3. *Linear Differential Equations*

A differential equation of type

$$y' + a(x)y = f(x),$$

where $a(x)$ and $f(x)$ are continuous functions of x, is called a *linear nonhomogeneous differential equation* of the first order.

Example 1. Solve the differential equation $y' - 2y = x$.

A. **First** we solve this problem using an integrating factor. The given equation is already written in the standard form. Therefore,

$$a(x) = -2.$$

Then the integrating factor is

$$u(x) = \exp\left(\int a(x)dx\right) = \exp\left(\int (-2)dx\right) = \left\{\int (-2)dx = -2x\right\} = e^{-2x}.$$

The general solution of the original differential equation has the form:

$$y = \frac{\int u(x)f(x)dx + C}{u(x)} = \frac{\int e^{-2x}xdx + C}{e^{-2x}}.$$

We calculate the last integral with help of integration by parts.

$$\int \underbrace{e^{-2x}}_{q'} \underbrace{x}_{p} dx = \left[\begin{array}{c} \int q'pdx = qp - \int q'pdx \\ p = x, \ \ p' = 1 \\ q' = e^{-2x}, \Rightarrow q = \int e^{-2x}dx = -\frac{1}{2}e^{-2x} \end{array}\right] =$$

$$= -\frac{x}{2}e^{-2x} - \int 1 \cdot (-\frac{1}{2}e^{-2x})dx = -\frac{x}{2}e^{-2x} + \frac{1}{2}\int e^{-2x}dx = -\frac{x}{2}e^{-2x} - \frac{1}{4}e^{-2x}$$

$$= -\frac{1}{4}e^{-2x}(1 + 2x).$$

Then,

$$y = \frac{-\frac{1}{4}e^{-2x}(1 + 2x) + C}{e^{-2x}} = -\frac{1}{4}(1 + 2x) + Ce^{2x}.$$

B. **Second**, we construct the solution by the method of variation of a constant. Consider the corresponding homogeneous equation:

$$y' - 2y = 0$$

and find its general solution.

$$\frac{dy}{dx} = 2y, \Rightarrow \frac{dy}{y} = 2dx, \Rightarrow \int \frac{dy}{y} = 2\int dx, \Rightarrow$$

$$\ln|y| = 2x + C, \Rightarrow |y| = e^{2x+C} = e^{2x}e^{C} = C_1e^{2x}, \Rightarrow$$

$$y = \pm C_1e^{2x} = Ce^{2x},$$

where C again denotes any real number. Notice that at $C = 0$, we get $y = 0$ that is also a solution of the homogeneous equation.

Next we suppose that C is a function of x and substitute the solution $y = C(x)e^{2x}$ into the initial nonhomogeneous equation. We can write

$$y' = [C(x)e^{2x}]' = C'(x)e^{2x} + C(x) \cdot 2e^{2x}.$$

Hence,

$$C'(x)e^{2x} + \cancel{2C(x)e^{2x}} - \cancel{2C(x)e^{2x}} = x, \Rightarrow C'(x) = e^{-2x}x, \Rightarrow C(x) = \int e^{-2x}xdx.$$

This integral was already found above in section *A*, so we obtain

$$C(x) = -\frac{1}{4}e^{-2x}(1 + 2x) + C.$$

As a result, the general solution of the nonhomogeneous differential equation is given by

$$y = C(x)e^{2x} = [-\frac{1}{4}e^{-2x}(1 + 2x) + C]e^{2x} = -\frac{1}{4}(1 + 2x) + Ce^{2x}.$$

As one can see, both methods give the same answer.

Example 2. Solve the differential equation $x^2y' + xy + 2 = 0$.

We solve this problem using the method of variation of a constant. For convenience, we write this equation in the standard form:

$$y' + \frac{y}{x} = -\frac{2}{x^2}.$$

Herein, we divided both sides by x^2. Obviously, that $x = 0$ is not the solution of the equation.

Consider the homogeneous equation:

$$y' + \frac{y}{x} = 0, \Rightarrow \frac{dy}{dx} = -\frac{y}{x}, \Rightarrow \frac{dy}{y} = -\frac{dx}{x}, \Rightarrow \int \frac{dy}{y} = -\int \frac{dx}{x}, \Rightarrow$$

$$\ln|y| = -\ln|x| + \ln C_1 (C_1 > 0), \Rightarrow \ln|y| = \ln\frac{C_1}{|x|}, \Rightarrow y = \frac{C_1}{|x|}.$$

After easy transformations we find the answer $y = \frac{C}{x}$ where C is any real number. The last expression includes the case $y = 0$, which is also a solution of the homogeneous equation.

Now we replace the constant C with the function $C(x)$ and substitute the solution $y = \frac{C(x)}{x}$ into the initial nonhomogeneous differential equation. As

$$y' = [\frac{C(x)}{x}]' = \frac{C'(x) \cdot x - C(x)}{x^2},$$

we obtain

$$\frac{C'(x) \cdot x - C(x)}{x^2} + \frac{C(x)}{x^2} = -\frac{2}{x^2}, \Rightarrow \frac{C'(x)}{x} - \frac{C(x)}{x^2} + \frac{C(x)}{x^2} = -\frac{2}{x^2}, \Rightarrow C'(x) = -\frac{2}{x},$$

$$\Rightarrow C(x) = -\int \frac{2}{x} dx = -2\ln|x| + C.$$

Thus, the general solution of the initial equation is given by

$$y = \frac{C(x)}{x} = -\frac{2\ln|x|}{x} + \frac{C}{x}.$$

Example 3. Solve the initial value problem $y' - y\tan x = \sin x, y(0) = 1$.

First of all we calculate the integrating factor, which is written as

$$u(x) = e^{\int (-\tan x) dx} = e^{-\int \tan x dx}.$$

In doing so, we need to find

$$\int \tan x dx = \int \frac{\sin x}{\cos x} dx = -\int \frac{d(\cos x)}{\cos x} = -\ln|\cos x|.$$

Hence, the integrating factor is given by

$$u(x) = e^{-\int \tan x dx} = e^{\ln|\cos x|} = |\cos x|.$$

We can take the function $u(x) = \cos x$ as the integrating factor. Make sure that the left side of the equation is the derivative of the product $y(x)u(x)$:

$$(y' - y\tan x)\cos x = y'\cos x - y\tan x \cos x = y'\cos x - y\sin x = (y\cos x)' = [y(x)u(x)]'.$$

Then the general solution of the equation is written in the form:

$$y(x) = \frac{1}{u(x)} \cdot [\int u(x)\sin x dx + C] = \frac{1}{\cos x} \cdot [\int \cos x \sin x dx + C] = \frac{1}{2\cos x}\int \sin 2x dx + \frac{C}{\cos x} = \frac{C}{\cos x} - \frac{\cos 2x}{4\cos x}.$$

Next, we determine the value of C, which satisfies the initial condition $y(0) = 1$

$$y(0) = \frac{C}{\cos 0} - \frac{\cos 0}{4\cos 0} = C - \frac{1}{4} = 1,$$

So,

$$C = \frac{5}{4}.$$

Hence, the solution for the initial value problem is given by

$$y(x) = \frac{5}{4\cos x} - \frac{\cos 2x}{4\cos x} = \frac{5 - \cos 2x}{4\cos x}.$$

Example 4. Solve the initial value problem $y' + \frac{3}{x}y = \frac{2}{x^2}$, $y(1) = 2$.

Determine the integrating factor:

$$u(x) = e^{\int \frac{3}{x}dx} = e^{3\int \frac{dx}{x}} = e^{3\ln|x|} = e^{\ln|x|^3} = |x|^3.$$

We can take the function $u(x) = x^3$ as the integrating factor. One can check that the left side of the equation is the derivative of the product $y(x)u(x)$, i.e.

$$\left(y' + \frac{3}{x}y\right)x^3 = y'x^3 + \frac{3}{x}yx^3 = y'x^3 + 3yx^2 = (yx^3)'.$$

The general solution of the differential equation is written as

$$y = \frac{\int u(x)f(x)dx + C}{u(x)} = \frac{\int x^3 \cdot \frac{2}{x^2}dx + C}{x^3} = \frac{\int 2xdx + C}{x^3} = \frac{x^2 + C}{x^3} = \frac{1}{x} + \frac{C}{x^3}.$$

Now we can find the constant C using the initial condition $y(1) = 2$. Substituting the general solution into this condition gives:

$$y(1) = \frac{1}{1} + \frac{C}{1^3} = 2, \Rightarrow C = 1.$$

Thus, the solution of the IVP is given by

$$y = \frac{1}{x} + \frac{1}{x^3}.$$

Example 5. Solve the differential equation $y = (2y^4 + 2x)y'$.

One can see that this equation is not linear with respect to the function $y(x)$ However, we can try to find the solution for the inverse function $x(y)$. We write the given equation in terms of differentials and make some transformations:

$$y = (2y^4 + 2x)\frac{dy}{dx}, \Rightarrow ydx = 2y^4dy + 2xdy, \Rightarrow$$

$$y\frac{dx}{dy} = 2y^4 + 2x, \Rightarrow \frac{dx}{dy} - \frac{2}{y}x = 2y^3.$$

Now we see that we have a linear differential equation with respect to the function $x(y)$. We can solve it with help of the integrating factor:

$$u(y) = e^{\int (-\frac{2}{y})dy} = e^{-2\int \frac{dy}{y}} = e^{-2\ln|y|} = e^{\ln\frac{1}{|y|^2}} = e^{\ln\frac{1}{y^2}} = \frac{1}{y^2}.$$

Then the general solution as the inverse function $x(y)$ is expressed in the form

$$x(y) = \frac{\int u(y)f(y)dy + C}{u(y)} = \frac{\int \frac{1}{y^2} \cdot 2y^3 dy + C}{\frac{1}{y^2}} = \frac{\int 2y dy + C}{\frac{1}{y^2}} = y^2(y^2 + C).$$