2. *Homogeneous Equations*

Definition. A first order differential equation $\frac{dy}{dx} = f(x,y)$ is called *homogeneous* if the right side satisfies the condition:

$$f(tx,ty) = f(x,y)$$

for all t. In other words, the right side is a homogeneous function (with respect to the variables x and y) of the zero order:

$$f(tx,ty) = t^0 f(x,y) = f(x,y).$$

Also, a homogeneous differential equation can be also written in the form:

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

or alternatively, in the differential form:

$$P(x,y)dx + Q(x,y)dy = 0,$$

where $P(x,y)$ and $Q(x,y)$ are homogeneous functions of the same degree.

Definition. A function $P_n(x,y)$ is called a homogeneous function of the degree n if the following relationship is valid for all $t > 0$:

$$P(tx,ty) = t^n P(x,y).$$

We can solve it using separation of variables but first we create a new variable

$$u = \frac{y}{x} \text{ which is also } y = ux\text{, then, } y' = u'x + u.$$

Using $y = ux$ and $\frac{dy}{dx} = \frac{du}{dx}x + u$ we can solve the homogeneous differential equation.

Example 1. Find solution of $\frac{dy}{dx} = \frac{x^2+y^2}{xy}$.

Start with the right-hand part: $\frac{x^2+y^2}{xy} = \frac{x}{y} + \frac{y}{x} = \frac{x}{y} + \left(\frac{x}{y}\right)^{-1}$. One can see we have a function of $\frac{y}{x}$.

Substitute $y = ux$ and $\frac{dy}{dx} = \frac{du}{dx}x + u$ into the equation: $\frac{du}{dx}x + u = u + \frac{1}{u}$

Now use separation of variables for $\frac{du}{dx}x = \frac{1}{u}$, we have integrals on both sides as $\int udu = \int \frac{dx}{x}$

So, integrating both sides gives: $\frac{u^2}{2} = \ln|x| + \ln C \Rightarrow u = \pm\sqrt{2\ln Cx}$

Then, $\frac{y}{x} = \pm\sqrt{2\ln Cx} \Rightarrow y = \pm x\sqrt{2\ln Cx}$

Example 2. $xy' = y\ln\frac{y}{x}$.

Rewrite the equation in the form: $y' = \frac{y}{x}\ln\frac{y}{x} = f(\frac{y}{x})$.

Make the substitution $y = ux$. Hence, $y' = (ux)' = u'x + u$.

Substituting this expression into the equation gives: $x(u'x + u) = ux\ln\frac{ux}{x}$.

Divide by $x \neq 0$ to get:

$$u'x + u = u\ln u, \Rightarrow \frac{du}{dx}x = u\ln u - u, \Rightarrow \frac{du}{dx}x = u(\ln u - 1).$$

We obtain the separable equation:

$$\frac{du}{u(\ln u - 1)} = \frac{dx}{x}.$$

The next step is to integrate the left and the right side of the equation:

$$\int \frac{du}{u(\ln u - 1)} = \int \frac{dx}{x}, \Rightarrow \int \frac{d(\ln u)}{\ln u - 1} = \int \frac{dx}{x}, \Rightarrow \int \frac{d(\ln u - 1)}{\ln u - 1} = \int \frac{dx}{x}.$$

Hence,

$$\ln|\ln u - 1| = \ln|x| + C.$$

Here the constant C can be written as $\ln C_1$. Then,

$$\ln|\ln u - 1| = \ln|x| + \ln C_1, \Rightarrow \ln|\ln u - 1| = \ln|C_1 x|, \Rightarrow \ln u - 1 = \pm C_1 x, \Rightarrow \ln u = 1 \pm C_1 x \text{ or } u = e^{1\pm C_1 x}.$$

Thus, we have got two solutions:

$$u = e^{1+C_1 x} \text{ and } u = e^{1-C_1 x}.$$

All the solutions can be represented by one formula: $y = xe^{1+Cx}$.

A differential equation of kind

$$(a_1 x + b_1 y + c_1)dx + (a_2 x + b_2 y + c_2)dy = 0$$

is converted into a separable equation by moving the origin of the coordinate system to the point of intersection of the given straight lines. If these straight lines are parallel, the differential equation is transformed into separable equation by using the change of variable:

$$u = ax + by.$$

Example 3. Find the general solution of the differential equation

$$(x + y + 1)dx + (4x + 4y + 10)dy = 0$$

We make the following substitution:

$$x + y = u, \Rightarrow y = u - x, dy = du - dx.$$

Substituting this in the equation gives

$$(u + 1)dx + (4u + 10)(du - dx) = 0.$$

Hence,

$$udx + dx + 4udu + 10du - 4udx - 10dx = 0, \Rightarrow$$

$$-3udx - 9dx + 4udu + 10du = 0,$$

$$-3(u + 3)dx + 2(2u + 5)du = 0,$$

$$\frac{3dx}{2} = \frac{2u + 5}{u + 3}du.$$

Integrate the last equation:

$$\frac{3}{2}\int dx = \int \frac{2u+5}{u+3}du + C, \Rightarrow \frac{3}{2}\int dx = \int \frac{2u+6-1}{u+3}du + C, \Rightarrow$$

$$\frac{3}{2}\int dx = \int (2 - \frac{1}{u+3})du + C, \Rightarrow \frac{3}{2}x = 2u - \ln|u+3| + C.$$

As, $u = x + y$, the final answer in the implicit form is written in the following way:

$$\frac{3}{2}x = 2(x+y) - \ln|x+y+3| + C \text{ or}$$

$$\frac{x}{2} + 2y - \ln|x+y+3| + C = 0.$$

3. *Linear Differential Equations of First Order*

Definition. A differential equation of type

$$y' + a(x)y = f(x),$$

where $a(x)$ and $f(x)$ are continuous functions of x, is called a *linear nonhomogeneous differential equation* of the first order.

We consider two methods of solving linear differential equations of first order:

- ✓ Using an integrating factor;
- ✓ Method of variation of a constant

Using an Integrating Factor

If a linear differential equation is written in the standard form:

$$y' + a(x)y = f(x)$$

the integrating factor is defined by the formula

$$u(x) = \exp(\int a(x)dx).$$

Multiplying the left side of the equation by the integrating factor $u(x)$ converts the left side into the derivative of the product $y(x) \cdot u(x)$, i.e.

$$u(x) \cdot (y' + a(x)y) = (y(x) \cdot u(x))'$$

Then, the general solution of the differential equation is expressed as follows:

$$y = \frac{\int u(x)f(x)dx + C}{u(x)},$$

where C is an arbitrary constant.

Method of Variation of a Constant

This method is similar to the previous approach. First it's necessary to find the general solution of the homogeneous equation:

$$y' + a(x)y = 0.$$

The general solution of the homogeneous equation contains a constant of integration C. We replace the constant C with a certain (still unknown) function $C(x)$. By substituting this solution into the nonhomogeneous differential equation, we can determine the function $C(x)$.

The described algorithm is called the method of variation of a constant. Of course, both methods lead to the same solution.

Initial Value Problem

If besides the differential equation, there is also an initial condition in the form of $y(x_0) = y_0$, such a problem is called the initial value problem (IVP) or Cauchy problem.

A particular solution for an IVP does not contain the constant C, which is defined by substitution of the general solution into the initial condition $y(x_0) = y_0$

Example 1. Solve the equation $y' - y - xe^x$

We rewrite this equation in standard form: $y' - y = xe^x$.

We will solve this equation using the integrating factor:

$$u(x) = e^{\int (-1)dx} = e^{-\int dx} = e^{-x}.$$

Then the general solution of the linear equation is given by

$$y(x) = \frac{\int u(x)f(x)dx + C}{u(x)} = \frac{\int \cancel{e^{-x}}x\cancel{e^{x}}dx + C}{e^{-x}} = \frac{\int xdx + C}{e^{-x}} = e^x(\frac{x^2}{2} + C).$$

Example 2. Solve the equation $xy' = y + 2x^3$.

We rewrite this equation in standard form: $y' - \frac{1}{x}y = 2x^2$

We will solve this problem by using the method of variation of a constant. First we find the general solution of the homogeneous equation:

$y' - \frac{1}{x}y = 0 \Rightarrow xy' = y,$

which can be solved by separating the variables:

$$x\frac{dy}{dx} = y, \Rightarrow \frac{dy}{y} = \frac{dx}{x}, \Rightarrow \int \frac{dy}{y} = \int \frac{dx}{x}, \Rightarrow \ln|y| = \ln|x| + \ln C, \Rightarrow y = Cx.$$

where C is a positive real number.

Now we replace C with a certain (still unknown) function $C(x)$ and will find a solution of the original nonhomogeneous equation in the form:

$$y = C(x)x.$$

Then the derivative is given by

$$y' = [C(x)x]' = C'(x)x + C(x).$$

Substituting this into the equation gives:

$$x[C'(x)x + C(x)] = C(x)x + 2x^3, \Rightarrow C'(x)x^2 + \cancel{C(x)x} = \cancel{C(x)x} + 2x^3, \Rightarrow$$

$$C'(x) = 2x.$$

Upon integration, we find the function $C(x)$:

$$C(x) = \int 2x dx = x^2 + C_1,$$

where C_1 is an arbitrary real number.

Thus, the general solution of the given equation is written in the form

$$y = C(x)x = (x^2 + C_1)x = x^3 + C_1 x.$$

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