x+3)dx$.

Integrating both sides and replacing $4C$ with C_1 gives: $\ln|\frac{y-2}{y+2}| = 4x^2 + 12x + C_1$.

Exponentiating both sides of the equation and define $C_2 = e^{C_1}$ yields:

$$|\frac{y-2}{y+2}| = C_2 e^{4x^2+12x}.$$

Finally,

$$y = \frac{2 + 2C_2 e^{4x^2+12x}}{1 - C_2 e^{4x^2+12x}}.$$

To determine the value of C_2 , substitute $x = 0$ and $y = -1$ into the general solution:

$$\frac{-1-2}{-1+2} = C_2 e^{4(0)^2+12(0)} \Rightarrow C_2 = -3.$$

Therefore the solution to the initial-value problem is

$$y = \frac{2 - 6e^{4x^2+12x}}{1 + 3e^{4x^2+12x}}.$$