$$y' = [C(x)x]' = C'(x)x + C(x).$$

Substituting this into the equation gives:

$$x[C'(x)x + C(x)] = C(x)x + 2x^3, \Rightarrow C'(x)x^2 + \cancel{C(x)x} = \cancel{C(x)x} + 2x^3, \Rightarrow$$

$$C'(x) = 2x.$$

Upon integration, we find the function $C(x)$:

$C(x) = \int 2xdx = x^2 + C_1$, where $C_1$ is an arbitrary real number.

Thus, the general solution of the given equation is written in the form

$$y = C(x)x = (x^2 + C_1)x = x^3 + C_1 x.$$

## 4. *Bernoulli Equation*

*Bernoulli equation* is one of the well-known nonlinear differential equations of the first order. It is written as

$$y' + a(x)y = b(x)y^m, \qquad (1)$$

where $a(x)$ and $b(x)$ are continuous functions.

*Note*. If $m = 0$ the equation (1) becomes a linear differential equation. In case $m = 1$, the equation (1) becomes separable.

In general case, when $m \neq 0, 1$ Bernoulli equation can be converted to a linear differential equation using the change of variable

$$z = y^{1-m}.$$

The new differential equation for the function $z(x)$ has the form:

$$z' + (1 - m)a(x)z = (1 - m)b(x)$$

and can be solved by the methods described on the page Linear Differential Equation of First Order.

*Example* 1. Find the general solution of the equation $y' - y = y^2 e^x$.

We set $m = 2$ for the given Bernoulli equation, so we use the substitution

$$z = y^{1-m} = y^{1-2} = \frac{1}{y}.$$

Differentiating both sides of the equation (we consider $y$ in the right side as a composite function of $x$), we obtain:

$$z' = \left(\frac{1}{y}\right)' = -\frac{1}{y^2}y'.$$

Divide both sides of the original differential equation by $y^2$

$$y' - y = y^2 e^x, \Rightarrow \frac{y'}{y^2} - \frac{1}{y} = e^x.$$

Substituting $z$ and $z'$ we find

$$-z - z = e^x, \Rightarrow z' + z = -e^x.$$

We get the linear equation for the function $z(x)$. To solve it, we use the integrating factor:

$$u(x) = e^{\int 1dx} = e^x.$$

Then the general solution of the linear equation is given by

$$z(x) = \frac{\int u(x)f(x)dx + C}{u(x)} = \frac{\int e^x(-e^x)dx + C}{e^x} = \frac{-\frac{e^{2x}}{2} + C}{e^x} = -\frac{e^x}{2} + Ce^{-x}$$
$$= \frac{2Ce^{-x} - e^x}{2}.$$

Since $C$ is an arbitrary constant, we can replace $2C$ with a constant $C_1$. Returning to the function $y(x)$, we obtain the implicit expression:

$$y = \frac{1}{z} = \frac{2}{C_1 e^{-x} - e^x}.$$

Note that we have lost the solution $y = 0$ when dividing the equation by $y^2$. Thus, the final answer is given by

$$y = \frac{2}{C_1 e^{-x} - e^x}, y = 0.$$

*Example* 2. Find the solution of the differential equation $4xyy' = y^2 + x^2$, satisfying the initial condition $y(1) = 1$.

First we should check whether this differential equation is a Bernoulli equation:

$$4xyy' = y^2 + x^2, \Rightarrow \frac{4xyy'}{4xy} - \frac{y^2}{4xy} = \frac{x^2}{4xy}, \Rightarrow y' - \frac{y}{4x} = \frac{x}{4y}.$$

As it can be seen, we have a Bernoulli equation with the parameter $m = -1$. Hence, we can make the substitution

$$z = y^{1-m} = y^2$$

The derivative of the function is $z' = 2yy'$.

Next, we multiply both sides of the differential equation by $2y$

$$2yy' - \frac{2y^2}{4x} = \frac{2xy}{4y}, \Rightarrow 2yy' - \frac{y^2}{2x} = \frac{x}{2}.$$

By replacing $y$ with $z$, we can convert the Bernoulli equation into the linear differential equation:

$$z' - \frac{z}{2x} = \frac{x}{2}.$$

Calculate the integrating factor:

$$u(x) = e^{\int (-\frac{1}{2x})dx} = e^{-\frac{1}{2}\int \frac{dx}{x}} = e^{-\frac{1}{2}\ln|x|} = e^{\ln\frac{1}{\sqrt{|x|}}} = \frac{1}{\sqrt{|x|}}.$$

Let's choose the function $u(x) = \frac{1}{\sqrt{x}}$ and make sure that the left side of the equation becomes the derivative of the product $z(x)u(x)$ after multiplying by $u(x)$

$$(z' - \frac{z}{2x})u(x) = z' \cdot \frac{1}{\sqrt{x}} - \frac{z}{2x} \cdot \frac{1}{\sqrt{x}} = z' \cdot \frac{1}{\sqrt{x}} - z \cdot \frac{1}{2x^{\frac{3}{2}}} = z' \cdot \frac{1}{\sqrt{x}} - z \cdot \frac{x^{-\frac{3}{2}}}{2}$$
$$= z' \cdot \frac{1}{\sqrt{x}} + z \cdot (x^{-\frac{1}{2}})' = z' \cdot \frac{1}{\sqrt{x}} + z \cdot (\frac{1}{\sqrt{x}})' = (z \cdot \frac{1}{\sqrt{x}})'$$

Find the general solution of the linear equation:

$$z = \frac{\int u(x)f(x)dx + C}{u(x)} = \frac{\int \frac{1}{\sqrt{x}} \cdot \frac{x}{2}dx + C}{\frac{1}{\sqrt{x}}} = \frac{\frac{1}{2}\int \sqrt{x}dx + C}{\frac{1}{\sqrt{x}}} = \sqrt{x}[\frac{1}{2} \cdot \frac{2x^{\frac{3}{2}}}{3} + C]$$
$$= \frac{x^2}{3} + C\sqrt{x}.$$

Taking into account that $z = y^2$, we obtain the following solution:

$$y = \pm\sqrt{\frac{x^2}{3} + C\sqrt{x}}.$$

Now we determine the value of the constant $C$ that matches the initial condition $y(1) = 1$. We see that only solution with the positive sign satisfies this condition. Hence,

$$y = \sqrt{\frac{1^2}{3} + C\sqrt{1}} = \sqrt{\frac{1}{3} + C} = 1.$$

This gives: $C = \frac{2}{3}$.

So the solution of the IVP is given by the function

$$y = \sqrt{\frac{x^2}{3} + \frac{2\sqrt{x}}{3}}.$$

*5. General Riccati Equations*

The *Riccati equation* is one of the most interesting nonlinear differential equations of first order. It's written in the form:

$$y' = a(x)y + b(x)y^2 + c(x),$$

where $a(x), b(x),\ c(x)$ are continuous functions of $x$.

The differential equation given above is called the *general Riccati equation*. It can be solved with help of the following theorem:

*Theorem*. If a particular solution $y_1$ of a Riccati equation is known, the general solution of the equation is given by

$$y = y_1 + u.$$

Indeed, substituting the solution $y = y_1 + u$ into Riccati equation, we have

$$(y_1 + u)' = a(x)(y_1 + u) + b(x)(y_1 + u)^2 + c(x),$$

$$\underline{y_1{}'} + u' = \underline{a(x)y_1} + a(x)u + \underline{b(x)y_1^2} + 2b(x)y_1u + b(x)u^2 + \underline{c(x)}.$$

The underlined terms in the left and in the right side can be canceled because $y_1$ is a particular solution satisfying the equation. As a result we obtain the differential equation for the function $u(x)$:

$$u' = b(x)u^2 + [2b(x)y_1 + a(x)]u,$$

which is a *Bernoulli equation*.

Substitution of $z = \frac{1}{u}$ converts the given Bernoulli equation into a linear differential equation that allows integration.

So, we can construct the general solution if a particular solution is known. Unfortunately, there is no strict algorithm to find the particular solution, which depends on the types of the functions $a(x), b(x),\ c(x)$.

*Special Case* 1: Coefficients $a$, $b$, $c$ are constants.

If the coefficients in the Riccati equation are constants, this equation can be reduced to a separable differential equation. The solution is described by the integral of a rational function with a quadratic function in the denominator:

$$y' = ay + by^2 + c, \Rightarrow \frac{dy}{dx} = ay + by^2 + c, \Rightarrow \int \frac{dy}{ay + by^2 + c} = \int dx.$$

This integral can be easily calculated at any values of $a, b, c$.

*Special Case* 2: Equation of type $y' = by^2 + cx^n$

That is the function $a(x)$ at the linear term is zero, the coefficient $b$ at $y^2$ is a constant, and $c(x)$ is a power function:

$$a(x) \equiv 0, b(x) = b, c(x) = cx^n.$$

First of all, if $n = 0$, we get the Case 1 where the variables are separated and the differential equation can be integrated.

If $n = -2$, the Riccati equation is converted into a homogeneous equation with help of the substitution $y = \frac{1}{z}$ and then also can be integrated.

This differential equation can be also solved at

$$n = \frac{4k}{1-2k}, \text{where} \quad k = \pm 1, \pm 2, \pm 3, \ldots$$

Here the general solution is expressed through *cylinder functions*.

At all other values of the power n, the solution of the Riccati equation can be expressed through integrals of elementary functions. This fact was discovered by the French mathematician Joseph Liouville

*Example* 1. Find the solution of the differential equation $y' = y + y^2 + 1$.

The given equation is a simple Riccati equation with *constant coefficients*. Here the variables $x, y$ can be easily separated, so the general solution of the equation is given by

$$\frac{dy}{dx} = y + y^2 + 1, \Rightarrow \frac{dy}{y + y^2 + 1} = dx, \Rightarrow \int \frac{dy}{y + y^2 + 1} = \int dx,$$

$$\Rightarrow \int \frac{dy}{y^2 + y + \frac{1}{4} + \frac{3}{4}} = \int dx, \Rightarrow \int \frac{dy}{(y + \frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} = \int dx,$$

$$\Rightarrow \frac{1}{\frac{\sqrt{3}}{2}} \arctan \frac{y + \frac{1}{2}}{\frac{\sqrt{3}}{2}} = x + C, \Rightarrow \frac{2}{\sqrt{3}} \arctan \frac{2y + 1}{\sqrt{3}} = x + C.$$

*Example* 2. Find the solution of the differential equation $y' + y^2 = \frac{2}{x^2}$.

We will seek for a particular solution in the form:

$$y = \frac{c}{x}, \Rightarrow y' = -\frac{c}{x^2}.$$

Substituting this into the equation, we obtain:

$$-\frac{c}{x^2}+\left(\frac{c}{x}\right)^2=\frac{2}{x^2} \quad \text{or} \quad -\frac{c}{x^2}+\frac{c^2}{x^2}=\frac{2}{x^2}.$$

We get a quadratic equation for $c$:

$$c^2-c-2=0, \Rightarrow D=1-4\cdot(-2)=9, \Rightarrow c_{1,2}=\frac{1\pm 3}{2}=-1,2.$$

We can take any value of $c$. For example, let $c=2$. Now, when the particular solution is known, we make the replacement:

$$y=z+\frac{2}{x}, \Rightarrow y'=z'-\frac{2}{x^2}.$$

Now substitute this into the original Riccati equation:

$$z'-\frac{2}{x^2}+(z+\frac{2}{x})^2=\frac{2}{x^2}, \Rightarrow z'-\cancel{\frac{2}{x^2}}+z^2+\frac{4}{x}z+\cancel{\frac{4}{x^2}}=\cancel{\frac{2}{x^2}}, \Rightarrow z'+\frac{4}{x}z=-z^2.$$

As it can be seen, we have a Bernoulli equation with the parameter $m=2$. Make one more substitution:

$$v=z^{1-m}=\frac{1}{z}, \Rightarrow v'=-\frac{z'}{z^2}.$$

Divide the Bernoulli equation by $z^2$ (assuming that $z\neq 0$) and rewrite it in terms of $v$:

$$\frac{z'}{z^2}+\frac{4z}{xz^2}=-1, \Rightarrow -\frac{z'}{z^2}-\frac{4}{xz}=1, \Rightarrow v'-\frac{4}{x}v=1.$$

The last equation is linear and can be easily solved using the integrating factor:

$$u=e^{\int(-\frac{4}{x})dx}=e^{-4\int\frac{dx}{x}}=e^{-4\ln|x|}=e^{\ln\frac{1}{|x|^4}}=\frac{1}{|x|^4}=\frac{1}{x^4}.$$

The general solution of the linear equation is given by

$$v=\frac{\int u(x)f(x)dx+C}{u(x)}=\frac{\int\frac{1}{x^4}\cdot 1dx+C}{\frac{1}{x^4}}=\frac{\int x^{-4}dx+C}{\frac{1}{x^4}}=(-\frac{1}{3}x^{-3}+C)x^4$$
$$=-\frac{x}{3}+Cx^4.$$

From now on we will subsequently return back to the previous variables. Since $z=\frac{1}{v}$, the general solution for $z$ is written as follows:

$$\frac{1}{z}=-\frac{x}{3}+Cx^4, \Rightarrow z=\frac{1}{-\frac{x}{3}+Cx^4}=-\frac{3}{x+3Cx^4}=-\frac{3}{x(1+3Cx^3)}.$$

Hence,

$$y = z + \frac{2}{x} = -\frac{3}{x(1+3Cx^3)} + \frac{2}{x} = \frac{-3+2(1+3Cx^3)}{x(1+3Cx^3)} = \frac{-3+2+6Cx^3}{x(1+3Cx^3)}$$
$$= \frac{6Cx^3-1}{x(1+3Cx^3)}.$$

We can take: $3C = C_1$ and write the answer in the form:

$$y = \frac{2C_1x^3-1}{x(1+C_1x^3)},$$

where $C_1$ is an arbitrary real number.

*Example* 3. Find the solution of the differential equation $y' + 6y^2 = \frac{1}{x^2}$.

As it can be seen, this is a special Riccati equation of type $y' = by^2 + cx^n$ with $n = -2$.

By making the substitution $y = \frac{1}{z}$ we can convert the equation to a homogeneous one and then integrate.

Let $z = \frac{1}{y}$, $z' = -\frac{y'}{y^2}$. Then

$$y' + 6y^2 = \frac{1}{x^2}, \Rightarrow y' = -6y^2 + \frac{1}{x^2}, \Rightarrow -\frac{y'}{y^2} = 6 - \frac{1}{y^2x^2}, \Rightarrow z' = 6 - \frac{z^2}{x^2}, \Rightarrow$$
$$z' = 6 - \left(\frac{z}{x}\right)^2.$$

To solve the homogeneous equation we make one more substitution: $z = tx, z' = t'x + t$.

Hence,

$$t'x + t = 6 - t^2, \Rightarrow x\frac{dt}{dx} = 6 - t - t^2, \Rightarrow \frac{dt}{t^2+t-6} = -\frac{dx}{x}, \Rightarrow$$
$$\int \frac{dt}{t^2+t-6} = -\int \frac{dx}{x}.$$

The trinomial in the denominator of the left side can be factored as follows: $t^2 + t - 6 = (t+3)(t-2)$, so we may use partial decomposition to simplify the integrand:

$$\frac{1}{t^2+t-6} = \frac{1}{(t+3)(t-2)} = \frac{A}{t+3} + \frac{B}{t-2}, \Rightarrow A(t-2) + B(t+3) = 1,$$
$$\Rightarrow At - 2A + Bt + 3B = 1, \Rightarrow (A+B)t + 3B - 2A = 1$$

$$\Rightarrow \begin{cases} A + B = 0 \\ 3B - 2A = 1 \end{cases}, \Rightarrow \begin{cases} A = -B \\ 5B = 1 \end{cases}, \Rightarrow \begin{cases} A = -\dfrac{1}{5} \\ B = \dfrac{1}{5} \end{cases}.$$

As a result, we have

$$\int \frac{dt}{(t+3)(t-2)} = -\int \frac{dx}{x}, \Rightarrow -\frac{1}{5}\int \frac{dt}{t+3} + \frac{1}{5}\int \frac{dt}{t-2} = -\int \frac{dx}{x},$$

$$\Rightarrow \frac{1}{5}\ln|t+3| - \frac{1}{5}\ln|t-2| = \ln|x| + \ln C_1 (C_1 > 0), \Rightarrow$$

$$\frac{1}{5}\ln\left|\frac{t+3}{t-2}\right| = \ln(C_1|x|), \Rightarrow \ln\left|\frac{t+3}{t-2}\right| = \ln(C_1^5|x|^5), \Rightarrow \left|\frac{t+3}{t-2}\right| = C_1^5|x|^5, \Rightarrow \frac{t+3}{t-2} = \pm C_1^5 x^5.$$

Rename the constant: $C = \pm C_1^5$, so the solution for the function $t(x)$ will have the form:

$$\frac{t+3}{t-2} = Cx^5.$$

Remember that $t = \frac{z}{x}$. Therefore,

$$\frac{\frac{z}{x} + 3}{\frac{z}{x} - 2} = Cx^5, \Rightarrow \frac{z + 3x}{z - 2x} = Cx^5.$$

Returning to the variable $y$, which is related to $z$ by the relationship $z = \frac{1}{y}$, we get

$$\frac{\frac{1}{y} + 3x}{\frac{1}{y} - 2x} = Cx^5, \Rightarrow \frac{1 + 3xy}{1 - 2xy} = Cx^5.$$

The last expression is the general solution of the Riccati equation in the implicit form. Here the constant $C$ is any real number. Indeed, substituting $C = 0$, we see that this value also satisfies the differential equation:

$$C = 0, \Rightarrow 1 + 3xy = 0, \Rightarrow y = -\frac{1}{3x}, \Rightarrow y' = \frac{1}{3x^2}.$$

Hence,

$$\frac{1}{3x^2} + 6\left(-\frac{1}{3x}\right)^2 = \frac{1}{x^2}, \Rightarrow \frac{1}{3x^2} + 6 \cdot \frac{1}{9x^2} = \frac{1}{x^2}, \Rightarrow \frac{1}{3x^2} + \frac{2}{3x^2} = \frac{1}{x^2}, \Rightarrow \frac{1}{x^2} \equiv \frac{1}{x^2}.$$

*6. Exact Differential Equations*

*Definition*. A differential equation of type

$$P(x,y)dx + Q(x,y)dy = 0$$

is called an exact differential equation if there exists a function of two variables $u(x,y)$ with continuous partial derivatives such that

$$du(x,y) = P(x,y)dx + Q(x,y)dy.$$

The general solution of an exact equation is given by

$$u(x,y) = C,$$

where $C$ is an arbitrary constant.

*Test for Exactness*

Let functions $P(x,y)$ and $Q(x,y)$ have continuous partial derivatives in a certain domain $D$. The differential equation $P(x,y)dx + Q(x,y)dy = 0$ is an exact equation if and only if

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}.$$

*Algorithm for Solving an Exact Differential Equation*

1. First it's necessary to make sure that the differential equation is exact using the test for exactness;

2. Then we write the system of two differential equations that define the function $u(x,y)$:

$$\begin{cases} \frac{\partial u}{\partial x} = P(x,y) \\ \frac{\partial u}{\partial y} = Q(x,y) \end{cases}.$$

3. Integrate the first equation over the variable $x$. Instead of the constant $C$, we write an unknown function of $y$:

$$u(x,y) = \int P(x,y)dx + \varphi(y).$$

4. Differentiating with respect to $y$, we substitute the function $u(x,y)$ into the second equation:

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left[\int P(x,y)dx + \varphi(y)\right] = Q(x,y).$$

From here we get expression for the derivative of the unknown function $\varphi(y)$:

$$\varphi'(y) = Q(x,y) - \frac{\partial}{\partial y}\left(\int P(x,y)dx\right).$$

5. By integrating the last expression, we find the function $\varphi(y)$ and, hence, the function $u(x,y)$:

$$u(x,y) = \int P(x,y)dx + \varphi(y).$$

6. The general solution of the exact differential equation is given by

$$u(x,y) = C,$$

where $C$ is an arbitrary constant.

*Note*: In Step 3, we can integrate the second equation over the variable $y$ instead of integrating the first equation over $x$. After integration we need to find the unknown function $\psi(x)$.

*Example* 1. Find the solution of the differential equation $2xydx + (x^2 + 3y^2)dy = 0$

1. The given equation is exact because the partial derivatives are the same:

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(x^2 + 3y^2) = 2x, \qquad \frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(2xy) = 2x.$$

2. We have the following system of differential equations to find the function $u(x,y)$:

$$\begin{cases} \frac{\partial u}{\partial x} = 2xy \\ \frac{\partial u}{\partial y} = x^2 + 3y^2 \end{cases}.$$

3. By integrating the first equation with respect to $x$, we obtain

$$u(x,y) = \int 2xydx = x^2y + \varphi(y).$$

4. Substituting this expression for $u(x,y)$ into the second equation gives us:

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}[x^2y + \varphi(y)] = x^2 + 3y^2, \Rightarrow x^2 + \varphi'(y) = x^2 + 3y^2, \Rightarrow \varphi'(y) = 3y^2.$$

5. By integrating the last equation, we find the unknown function $\varphi(y)$:

$$\varphi(y) = \int 3y^2dy = y^3,$$

so that the general solution of the exact differential equation is given by

$$x^2y + y^3 = C,$$

where $C$ is an arbitrary constant.

*Example* 2. Solve the differential equation $\frac{1}{y^2} - \frac{2}{x} = \frac{2xy'}{y^3}$ with the initial condition $y(1) = 1$.

Check the equation for exactness by converting it into standard form:

$$\frac{1}{y^2} - \frac{2}{x} = \frac{2x}{y^3}\frac{dy}{dx}, \Rightarrow \left(\frac{1}{y^2} - \frac{2}{x}\right)dx = \frac{2x}{y^3}dy, \Rightarrow \left(\frac{1}{y^2} - \frac{2}{x}\right)dx - \frac{2x}{y^3}dy = 0.$$

The partial derivatives are

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\left(-\frac{2x}{y^3}\right) = -\frac{2}{y^3}, \frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(\frac{1}{y^2} - \frac{2}{x}\right) = -\frac{2}{y^3}.$$

Hence, the given equation is exact. Therefore, we can write the following system of equations to determine the function $u(x, y)$

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{1}{y^2} - \frac{2}{x} \\ \frac{\partial u}{\partial y} = -\frac{2x}{y^3} \end{cases}.$$

In the given case, it is more convenient to integrate the second equation with respect to the variable $y$:

$$u(x, y) = \int \left(-\frac{2x}{y^3}\right)dy = \frac{x}{y^2} + \psi(x).$$

Now we differentiate this expression with respect to the variable $x$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left[\frac{x}{y^2} + \psi(x)\right] = \frac{1}{y^2} - \frac{2}{x}, \Rightarrow \cancel{\frac{1}{y^2}} + \psi'(x) = \cancel{\frac{1}{y^2}} - \frac{2}{x}, \Rightarrow \psi'(x) = -\frac{2}{x}, \Rightarrow$$

$$\psi(x) = -2\ln|x| = \ln\frac{1}{x^2}.$$

Thus, the general solution of the differential equation in implicit form is given by the expression:

$$\frac{x}{y^2} + \ln\frac{1}{x^2} = C.$$

The particular solution can be found using the initial condition $y(1) = 1$. By substituting the initial values, we find the constant

$$\frac{1}{1^2} + \ln\frac{1}{1^2} = C, \Rightarrow 1 + 0 = C, \Rightarrow C = 1.$$

Hence, the solution of the given initial value problem is

$$\frac{1}{y^2} + \ln\frac{1}{x^2} = 1.$$

*7. Using an Integrating Factor*

Consider a differential equation of type

$$P(x,y)dx + Q(x,y)dy = 0,$$

where $P(x,y)$ and $Q(x,y)$ are functions of two variables $x$ and $y$ continuous in a certain region $D$. If

$$\frac{\partial Q}{\partial x} \neq \frac{\partial P}{\partial y},$$

the equation is *not exact*. However, we can try to find so-called *integrating factor*, which is a function $\mu(x,y)$ such that the equation becomes exact after multiplication by this factor. If so, then the relationship

$$\frac{\partial(\mu Q(x,y))}{\partial x} = \frac{\partial(\mu P(x,y))}{\partial y}$$

is valid. This condition can be written in the form:

$$Q\frac{\partial\mu}{\partial x} + \mu\frac{\partial Q}{\partial x} = P\frac{\partial\mu}{\partial y} + \mu\frac{\partial P}{\partial y}, \Rightarrow Q\frac{\partial\mu}{\partial x} - P\frac{\partial\mu}{\partial y} = \mu(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}).$$

The last expression is the *partial differential equation of first order* that defines the integrating factor $\mu(x,y)$.

Unfortunately, there is no general method to find the integrating factor. However, one can mention some particular cases for which the partial differential equation can be solved and as a result we can construct the integrating factor.

*Integrating factor depends on the variable $x$: $\mu = \mu(x)$*

In this case we have $\frac{\partial\mu}{\partial y} = 0$, so the equation for $\mu(x,y)$ can be written in the form:

$$\frac{1}{\mu}\frac{d\mu}{dx} = \frac{1}{Q}(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}).$$

The right side of this equation must be a function of only $x$. We can find the function $\mu(x)$ by integrating the last equation.

*Integrating factor depends on the variable $y$: $\mu = \mu(y)$*

Similarly, if $\frac{\partial\mu}{\partial x} = 0$, we get the following ordinary differential equation for the integrating factor $\mu(x,y)$:

$$\frac{1}{\mu}\frac{d\mu}{dy} = -\frac{1}{P}(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}),$$

where the right side depends only on $y$. The function $\mu(y)$ can be found by integrating the given equation.

*Integrating factor depends on a certain combination of the variables x and* $y$*:* $\mu = \mu(z(x,y))$.

The new function $z(x,y)$ can be, for example, of the following type:

$$z = \frac{x}{y}, z = xy, z = x^2 + y^2, z = x + y,$$

and so on.

Here it is important that the integrating factor $\mu(x,y)$ becomes a function of one variable $z$: $\mu(x,y) = \mu(z)$ and can be found from the differential equation:

$$\frac{1}{\mu}\frac{d\mu}{dz} = \frac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{Q\frac{\partial z}{\partial x} - P\frac{\partial z}{\partial y}}.$$

We assume that the right side of the equation depends only on $z$ and the denominator is not zero.

*Example* 1. Solve the differential equation $(1 + y^2)dx + xydy = 0$.

First we test this differential equation for exactness:

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(xy) = y, \frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(1 + y^2) = 2y.$$

As one can see, this equation is not exact. We try to find an integrating factor to convert the equation into exact. Calculate the function

$$\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = 2y - y = y.$$

One can notice that the expression

$$\frac{1}{Q}(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}) = \frac{1}{xy} \cdot y = \frac{1}{x}$$

depends only on the variable $x$. Hence, the integrating factor will also depend only on $x$: $\mu = \mu(x)$. We can get it from the equation

$$\frac{1}{\mu}\frac{d\mu}{dx} = \frac{1}{x}.$$

Separating variables and integrating, we obtain:

$$\int \frac{d\mu}{\mu} = \int \frac{dx}{x}, \Rightarrow \ln|\mu| = \ln|x|, \Rightarrow \mu = \pm x.$$

We choose $\mu = x$. Multiplying the original differential equation by $\mu = x$, produces the exact equation:

$$(x + xy^2)dx + x^2ydy = 0.$$

Indeed, now we have

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(x^2y) = 2xy, \quad \frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(x + xy^2) = 2xy.$$

Solve the resulting equation. The function $u(x, y)$ can be found from the system of equations:

$$\begin{cases} \frac{\partial u}{\partial x} = x + xy^2 \\ \frac{\partial u}{\partial y} = x^2y \end{cases}.$$

It follows from the first equation that

$$u(x, y) = \int (x + xy^2)dx = \frac{x^2}{2} + \frac{x^2y^2}{2} + \varphi(y).$$

Substitute this in the second equation to determine $\varphi(y)$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}[\frac{x^2}{2} + \frac{x^2y^2}{2} + \varphi(y)] = x^2y, \Rightarrow x^2y + \varphi'(y) = x^2y, \Rightarrow \varphi'(y) = 0.$$

It follows from here that $\varphi(y) = C$, where $C$ is a constant.

Thus, the general solution of the original differential equation is given by

$$\frac{x^2}{2} + \frac{x^2y^2}{2} + C = 0.$$

*Example* 2. Solve the differential equation $(xy^2 - 2y^3)dx + (3 - 2xy^2)dy = 0$

The given equation is not exact, because

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(3 - 2xy^2) = -2y^2 \neq \frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(xy^2 - 2y^3) = 2xy - 6y^2.$$

We try to find the general solution of the equation using an integrating factor. Calculate the difference

$$\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = 2xy - 6y^2 - (-2y^2) = 2xy - 4y^2.$$

Notice that the expression

$$\frac{1}{P}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = \frac{2xy - 4y^2}{xy^2 - 2y^3} = \frac{2\cancel{(xy - 2y^2)}}{y\cancel{(xy - 2y^2)}} = \frac{2}{y}$$

depends only on $y$. Therefore, the integrating factor $\mu$ is also a function only of the variable $y$. We can find it from the equation

$$\frac{1}{\mu}\frac{d\mu}{dy} = -\frac{1}{P}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = -\frac{2}{y}.$$

Integrating, we get:

$$\int \frac{d\mu}{\mu} = -2\int \frac{dy}{y}, \Rightarrow \ln|\mu| = -2\ln|\mu|, \Rightarrow \mu = \pm\frac{1}{y^2}.$$

By choosing $\mu = \frac{1}{y^2}$ as the integrating factor and then multiplying the original differential equation by it, we get the exact equation:

$$(x - 2y)dx + \left(\frac{3}{y^2} - 2x\right)dy = 0.$$

Indeed, we see that

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\left(\frac{3}{y^2} - 2x\right) = -2 = \frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(x - 2y) = -2.$$

Notice that when we multiplied by the integrating factor we lost the solution $y = 0$. This can be proved by direct substitution of the solution $y = 0$ in the original differential equation.

Now we find the function $u(x, y)$ from the system of equations:

$$\begin{cases} \frac{\partial u}{\partial x} = x - 2y \\ \frac{\partial u}{\partial y} = \frac{3}{y^2} - 2x \end{cases}.$$

It follows from the first equation that

$$u(x, y) = \int (x - 2y)dx = \frac{x^2}{2} - 2yx + \varphi(y).$$

Then we get from the second equation:

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left[\frac{x^2}{2} - 2yx + \varphi(y)\right] = \frac{3}{y^2} - 2x, \Rightarrow -2x + \varphi'(y) = \frac{3}{y^2} - 2x, \Rightarrow$$

$$\varphi'(y) = \frac{3}{y^2}, \Rightarrow \varphi(y) = \int \frac{3}{y^2} dy = -\frac{3}{y}.$$

Thus, the original differential equation has the following solutions:

$$\frac{x^2}{2} - 2yx - \frac{3}{y} = C, y = 0,$$

where $C$ is an arbitrary real number.

*Example* 3. Solve the differential equation $ydx + (x^2 + y^2 - x)dy = 0$ using the integrating factor $\mu(x, y) = x^2 + y^2$.

We can make sure that this equation is not exact:

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(x^2 + y^2 - x) = 2x - 1 \neq \frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(y) = 1.$$

The difference of the partial derivatives is

$$\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = 1 - (2x - 1) = 2 - 2x.$$

Using the integrating factor $\mu(x, y) = z = x^2 + y^2$, we find that

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(x^2 + y^2) = 2x, \frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(x^2 + y^2) = 2y.$$

Calculate the following expression:

$$Q\frac{\partial z}{\partial x} - P\frac{\partial z}{\partial y} = (x^2 + y^2 - x) \cdot 2x - y \cdot 2y = 2x^3 + 2xy^2 - 2x^2 - 2y^2$$
$$= 2x(x^2 + y^2) - 2(x^2 + y^2) = (x^2 + y^2)(2x - 2).$$

As a result, we obtain the differential equation for the function $\mu(z)$:

$$\frac{1}{\mu}\frac{d\mu}{dz} = \frac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{Q\frac{\partial z}{\partial x} - P\frac{\partial z}{\partial y}} = \frac{2 - 2x}{(x^2 + y^2)(2x - 2)} = -\frac{\cancel{2x - 2}}{z\cancel{(2x - 2)}} = -\frac{1}{z}.$$

By integrating we get the function $\mu(z)$:

$$\int \frac{d\mu}{\mu} = -\int \frac{dz}{z}, \Rightarrow \ln|\mu| = -\ln|z|, \Rightarrow \mu = \pm\frac{1}{z}.$$

We can choose the integrating factor $\mu = \frac{1}{z} = \frac{1}{x^2+y^2}$. After multiplication by $\frac{1}{x^2+y^2}$ the original differential equation is converted into exact:

$$\frac{y}{x^2+y^2}dx+\frac{x^2+y^2-x}{x^2+y^2}dy=0 \text{or} \frac{y}{x^2+y^2}dx+(1-\frac{x}{x^2+y^2})dy=0.$$

The general solution $u(x,y)=C$ is defined by the following system of equations:

$$\begin{cases}\frac{\partial u}{\partial x}=\frac{y}{x^2+y^2}\\ \frac{\partial u}{\partial y}=1-\frac{x}{x^2+y^2}\end{cases}.$$

Integrating the first equation with respect to $x$ produces:

$$u(x,y)=\int\frac{y}{x^2+y^2}dx=y\int\frac{dx}{x^2+y^2}=y\cdot\frac{1}{y}\arctan\frac{x}{y}+\varphi(y)$$
$$=\arctan\frac{x}{y}+\varphi(y).$$

Substituting this in the second equation, we have:

$$\frac{\partial u}{\partial y}=\frac{\partial}{\partial y}\left[\arctan\frac{x}{y}+\varphi(y)\right]=1-\frac{x}{x^2+y^2},\Rightarrow\frac{1}{1+\left(\frac{x}{y}\right)^2}\cdot\left(-\frac{x}{y^2}\right)+\varphi'(y)$$
$$=1-\frac{x}{x^2+y^2},\Rightarrow-\frac{\cancel{y^2}x}{(x^2+y^2)\cancel{y^2}}+\varphi'(y)=1-\frac{x}{x^2+y^2},\Rightarrow$$
$$\varphi'(y)=1,\Rightarrow\varphi(y)=y.$$

Thus, the general solution of differential equation in implicit form is defined by the formula:

$$\arctan\frac{x}{y}+y=C,$$

where $C$ is a constant.