4. *Bernoulli Equations*

Bernoulli equation is one of the well-known nonlinear differential equations of the first order. It is written as

$$y' + a(x)y = b(x)y^m,$$

where $a(x)$ and $b(x)$ are continuous functions.

Example 1. Solve the differential equation $y' + \frac{y}{x} = y^2$.

As it can be seen, this differential equation is a Bernoulli equation. To solve it, we make the substitution

$$z = y^{1-m} = \frac{1}{y}.$$

Differentiating, we find:

$$z' = \left(\frac{1}{y}\right)' = -\frac{y'}{y^2}.$$

Divide the original equation by y^2 and replace y with z

$$\frac{y'}{y^2} + \frac{1}{yx} = 1.$$

When dividing by y^2 we have lost the solution $\mathrm{y} = 0$.

In terms of z the differential equation is written in the form:

$$-z' + \frac{z}{x} = 1 \quad \text{or} \quad z' - \frac{z}{x} = -1.$$

We get the linear equation for the function $z(x)$, so we can solve it using the integrating factor technique:

$$u(x) = e^{\int (-\frac{1}{x})dx} = e^{-\int \frac{dx}{x}} = e^{-\ln|x|} = e^{\ln\frac{1}{|x|}} = \frac{1}{|x|}.$$

We can make sure that the function $\frac{1}{x}$ is the integrating factor. Indeed:

$$z' \cdot \frac{1}{x} - \frac{z}{x} \cdot \frac{1}{x} = z' \cdot \frac{1}{x} - \frac{z}{x^2} = \left(z \cdot \frac{1}{x}\right)'.$$

We see that the left side of the equation becomes the derivative of the product $z(x)u(x)$ after multiplying by $\frac{1}{x}$.

Then the general solution of the linear equation for $z(x)$ is given by

$$z = \frac{\int u(x)f(x)dx + C}{u(x)} = \frac{\int \frac{1}{x} \cdot (-1)dx + C}{\frac{1}{x}} = \frac{-\ln|x| + C}{\frac{1}{x}} = x(C - \ln|x|).$$

Taking into account that $y = \frac{1}{z}$, we can write the answer:

$$y = \frac{1}{x(C - \ln|x|)},$$

or in the implicit form:

$$yx(C - \ln|x|) = 1.$$

Thus, the final answer is

$$yx(C - \ln|x|) = 1, y = 0.$$

Example 2. Solve the differential equation $y' + y\cot x = y^4 \sin x$

This is a Bernoulli equation with the parameter $m = 4$. Therefore, we make the substitution

$$z = y^{1-m} = y^{-3}.$$

The derivative is given by

$$z' = (y^{-3})' = -3y^{-4}y' = -\frac{3y'}{y^4}$$

Multiply both sides of the original equation by (–3) and divide by y^4:

$$y' + y\cot x = y^4 \sin x, \Rightarrow -\frac{3y'}{y^4} - \frac{3\cot x}{y^3} = -3\sin x.$$

Notice that in dividing by y^4 we have lost the solution $y = 0$. (You can check this by direct substitution.)

Rewriting the last equation in terms of z, we get

$$z' - 3\cot x \cdot z = -3\sin x.$$

This differential equation is linear, so we can solve it using the integrating factor:

$$u(x) = e^{\int(-3)\cot x dx} = e^{-3\int \cot x dx} = e^{-3\int \frac{\cos x dx}{\sin x}} = e^{-3\int \frac{d(\sin x)}{\sin x}} = e^{-3\ln|\sin x|}$$

$$= e^{\ln\frac{1}{|\sin x|^3}} = \frac{1}{|\sin x|^3}.$$

We can take the function $u(x) = \frac{1}{\sin^3 x}$ as the integrating factor. In fact, the left side of the equation becomes the derivative of the product $z(x)u(x)$ after multiplying by $u(x)$

$$z'\cdot\frac{1}{\sin^3 x}-3\cot x\cdot z\cdot\frac{1}{\sin^3 z}=z'\frac{1}{\sin^3 z}-\frac{3z\cos x}{\sin^4 x}=\left(z\frac{1}{\sin^3 x}\right)'.$$

Hence, the general solution of the linear differential equation for $z(x)$ can be presented in the form:

$$z=\frac{\int u(x)f(x)dx+C}{u(x)}=\frac{\int\frac{1}{\sin^3 x}(-3\sin x)dx+C}{\frac{1}{\sin^3 x}}=\frac{-3\int\frac{dx}{\sin^2 x}+C}{\frac{1}{\sin^3 x}}$$
$$=(3\cot x+C)\sin^3 x.$$

Since $z=y^{-3}$, we obtain the following solutions of the given Bernoulli equation:

$$\frac{1}{y^3}=(3\cot x+C)\sin^3 x, y=0.$$

Example 4. Find all solutions of the differential equation $y'+\frac{2y}{x}=2x\sqrt{y}$.

This equation is also a Bernoulli equation with the fractional parameter $m=\frac{1}{2}$. It can be reduced to the linear equation by making the replacement $z=y^{1-m}=\sqrt{y}$. The derivative of the new function $z(x)$ is given by

$$z'=(\sqrt{y})'=\frac{y'}{2\sqrt{y}}$$

Divide the original Bernoulli equation by $2\sqrt{y}$. Like in other examples on this page, the root $y=0$ is also the trivial solution of the differential equation. So we have

$$y'+\frac{2y}{x}=2x\sqrt{y},\Rightarrow\frac{y'}{2\sqrt{y}}+\frac{\cancel{2}y}{\cancel{2}x\sqrt{y}}=\frac{\cancel{2}x\sqrt{y}}{\cancel{2}\sqrt{y}},\Rightarrow\frac{y'}{2\sqrt{y}}+\frac{\sqrt{y}}{x}=x.$$

Replacing y with z, we get

$$z'+\frac{z}{x}=x.$$

We obtain a simple linear equation for the function $z(x)$. The integrating factor here is

$$u(x)=e^{\int\frac{1}{x}dx}=e^{\ln|x|}=|x|.$$

We choose the function $u(x)=x$. One can check that the left side of the equation becomes the derivative of the product $z(x)u(x)$ after multiplying by $u(x)$

$$z'\cdot x+\frac{z}{x}\cdot x=z'x+z=(zx)'.$$

Then the general solution of the linear differential equation is given by

$$z = \frac{\int u(x)f(x)dx + C}{u(x)} = \frac{\int x \cdot xdx + C}{x} = \frac{\int x^2 dx + C}{x} = \frac{\frac{x^3}{3} + C}{x}.$$

Returning to the original function $y(x)$, we get the solution in the implicit form:

$$\sqrt{y} = \frac{\frac{x^3}{3} + C}{x} \quad \text{or} \quad x\sqrt{y} = \frac{x^3}{3} + C.$$

Thus, the full answer is written as follows:

$$x\sqrt{y} = \frac{x^3}{3} + C, y = 0.$$

5. ***Exact Differential Equations***

Definition. A differential equation of type

$$P(x,y)dx + Q(x,y)dy = 0$$

is called an exact differential equation if there exists a function of two variables $u(x,y)$ with continuous partial derivatives such that

$$du(x,y) = P(x,y)dx + Q(x,y)dy.$$

The general solution of an exact equation is given by

$$u(x,y) = C,$$

where C is an arbitrary constant.

Example 1. Find the solution of the differential equation $(6x^2 - y + 3)dx + (3y^2 - x - 2)dy = 0$

We check this equation for exactness:

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(3y^2 - x - 2) = -1, \frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(6x^2 - y + 3) = -1.$$

Hence, the given differential equation is exact. Write the system of equations to determine the function $u(x,y)$:

$$\begin{cases} \dfrac{\partial u}{\partial x} = P(x,y) = 6x^2 - y + 3 \\ \dfrac{\partial u}{\partial y} = Q(x,y) = 3y^2 - x - 2 \end{cases}$$

Integrate the first equation with respect to the variable x assuming that y is a constant. This produces:

$$u(x,y) = \int (6x^2 - y + 3)dx = \frac{6x^3}{3} - xy + 3x + \varphi(y) = 2x^3 - xy + 3x + \varphi(y).$$

Here we introduced a continuous differentiable function $\varphi(y)$ instead of the constant C.

Plug in the function $u(x,y)$ into the second equation:

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}[2x^3 - xy + 3x + \varphi(y)] = -x + \varphi'(y) = 3y^2 - x - 2.$$

We get equation for the derivative $\varphi'(y)$:

$$\varphi'(y) = 3y^2 - 2.$$

Integrating gives the function $\varphi(y)$

$$\varphi(y) = \int (3y^2 - 2)dy = y^3 - 2y.$$

So, the function $u(x, y)$ is given by

$$u(x, y) = 2x^3 - xy + 3x + y^3 - 2y.$$

Hence, the general solution of the equation is defined by the following implicit expression:

$$2x^3 - xy + 3x + y^3 - 2y = C,$$

where C is an arbitrary constant.

Example 2. Find the solution of the differential equation $e^y dx + (2y + xe^y)dy = 0$

First, we check this equation for exactness:

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(2y + xe^y) = e^y, \frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(e^y) = e^y.$$

We see that $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ o that this equation is exact. Find the function $u(x, y)$ from the system of equations:

$$\begin{cases} \dfrac{\partial u}{\partial x} = e^y \\ \dfrac{\partial u}{\partial y} = 2y + xe^y \end{cases}.$$

Integrate the first equation with respect to the variable x assuming that y is a constant, it produces:

$$u(x, y) = \int P(x, y)dx = \int e^y dx = xe^y + \varphi(y).$$

Now, by differentiating this expression with respect to y and equating it to $\frac{\partial u}{\partial y}$ we find the derivative $\varphi'(y)$:

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}[xe^y + \varphi(y)] = 2y + xe^y, \Rightarrow \cancel{xe^y} + \varphi'(y) = 2y + \cancel{xe^y}, \Rightarrow \varphi'(y) = 2y.$$

As a result, we find $\varphi(y)$ and the entire function $u(x, y)$:

$$\varphi(y) = \int 2y dy = y^2, \Rightarrow u(x, y) = xe^y + \varphi(y) = xe^y + y^2.$$

Hence, the general solution of the equation is defined by the following implicit expression:

$$xe^y + y^2 = C.$$

where C is an arbitrary constant.

Example 3. Find the solution of the differential equation $(2xy - \sin x)dx + (x^2 - \cos y)dy = 0$

This differential equation is exact because

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(x^2 - \cos y) = 2x = \frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(2xy - \sin x) = 2x.$$

We find the function $u(x, y)$ from the system of two equations:

$$\begin{cases} \frac{\partial u}{\partial x} = 2xy - \sin x \\ \frac{\partial u}{\partial y} = x^2 - \cos y \end{cases}.$$

By integrating the 1st equation with respect to the variable x, we have

$$u(x, y) = \int (2xy - \sin x)dx = x^2y + \cos x + \varphi(y).$$

Plugging in the 2nd equation, we obtain

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}[x^2y + \cos x + \varphi(y)] = x^2 - \cos y, \Rightarrow \cancel{x^2} + \varphi'(y) = \cancel{x^2} - \cos y, \Rightarrow \varphi'(y) = -\cos y.$$

Hence,

$$\varphi(y) = \int (-\cos y)dy = -\sin y.$$

Thus, the function $u(x, y)$ is

$$u(x, y) = x^2y + \cos x - \sin y,$$

so that the general solution of the differential equation is given by the implicit formula:

$$x^2y + \cos x - \sin y = C.$$

Example 4. Solve the equation $\left(1 + 2x\sqrt{x^2 - y^2}\right)dx - 2y\sqrt{x^2 - y^2}dy = 0$

First of all we determine whether this equation is exact:

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(-2y\sqrt{x^2 - y^2}) = -2y \cdot \frac{2x}{2\sqrt{x^2 - y^2}} = -\frac{2xy}{\sqrt{x^2 - y^2}},$$

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(1 + 2x\sqrt{x^2 - y^2}) = 2x \cdot \frac{(-2y)}{2\sqrt{x^2 - y^2}} = -\frac{2xy}{\sqrt{x^2 - y^2}}.$$

As you can see, $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$. Hence, this equation is exact. Find the function $u(x,y)$ satisfying the system of equations:

$$\begin{cases} \dfrac{\partial u}{\partial x} = 1 + 2x\sqrt{x^2 - y^2} \\ \dfrac{\partial u}{\partial y} = -2y\sqrt{x^2 - y^2} \end{cases}.$$

Integrating the first equation gives:

$$u(x,y) = \int (1 + 2x\sqrt{x^2 - y^2})dx = x + \frac{(x^2 - y^2)^{\frac{3}{2}}}{\frac{3}{2}} + \varphi(y)$$
$$= x + \frac{2}{3}(x^2 - y^2)^{\frac{3}{2}} + \varphi(y),$$

where $\varphi(y)$ is a certain unknown function of y that will be defined later.

We substitute the result into the second equation of the system:

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}[x + \frac{2}{3}(x^2 - y^2)^{\frac{3}{2}} + \varphi(y)] = -2y\sqrt{x^2 - y^2}, \Rightarrow -\cancel{2y\sqrt{x^2 - y^2}} + \varphi'(y)$$
$$= -\cancel{2y\sqrt{x^2 - y^2}}, \Rightarrow \varphi'(y) = 0.$$

By integrating the last expression, we find the function $\varphi(y) = C$, where C is a constant.

Thus, the general solution of the given differential equation has the form:

$$x + \frac{2}{3}(x^2 - y^2)^{\frac{3}{2}} + C = 0.$$

6. *Using an Integrating Factor*

Consider a differential equation of type

$$P(x,y)dx + Q(x,y)dy = 0,$$

where $P(x,y)$ and $Q(x,y)$ are functions of two variables x and y continuous in a certain region D. If

$$\frac{\partial Q}{\partial x} \neq \frac{\partial P}{\partial y},$$

the equation is *not exact*. However, we can try to find so-called *integrating factor*, which is a function $\mu(x,y)$ such that the equation becomes exact after multiplication by this factor. If so, then the relationship

$$\frac{\partial(\mu Q(x,y))}{\partial x} = \frac{\partial(\mu P(x,y))}{\partial y}$$

is valid. This condition can be written in the form:

$$Q\frac{\partial \mu}{\partial x} + \mu\frac{\partial Q}{\partial x} = P\frac{\partial \mu}{\partial y} + \mu\frac{\partial P}{\partial y}, \Rightarrow Q\frac{\partial \mu}{\partial x} - P\frac{\partial \mu}{\partial y} = \mu(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}).$$

The last expression is the *partial differential equation of first order* that defines the integrating factor $\mu(x,y)$.

Example 1. Solve the differential equation $(x - \cos y)dx \; - \sin y dy = 0$

If we test this equation for exactness, we find that

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(-\sin y) = 0, \qquad \frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(x - \cos y) = \sin y.$$

Hence, this equation is not exact. We try to construct an integrating factor. Notice that

$$\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = \sin y,$$

and the expression $\frac{1}{Q}(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}) = \frac{\sin y}{(-\sin y)} = -1$ is constant.

Hence, we can find the integrating factor as a function $\mu(x)$ by solving the following equation:

$$\frac{1}{\mu}\frac{d\mu}{dx} = -1, \Rightarrow \int \frac{d\mu}{\mu} = -\int dx, \Rightarrow \ln|\mu| = -x, \Rightarrow \mu = e^{\pm x}.$$

We choose the function $\mu = e^{-x}$ and make sure that the equation becomes exact after multiplication by $\mu = e^{-x}$

$$e^{-x}(x - \cos y)dx - e^{-x}\sin y dy = 0$$

So,

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(-e^{-x}\sin y) = e^{-x}\sin y, \qquad \frac{\partial P}{\partial y} = \frac{\partial}{\partial}(e^{-x}(x - \cos y)) = e^{-x}\sin y.$$

Its general solution can be found from the system of equations:

$$\begin{cases} \frac{\partial u}{\partial x} = e^{-x}(x - \cos y) \\ \frac{\partial u}{\partial y} = -e^{-x}\sin y \end{cases}.$$

Here it is more convenient to integrate the second equation with respect to y:

$$u(x,y) = \int(-e^{-x}\sin y)dy = e^{-x}\cos y + \psi(x).$$

Substituting this in the first equation, we have

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}[e^{-x}\cos y + \psi(x)] = e^{-x}(x - \cos y), \Rightarrow -\cancel{e^{-x}\cos y} + \psi'(x)$$
$$= xe^{-x} - \cancel{e^{-x}\cos y}, \Rightarrow \psi'(x) = xe^{-x}.$$

Integrating by parts gives:

$$\psi(x) = \int xe^{-x}dx = \left[\begin{array}{c} u = x \\ v' = e^{-x} \\ u' = 1 \\ v = -e^{-x} \end{array}\right] = -xe^{-x} - \int (-e^{-x})dx = -xe^{-x} + \int e^{-x}dx$$
$$= -xe^{-x} - e^{-x}.$$

Thus, the general solution of the equation is given by

$$e^{-x}\cos y - xe^{-x} - e^{-x} = C \quad \text{or} \quad e^{-x}(\cos y - x - 1) = C,$$

where C is an arbitrary real number.

Example 2. Solve the differential equation $(xy + 1)dx + x^2dy = 0$

First of all we check this equation for exactness:

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(x^2) = 2x \neq \frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(xy + 1) = x.$$

The partial derivatives are not equal to each other. Therefore, this equation is not exact. Calculate the difference of the derivatives:

$$\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = x - 2x = -x.$$

Now we try to use the integrating factor in the form $z = xy$. Here we have

$$\frac{\partial z}{\partial x} = y, \frac{\partial z}{\partial y} = x.$$

Then,

$$Q\frac{\partial z}{\partial x} - P\frac{\partial z}{\partial y} = x^2 \cdot y - (xy + 1) \cdot x = \cancel{x^2y} - \cancel{x^2y} - x = -x,$$

and hence we get

$$\frac{1}{\mu}\frac{d\mu}{dz} = \frac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{Q\frac{\partial z}{\partial x} - P\frac{\partial z}{\partial y}} = \frac{-x}{-x} = 1.$$

We see that the integrating factor depends only on z:

$$\mu(x, y) = \mu(z) = \mu(xy).$$

We can find it by integrating the last equation:

$$\frac{1}{\mu}\frac{d\mu}{dz} = 1, \Rightarrow \int \frac{d\mu}{\mu} = \int dz, \Rightarrow \ln|\mu| = z, \Rightarrow \mu = e^{\pm z} = e^{\pm xy}.$$

By choosing the function $\mu = e^{xy}$ we can convert the original differential equation into exact:

$$(xy+1)e^{xy}dx + x^2 e^{xy}dy = 0.$$

Check this using again the test for exactness:

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}[x^2 e^{xy}] = 2xe^{xy} + x^2 y e^{xy},$$

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}[(xy+1)e^{xy}] = xe^{xy} + (xy+1)xe^{xy} = xe^{xy} + x^2 y e^{xy} + xe^{xy}$$
$$= 2xe^{xy} + x^2 y e^{xy}.$$

As one can see, now this equation is exact. We find its general solution form the system of equations:

$$\begin{cases} \dfrac{\partial u}{\partial x} = (xy+1)e^{xy} \\ \dfrac{\partial u}{\partial y} = x^2 e^{xy} \end{cases}.$$

Integrate the second equation with respect to the variable y (considering x as a constant):

$$u(x,y) = \int x^2 e^{xy} dy = x^2 \int e^{xy} dy = x^2 \cdot \frac{1}{x} e^{xy} + \psi(x) = xe^{xy} + \psi(x).$$

Substitute this in the first equation of the system to get:

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}[xe^{xy} + \psi(x)] = (xy+1)e^{xy}, \Rightarrow 1 \cdot e^{xy} + xye^{xy} + \psi'(x)$$
$$= (xy+1)e^{xy}, \Rightarrow (xy+1)e^{xy} + \psi'(x) = (xy+1)e^{xy}, \Rightarrow \psi'(x) = 0,$$
$$\Rightarrow \psi(x) = C.$$

Hence, the general solution of the given differential equation is written in the form:

$$xe^{xy} + C = 0,$$

where C is an arbitrary real number.