First Order ODEs

1. *Separable Equations*

A *separable differential equation* is any equation that can be written in the form:

$$y' = f(x)g(y).$$

Example 1. Solve the differential equation $\frac{dy}{dx} = y(y+2)$.

In the given case $f(x) = 1$ and $g(y) = y(y+2)$. We divide the equation by $g(y)$ and move dx to the right side:

$$\frac{dy}{y(y+2)} = dx.$$

One can notice that after dividing we can lose solutions when becomes zero, i.e. $y(y+2) = 0, \Rightarrow y = 0$ and $y = -2$ are the constant solutions of the Equation.

Next, we integrate the expression we got above:

$$\int \frac{dy}{y(y+2)} = \int dx$$

Here, we can calculate the left integral by using the decomposition of the integrand:

$$\frac{1}{y(y+2)} = \frac{1}{2} \cdot \frac{y+2-y}{y(y+2)} = \frac{1}{2}\left(\frac{1}{y} - \frac{1}{y+2}\right)$$

Hence,

$$\frac{1}{2}\int \left(\frac{1}{y} - \frac{1}{y+2}\right) dy = \int dx, \Rightarrow$$

$$\frac{1}{2}\left(\int \frac{dy}{y} - \int \frac{dy}{y+2}\right) = \int dx, \Rightarrow$$

$$\frac{1}{2}(\ln|y| - \ln|y+2|) = x + C, \Rightarrow \frac{1}{2}\ln\left|\frac{y}{y+2}\right| = x + C, \Rightarrow \ln|\frac{y}{y+2}| = 2x + 2C.$$

We can rename the constant: $2C = C_1$. Thus, the final solution of the equation is written in the form

$$\ln|\frac{y}{y+2}| = 2x + C_1, y = 0, y = -2.$$

Here the general solution is expressed in implicit form. In the given case we can transform the expression to obtain the answer as an explicit function $y = f(x, C_1)$, where C_1 is a constant. However, it is possible to do not for all differential equations.

Example 2. Solve the differential equation $(x^2 + 4)y' = 2xy$.

We can rewrite this equation in the following way:

$$(x^2+4)dy = 2xydx.$$

Divide both sides by $(x^2+4)y$ to get

$$\frac{dy}{y} = \frac{2xdx}{(x^2+4)}.$$

Obviously, that $x^2+4 \neq 0$ for all real x. Check if $y=0$ is a solution of the equation. Substituting $y=0$ and $dy=0$ into the differential equation,

$$(x^2+4)\cdot 0 = 2x\cdot 0\cdot dx.$$

$$0=0$$

we see that the function $y=0$ is one of the solutions of the equation.

Now we can integrate both the sides:

$$\int \frac{dy}{y} = \int \frac{2xdx}{(x^2+4)} + C, \Rightarrow \ln|y| = \int \frac{d(x^2)}{x^2+4} + C.$$

Notice that $d(x^2) = d(x^2+4)$. Hence,

$$\ln|y| = \int \frac{d(x^2+4)}{x^2+4} + C, \Rightarrow \ln|y| = \ln(x^2+4) + C.$$

We can represent the constant C as $\ln C_1$, where $C_1 > 0$. Then

$$\ln|y| = \ln(x^2+4) + \ln C_1, \Rightarrow \ln|y| = \ln\big(C_1(x^2+4)\big), \Rightarrow$$

$$|y| = C_1(x^2+4), \Rightarrow y = \pm C_1(x^2+4).$$

Thus, the given differential equation has the following solutions:

$$y = \pm C_1(x^2+4), y=0, \text{where} C_1 > 0.$$

This answer can be simplified. Indeed, if using an arbitrary constant C, which takes values from $-\infty$ to ∞, the solution can be written in the form:

$$y = C(x^2+4).$$

When $C=0$, it becomes $y=0$.

Example 3. Find all solutions of the differential equation $y' = -xe^y$.

We transform this equation in following way:

$$\frac{dy}{dx} = -xe^y, \Rightarrow \frac{dy}{e^y} = -xdx, \Rightarrow e^{-y}dy = -xdx.$$

Obviously, that division by e^y does not cause the loss of solutions as $e^y > 0$. After integrating we have

$$\int e^{-y}dy = \int (-x)dx + C, \Rightarrow -e^{-y} = -\frac{x^2}{2} + C \Rightarrow \text{or} e^{-y} = \frac{x^2}{2} + C.$$

This answer can be expressed in the explicit form:

$$-y = \ln\left(\frac{x^2}{2} + C\right) \text{ or } y = -\ln\left(\frac{x^2}{2} + C\right)$$

We assume in the latter expression that $C > 0$ in order to satisfy the domain of the logarithmic function.

Example 4. Solve the differential equation $y'\cot^2 x + \tan y = 0$.

We write this equation as follows:

$$\frac{dy}{dx}\cot^2 x = -\tan y, \Rightarrow \cot^2 x dy = -\tan y dx.$$

Divide both sides of the equation by $\tan y \cot^2 x$:

$$\frac{\cancel{\cot^2} x dy}{\tan y \cancel{\cot^2} x} = -\frac{\cancel{\tan y} dx}{\cancel{\tan y}\cot^2 x}, \Rightarrow \frac{dy}{\tan y} = -\frac{dx}{\cot^2 x}.$$

Check for possible missed solutions when dividing. There might be the following two roots:

$$\tan y \cot^2 x = 0.$$

1) $\tan y = 0, \Rightarrow y = \frac{\pi}{2} + \pi n, n \in Z, dy = 0$

Substituting this into the initial equations, we see that $y = \frac{\pi}{2} + \pi n, n \in Z$ is a solution.

The second possible solution is given by

2) $\cot^2 x = 0$.

Here we get the answer $x = \pi n, n \in Z, dx = 0$, which does not satisfy the initial differential equation.

Now we can integrate the given equation and find its general solution:

$$\int \frac{dy}{\tan y} = -\int \frac{dx}{\cot^2 x} + C, \Rightarrow \int \frac{dy}{\frac{\sin y}{\cos y}} = -\int \frac{dx}{\frac{\cos^2 x}{\sin^2 x}} + C, \Rightarrow$$

$$\int \frac{\cos y dy}{\sin y} = -\int \frac{\sin^2 x dx}{\cos^2 x} + C, \Rightarrow \int \frac{d(\sin y)}{\sin y} = -\int \frac{1 - \cos^2 x}{\cos^2 x} dx + C, \Rightarrow$$

$$\ln|\sin y| = -\int (\frac{1}{\cos^2 x} - 1) dx + C, \Rightarrow \ln|\sin y| = -(\tan x - x) + C, \Rightarrow$$

$$\ln|\sin y| = -\tan x + x + C.$$

The final answer is given by

$$\ln|\sin y| + \tan x - x = C,$$

$$y = \frac{\pi}{2} + \pi n, n \in Z.$$

Example 5. Solve the equation $y(1+xy)dx = x(1-xy)dy$.

The product xy in each side does not allow separating the variables. Therefore, we make the replacement:

$$xy = t \text{ or } y = \frac{t}{x}.$$

The relationship for differentials is given by

$$dy = \frac{xdt - tdx}{x^2}.$$

Substituting this into the equation, we can write:

$$\frac{t}{x}(1+t)dx = x(1-t)\frac{xdt - tdx}{x^2}.$$

By multiplying both sides by x and then canceling the corresponding fractions in the left and right side, we get

$$t(1+t)dx = (1-t)(xdt - tdx).$$

Take into account that $x = 0$ is a solution of the equation, which can be verified by direct substitution.

Simplify the latter expression:

$$tdx + \cancel{t^2dx} = xdt - tdx - xtdt + \cancel{t^2dx}, \Rightarrow 2tdt = x(1-t)dt.$$

Now the variables x and t are separated:

$$\frac{2dx}{x} = \frac{(1-t)dt}{t} \text{or} 2\frac{dx}{x} = (\frac{1}{t} - 1)dt.$$

After integrating we have

$$2\int \frac{dx}{x} = \int (\frac{1}{t} - 1)dt + C, \Rightarrow 2\ln|x| = \ln|t| - t + C, \Rightarrow \ln x^2 = \ln|t| - t + C.$$

By making the reverse substitution $t = xy$, we find the general solution of the equation:

$$\ln x^2 = \ln|xy| - xy + C, \Rightarrow \ln|\frac{xy}{x^2}| - xy + C = 0, \Rightarrow \ln|\frac{y}{x}| - xy + C = 0.$$

The complete answer is written in the form:

$$\ln|\frac{y}{x}| - xy + C = 0, x = 0.$$

Example 6. Find a particular solution of the differential equation $x(y+2)y' = \ln x + 1$ provided by the condition $y(1) = -1$, other words find a solution of the initial-value problem.

We divide both sides of the equation by x

$$x(y+2)\frac{dy}{dx} = \ln x + 1, \Rightarrow (y+2)dy = \frac{(\ln x + 1)dx}{x}.$$

We suppose that $x \neq 0$, because the domain of the given equation is $x > 0$.

Integrating this equation yields:

$$\int (y+2)dy = \int \frac{(\ln x + 1)dx}{x} + C.$$

The integral in the right side is calculated as follows:

$$\int \frac{(\ln x + 1)dx}{x} = \int (\ln x + 1)d(\ln x) = \int (\ln x + 1)d(\ln x + 1) = \frac{(\ln x + 1)^2}{2}.$$

Hence, the *general solution* in the implicit form is given by

$$\frac{1}{2}y^2 + 2y = \frac{(\ln x + 1)^2}{2} + C, \Rightarrow y^2 + 4y = (\ln x + 1)^2 + C_1,$$

where $C_1 = 2C$ is an integration constant. Next, we find the values of C_1 to satisfy the initial condition $y(1) = -1$:

$$(-1)^2 + 4(-1) = (\ln 1 + 1)^2 + C_1, \Rightarrow C_1 = -4.$$

Thus, the particular solution satisfying the initial condition is written in the following way:

$$y^2 + 4y = (\ln x + 1)^2 - 4.$$

Example 7. Find a particular solution of the equation $(1 + e^x)y' = e^x$ satisfying the initial condition $y(0) = 0$.

We write this equation in the following way:

$$(1 + e^x)dy = e^x dx.$$

Divide both sides by $1 + e^x$:

$$dy = \frac{e^x}{1 + e^x}dx.$$

Since $1 + e^x > 0$, then we did not miss solutions of the original equation. Integrating this equation yields:

$$\int dy = \int \frac{e^x}{1 + e^x}dx + C, \Rightarrow y = \int \frac{d(e^x)}{1 + e^x} + C, \Rightarrow$$

$$y = \int \frac{d(e^x + 1)}{1 + e^x} + C, \Rightarrow y = \ln(e^x + 1) + C.$$

Determine the constant C from the initial condition $y(0) = 0$

$$0 = \ln(e^0 + 1) + C, \Rightarrow 0 = \ln 2 + C, \Rightarrow C = -\ln 2.$$

So the final answer is $\quad y = \ln(e^x + 1) - \ln 2 = \ln\frac{e^x + 1}{2}.$

2. 2. Homogeneous Equations

A *homogeneous differential equation* is any equation $\frac{dy}{dx} = f(x, y)$ if the right side satisfies the condition:

$$f(tx, ty) = f(x, y)$$

for all t.

Example 1. Solve the differential equation $(2x + y)dx - xdy = 0$.

It is easy to see that the polynomials $(2x + y)$ and x at dx and dy are homogeneous functions of the first order. Therefore, the original differential equation is also homogeneous.

Suppose that $y = ux$, where u is a new function depending on x. Then

$$dy = d(ux) = udx + xdu.$$

Substituting this into the differential equation, we obtain

$$(2x + ux)dx - x(udx + xdu) = 0.$$

Hence,

$$2xdx + \cancel{uxdx} - \cancel{xudx} - x^2du = 0.$$

Dividing both sides by x yields:

$$xdu = 2dx \quad \text{or} \quad du = 2\frac{dx}{x}.$$

When dividing by x, we could lose the solution $x = 0$. The direct substitution shows that $x = 0$ is indeed a solution of the given differential equation.

Integrate the latter expression to obtain:

$$\int du = 2\int \frac{dx}{x} \quad \text{or} \quad u = 2\ln|x| + C,$$

where C is a constant of integration.

Returning to the old variable y, we can write:

$$y = ux = x(2\ln|x| + C).$$

Thus, the equation has two solutions:

$$y = x(2\ln|x| + C), x = 0.$$

Example 2. Solve the differential equation $xy' = y\ln\frac{y}{x}$.

We notice that the root $x = 0$ does not belong to the domain of the differential equation.

Rewrite the equation in the form:

$$y' = \frac{y}{x}\ln\frac{y}{x} = f\left(\frac{y}{x}\right).$$

As you can see, this equation is homogeneous.

Make the substitution $y = ux$, where u is a new function depending on x. Hence,

$$y' = (ux)' = u'x + u.$$

Substituting this into the differential equation gives:

$$x(u'x + u) = ux\ln\frac{ux}{x}.$$

Dividing by $x \neq 0$ to get:

$$u'x + u = u\ln u, \Rightarrow \frac{du}{dx}x = u\ln u - u, \Rightarrow \frac{du}{dx}x = u(\ln u - 1).$$

We obtain the separable equation:

$$\frac{du}{u(\ln u - 1)} = \frac{dx}{x}.$$

The next step is to integrate the left and the right side of the equation:

$$\int \frac{du}{u(\ln u - 1)} = \int \frac{dx}{x}, \Rightarrow \int \frac{d(\ln u)}{\ln u - 1} = \int \frac{dx}{x}, \Rightarrow \int \frac{d(\ln u - 1)}{\ln u - 1} = \int \frac{dx}{x}.$$

Hence,

$$\ln|\ln u - 1| = \ln|x| + C.$$

Here the constant C can be written as $\ln C_1$, where $C_1 > 0$. Then

$$\ln|\ln u - 1| = \ln|x| + \ln C_1, \Rightarrow \ln|\ln u - 1| = \ln|C_1 x|, \Rightarrow \ln u - 1 = \pm C_1 x \Rightarrow$$
$$\ln u = 1 \pm C_1 x \ \text{ or } \ u = e^{1 \pm C_1 x}.$$

Thus, we have got two solutions:

$$u = e^{1 + C_1 x} \ \text{ and } \ u = e^{1 - C_1 x}.$$

If $C_1 = 0$, the answer is $y = xe$ and we can make sure that it is also a solution to the equation. Indeed, substituting

$$y = xe, y' = e$$

into the differential equation, we obtain:

$$xe = xe\ln\frac{xe}{x}, \Rightarrow xe = xe\ln e, \Rightarrow xe = xe.$$

Then all the solutions can be represented by one formula:

$$y = xe^{1 + Cx},$$

where C is an arbitrary real number.

Example 3. Solve the differential equation $(xy + y^2)y' = y^2$.

Here we deal with a homogeneous equation. In fact, we can rewrite it in the form:

$$y' = \frac{y^2}{xy + y^2} = \frac{\frac{y^2}{x^2}}{\frac{xy}{x^2} + \frac{y^2}{x^2}} = \frac{\left(\frac{y}{x}\right)^2}{\frac{y}{x} + \left(\frac{y}{x}\right)^2} = f\left(\frac{y}{x}\right).$$

Make the substitution $y = ux$. Then $y' = (ux)' = u'x + u$. Substituting y and y' into the original equation, we have:

$$(xux + u^2x^2)(u'x + u) = u^2x^2, \Rightarrow$$

$$ux^2(u + 1)(u'x + u) = u^2x^2.$$

Divide both sides by ux^2. We notice that $x = 0$ is not the solution of the equation. However, one can check that $u = 0$ or $y = 0$ is one of the solutions of the differential equation.

As a result, we have a separable equation:

$$(u + 1)(u'x + u) = u, \Rightarrow u'x(u + 1) + u^2 + u = u, \Rightarrow$$

$$u'x(u + 1) = -u^2, \Rightarrow \left(\frac{1}{u} + \frac{1}{u^2}\right) du = -\frac{dx}{x}.$$

Integrating the left and the right side of the equation, we find a general solution as follows:

$$\int \left(\frac{1}{u} + \frac{1}{u^2}\right) du = -\int \frac{dx}{x}, \Rightarrow \ln|u| - \frac{1}{u} = -\ln|x| + C.$$

Taking into account that $u = \frac{y}{x}$, we can write the last expression in the form

$$\ln\left|\frac{y}{x}\right| - \frac{1}{\frac{y}{x}} = -\ln|x| + C, \Rightarrow \ln|y| - \cancel{\ln|x|} - \frac{x}{y} = -\cancel{\ln|x|} + C, \Rightarrow$$

$$y\ln|y| = Cy + x.$$

The given expression can be represented as an explicit inverse function $x(y)$:

$$x = y\ln|y| - Cy.$$

Since C is an arbitrary real number, we can replace the “minus” sign before the constant to the “plus” sign. Then

$$x = y\ln|y| + Cy.$$

Thus, the given differential equation has the solutions:

$$x = y\ln|y| + Cy, y = 0.$$

Example 4. Solve the differential equation $y' = \frac{y}{x} - \frac{x}{y}$.

As it follows from the right side of the equation, $x \neq 0$ and $y \neq 0$. We can make the substitution $y = ux$ and $y' = u'x + u$. This yields:

$$u'x + u = \frac{ux}{x} - \frac{x}{ux}, \Rightarrow u'x + \cancel{u} = \cancel{u} - \frac{1}{u}, \Rightarrow \frac{du}{dx}x = -\frac{1}{u}, \Rightarrow udu = -\frac{dx}{x}.$$

Integrating this separable equation, we obtain:

$$\int udu = -\int \frac{dx}{x}, \Rightarrow \frac{u^2}{2} = -\ln|x| + C, \Rightarrow u^2 = 2C - 2\ln|x|.$$

Let the constant $2C$ be denoted by just C. Hence,

$$u^2 = C - 2\ln|x| \quad \text{or} \quad u = \pm\sqrt{C - 2\ln|x|}.$$

Thus, the general solution is written in the form

$$y = ux = \pm x\sqrt{C - 2\ln|x|}.$$

Example 5. Solve the differential equation $(x^3 + xy^2)y'$

It is easy to see that the given equation is homogeneous. Therefore, we can use the substitution $y = ux$ and $y' = u'x + u$. As a result, the equation is converted into the separable differential equation:

$$(x^3 + x(ux)^2)(u'x + u) = (ux)^3, \Rightarrow$$

$$(x^3 + x^3u^2)(u'x + u) = u^3x^3, \Rightarrow$$

$$x^3(1 + u^2)(u'x + u) = u^3x^3.$$

Divide both sides by x^3: (we notice that $x = 0$ is not the solution of the equation)

$$(1 + u^2)(u'x + u) = u^3, \Rightarrow (1 + u^2)u'x + u + \cancel{u^3} = \cancel{u^3}, \Rightarrow$$

$$(1 + u^2)u'x = -u, \Rightarrow \frac{(1 + u^2)du}{u} = -\frac{dx}{x}, \Rightarrow$$

$$\left(\frac{1}{u} + u\right)du = -\frac{dx}{x}.$$

Now we can integrate the last equation:

$$\int \left(\frac{1}{u} + u\right)du = -\int \frac{dx}{x}, \Rightarrow \ln|u| + \frac{u^2}{2} = -\ln|x| + C.$$

Since $u = \frac{y}{x}$, the solution can be written in the form:

$$\ln\left|\frac{y}{x}\right| + \frac{y^2}{2x^2} = -\ln|x| + C, \Rightarrow \ln|y| - \cancel{\ln|x|} + \frac{y^2}{2x^2} = -\cancel{\ln|x|} + C, \Rightarrow$$

$$\ln|y| = C - \frac{y^2}{2x^2}.$$

It follows from here that

$$y = e^{C - \frac{y^2}{2x^2}} = e^C e^{-\frac{y^2}{2x^2}}.$$

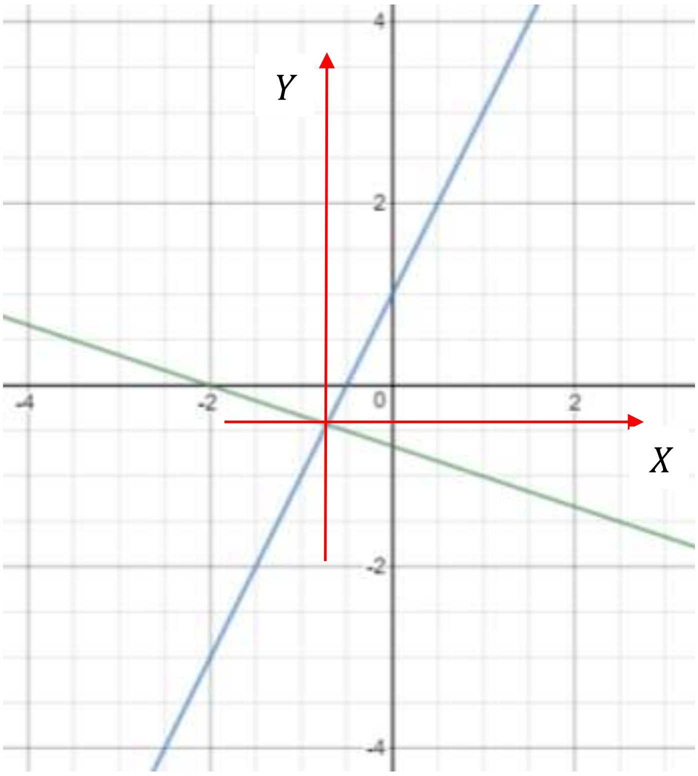
We can denote $e^C = C_1, C_1 > 0$ Then the solution in the implicit form is given by the equation

$$y = C_1 e^{-\frac{y^2}{2x^2}},$$

where the constant $C_1 > 0$.

Example 6. Solve the differential equation $y' = \frac{2x+1}{3y+x+2}$.

Here the numerator and denominator are the equations of intersecting straight lines. This differential equation can be converted into homogeneous after transformation of coordinates. Let the new and the old coordinates be connected by the relations



$$x = X + \alpha, y = Y + \beta.$$

We determine the constants α and β later. By substituting these relations into the equation, we get

$$y' = \frac{dy}{dx} = \frac{d(Y+\beta)}{d(X+\alpha)} = \frac{dY}{dX}$$

The differential equation in the new coordinates becomes

$$\frac{dY}{dX} = \frac{2(X+\alpha)+1}{3(Y+\beta)+X+\alpha+2} = \frac{2X+2\alpha+1}{3Y+X+\alpha+3\beta+2}.$$

This equation will be homogeneous if we choose the constants α and β such that

$$\begin{cases} 2\alpha + 1 = 0 \\ \alpha + 3\beta + 2 = 0 \end{cases}.$$

Solving the last system, we find $\begin{cases} \alpha = -\frac{1}{2} \\ \beta = -\frac{1}{2} \end{cases}$

At these values, the differential equation is written in the following way:

$$\frac{dY}{dX} = \frac{2X}{3Y + X}.$$

This equation is homogeneous, so we can make the replacement $Y = uX,$ where u is a function of X . Hence, $dY = Xdu + udX$. As a result, we have

$$\frac{Xdu + udX}{dX} = \frac{2X}{3Y + X}, \Rightarrow X\frac{du}{dX} + u = \frac{2}{3u + 1}.$$

We divided the numerator and denominator in the right side by X. We can check that $X = 0$ or $x = X + \alpha = -\frac{1}{2}$ is not the solution of the differential equation.

Some easy transformations give

$$X\frac{du}{dX} = \frac{2}{3u + 1} - u, \Rightarrow X\frac{du}{dX} = \frac{2 - 3u^2 - u}{3u + 1}.$$

We can factor the quadratic function in the numerator of the right side into the product of monomials:

$$2 - 3u^2 - u = 0, \Rightarrow D = 1 - 4 \cdot (-3) \cdot 2 = 25, \Rightarrow$$

$$u_{1,2} = \frac{1 \pm \sqrt{25}}{-6} = \frac{1 \pm 5}{-6} = -1 \text{ and } \frac{2}{3}.$$

Hence,

$$2 - 3u^2 - u = -3(u + 1)(u - \frac{2}{3}) = (u + 1)(2 - 3u).$$

Then,

$$X\frac{du}{dX} = \frac{(u + 1)(2 - 3u)}{3u + 1}.$$

By separating the variables, we can write:

$$\frac{3u + 1}{(u + 1)(2 - 3u)}du = \frac{dX}{X}.$$

Integrate the given equation:

$$\int \frac{3u + 1}{(u + 1)(2 - 3u)}du = \int \frac{dX}{X}.$$

Now we should transform the integrand in the left side. We use the method of uncertain coefficients (partial fraction decomposition) to represent the integrand as the sum of the right rational fractions:

$$\frac{3u+1}{(u+1)(2-3u)}=\frac{A}{u+1}+\frac{B}{2-3u},\Rightarrow$$

$$3u+1=A(2-3u)+B(u+1),\Rightarrow 3u+1=2A-3Au+Bu+B,\Rightarrow$$

$$3u+1=(B-3A)u+2A+B.$$

Hence,

$$\begin{cases} B-3A=3 \\ 2A+B=1 \end{cases},\Rightarrow \begin{cases} A=-\dfrac{2}{5} \\ B=\dfrac{9}{5} \end{cases}.$$

As a result, the differential equation is written as follows:

$$-\frac{2}{5}\int\frac{du}{u+1}+\frac{9}{5}\int\frac{du}{2-3u}=\int\frac{dX}{X}.$$

Upon integrating both sides, we get

$$-\frac{2}{5}\ln|u+1|+\frac{9}{5}\cdot\left(-\frac{1}{3}\right)\ln|2-3u|=\ln|X|+\ln C,$$

$$\Rightarrow -\frac{2}{5}\ln|u+1|-\frac{3}{5}\ln|2-3u|=\ln|X|+\ln C,$$

where the constant C is a positive real number.

Re-write the solution in terms of the variables X and Y:

$$-\frac{2}{5}\ln\left|\frac{Y}{X}+1\right|-\frac{3}{5}\ln\left|2-3\frac{Y}{X}\right|=\ln|X|+\ln C,$$

$$\Rightarrow -\frac{2}{5}\ln\left|\frac{Y+X}{X}\right|-\frac{3}{5}\ln\left|\frac{2X-3Y}{X}\right|=\ln|X|+\ln C,$$

$$\Rightarrow -\frac{2}{5}\ln|Y+X|+\cancel{\frac{2}{5}\ln|X|}-\frac{3}{5}\ln|2X-3Y|+\cancel{\frac{3}{5}\ln|X|}=\cancel{\ln|X|}+\ln C,$$

$$\Rightarrow 2\ln|Y+X|+3\ln|2X-3Y|=-5\ln C.$$

Further, it is convenient to denote $-5\ln C=\ln C_1$, where C_1 is an arbitrary positive number. Thus, we can write the solution in the form:

$$2\ln|Y+X|+3\ln|2X-3Y|=\ln C_1.$$

Now we return to the initial variables x, y. As

$$X=x-\alpha=x+\frac{1}{2}, Y=y-\beta=y+\frac{1}{2},$$

we obtain

$$2\ln\left|y+\frac{1}{2}+x+\frac{1}{2}\right|+3\ln\left|2\left(x+\frac{1}{2}\right)-3\left(y+\frac{1}{2}\right)\right|=\ln C_1,$$

$$\Rightarrow 2\ln|x+y+1| + 3\ln|2x-3y-\frac{1}{2}| = \ln C_1,$$
$$\Rightarrow 2\ln|x+y+1| + 3\ln|\frac{4x-6y-1}{2}| = \ln C_1,$$
$$\Rightarrow 2\ln|x+y+1| + 3\ln|4x-6y-1| - 3\ln 2 = \ln C_1,$$
$$\Rightarrow \ln|(x+y+1)^2 \cdot (4x-6y-1)^3| = \ln C_1 + 3\ln 2.$$

The right side can be written again in simpler form:

$$\ln C_1 + 3\ln 2 = \ln C_2 \ \ (C_2 > 0).$$

Then the final general solution of the original differential equation in the implicit form is given by

$$(x+y+1)^2 \cdot (4x-6y-1)^3 = \pm C_2 = C_3.$$

where C_3 is any nonzero number.

Example 7. Find the general solution of the differential equation $y' = \frac{x-y+3}{x-y}$.

We can notice that the equations of the lines in the numerator and denominator correspond to the parallel straight lines. Therefore we make the following change of variables:

$$z = x-y, \Rightarrow y = x-z, y' = 1-z'.$$

As a result, the differential equation becomes

$$1-z' = \frac{z+3}{z}, \Rightarrow 1-z' = 1+\frac{3}{z}, \Rightarrow z' = -\frac{3}{z}.$$

We get the simple separable equation. By solving it, we find the answer:

$$\frac{dz}{dx} = -\frac{3}{z}, \Rightarrow zdz = -3dx, \Rightarrow$$
$$\int zdz = -3\int dx, \Rightarrow$$
$$\frac{z^2}{2} = -3x + C, \Rightarrow (x-y)^2 = 2C - 6x.$$

We can derive the explicit function $y(x)$ from the last expression:

$$x - y = \pm\sqrt{2C-6x}.$$

Thus,

$$y = x \pm \sqrt{C-6x}.$$